

Uniqueness of meromorphic functions concerning derivatives and fixed points

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ABSTRACT: In this paper, we study the uniqueness of meromorphic function concerning derivatives and fixed points. We mainly prove: Let n, k be two positive integers with $n > 3k + 8$, and let f and g be two meromorphic functions all whose zeros and poles have multiplicity at least n . If $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ IM, then

- (1) $k = 1$, either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4c_1c_2c^2 = -1$, or $f \equiv g$;
- (2) $k \geq 2$, $f \equiv g$.

The result improves some results due to Fang-Qiu [J Math Anal Appl, 2002], Zhang [J Southeast Univ, 2004], Xu-Lü-Yi [Comput Math Appl, 2010] and Zhang [Comput Math Appl, 2008].

KEYWORDS: meromorphic functions, derivatives, fixed points, multiplicity

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INTRODUCTION AND MAIN RESULTS

In this paper, meromorphic always means meromorphic in the whole complex plane. We use the following standard notations in value distribution theory [1–4]: $T(r, f), N(r, f), m(r, f), \dots$

We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ possible outside of an exceptional set $E \in (0, \infty)$ with finite measure. A meromorphic function α is said to be a small function of f if it satisfies $T(r, \alpha) = S(r, f)$.

Let f and g be two nonconstant meromorphic functions, and let α be a small function of both f and g . If $f - \alpha$ and $g - \alpha$ have the same zeros counting multiplicities (ignoring multiplicities), then we call that f and g share α CM (IM). Moreover, we denote by $N(r, \alpha)$ be the counting function for common zeros of both $f - \alpha$ and $g - \alpha$ with the same multiplicities and the multiplicity is counted. If

$$N\left(r, \frac{1}{f-\alpha}\right) + N\left(r, \frac{1}{g-\alpha}\right) - 2N(r, \alpha) \leq S(r, f) + S(r, g),$$

then we call that f and g share α CM almost. If

$$\bar{N}\left(r, \frac{1}{f-\alpha}\right) + \bar{N}\left(r, \frac{1}{g-\alpha}\right) - 2\bar{N}(r, \alpha) \leq S(r, f) + S(r, g),$$

then we call that f and g share α IM almost.

Let f and g share 1 IM almost. We denote by $\bar{N}_L\left(r, \frac{1}{f-1}\right)$ the counting function for 1-points of both f and g about which f has larger multiplicity than g , with multiplicity being not counted. Similarly, we have the notation $\bar{N}_L\left(r, \frac{1}{g-1}\right)$. Especially, if f and g share 1 CM, then $\bar{N}_L\left(r, \frac{1}{f-1}\right) = \bar{N}_L\left(r, \frac{1}{g-1}\right) = 0$.

We denote by $N_{(k)}(r, f)$ the counting function for poles of f with multiplicity at least k , and by $\bar{N}_{(k)}(r, f)$ the corresponding one for which multiplicity is not counted. Set $N_k(r, f) = \bar{N}_{(k)}(r, f) + \bar{N}_{(2k)}(r, f) + \dots + \bar{N}_{(k)}(r, f)$.

Fang-Hua [5], Yang-Hua [6] obtained the following unicity theorem.

Theorem A Let f and g be two nonconstant entire functions, and let $n(\geq 6)$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv t g$, where t is a constant such that $t^{n+1} = 1$.

In 2002, Fang-Qiu [7] proved the following theorem.

Theorem B Let f and g be two nonconstant entire functions, and let $n(\geq 6)$ be a positive integer. If $f^n f'$ and $g^n g'$ share z CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv t g$, where t is a constant such that $t^{n+1} = 1$.

In 2002, Fang [8] obtained the following result.

Theorem C Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv t g$, where t is a constant such that $t^n = 1$.

In 2004, Zhang [9] proved the following theorem.

Theorem D Let n, k be two positive integers with $n > 2k + 4$, and let f and g be two nonconstant entire functions all whose zeros have multiplicity at least n . If $f^{(k)}$ and $g^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1 c_2 c^{2k} = 1$, or $f \equiv g$.

In 2008, Zhang [10] proved the following theorem.

Theorem E Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then

- (1) $k = 1$, either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^n (nc)^2 = -1$, or $f \equiv g$;
- (2) $k \geq 2$, $f \equiv tg$, where t is a constant such that $t^n = 1$.

From above theorems, we naturally ask the following problem.

Problem 1. Are Theorems A–E valid or not for meromorphic functions?

In 2010, Xu-Lú-Yi [11] studied the case of meromorphic functions and obtained the following theorems.

Theorem F Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 8$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv tg$, where t is a constant such that $t^n = 1$.

Theorem G Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^n (nc)^2 = -1$, or $f \equiv tg$, where t is a constant such that $t^n = 1$.

In this paper, we extend and improve Theorem F and Theorem G, and prove the following results.

Theorem 1 Let n, k be two positive integers with $n > 3k + 6$, and let f and g be two meromorphic functions all whose zeros and poles have multiplicity at least n . If $f^{(k)}$ and $g^{(k)}$ share 1 CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1 c_2 c^{2k} = 1$, or $f \equiv g$.

Remark 1 By Theorem 1, we get Theorem F, and improve the condition that $n > 3k + 8$ to $n > 3k + 6$.

Theorem 2 Let n, k be two positive integers with $n > 3k + 5$, and let f and g be two nonconstant meromorphic

functions all whose zeros and poles have multiplicity at least n . If $f^{(k)}$ and $g^{(k)}$ share 1 CM, f and g share ∞ CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2) c^{2k} = 1$, or $f \equiv g$.

Theorem 3 Let n, k be two positive integers with $n > 3k + 8$, and let f and g be two nonconstant meromorphic functions all whose zeros and poles have multiplicity at least n . If $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ IM, then

- (1) $k = 1$, either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4c_1 c_2 c^2 = -1$, or $f \equiv g$;
- (2) $k \geq 2$, $f \equiv g$.

Remark 2 By Theorem 3, we get Theorem G, and improve the condition that $n > 3k + 10$ to $n > 3k + 8$.

Theorem 4 Let n, k be two positive integers with $n > 3k + 7$, and let f and g be two nonconstant meromorphic functions all whose zeros and poles have multiplicity at least n . If $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ CM, then

- (1) $k = 1$, either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4c_1 c_2 c^2 = -1$, or $f \equiv g$;
- (2) $k \geq 2$, $f \equiv g$.

LEMMAS

Lemma 1 ([1]) Let f be a nonconstant meromorphic function, let k be a positive integer, and let c be a nonzero constant. Then

$$\begin{aligned} T(r, f) &\leq \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) \\ &\quad - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \\ &\leq \bar{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - c}\right) \\ &\quad - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f), \end{aligned}$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function for which $f^{(k+1)} = 0$ and $f(f^k - c) \neq 0$.

Lemma 2 ([1]) Let f be a nonconstant meromorphic function, and let k be a positive integer. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

Lemma 3 ([1]) Let f be a nonconstant meromorphic function, and let $\alpha_1, \alpha_2, \alpha_3$ (one may be ∞) be three distinct small functions of f . Then

$$\begin{aligned} T(r, f) &\leq \bar{N}\left(r, \frac{1}{f - \alpha_1}\right) + \bar{N}\left(r, \frac{1}{f - \alpha_2}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{f - \alpha_3}\right) + S(r, f). \end{aligned}$$

Lemma 4 ([2]) Let k be a positive integer, let f be a meromorphic function such that $f^{(k)} \not\equiv 0$. Then

$$T(r, f^{(k)}) \leq T(r, f) + k\bar{N}(r, f) + S(r, f),$$

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f).$$

Lemma 5 ([2]) Let f be a nonconstant meromorphic function, and let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$, where $a_n (\neq 0)$, a_{n-1}, \dots, a_0 are constants. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 6 ([2]) Let $f_i (i = 1, 2, 3)$ be nonconstant meromorphic functions such that $f_1 + f_2 + f_3 \equiv 1$. If $f_i (i = 1, 2, 3)$ are linearly independent, then

$$T(r, f_i) \leq \sum_{i=1}^3 N_2\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^3 \bar{N}(r, f_i) + o(T(r)),$$

where $T(r) = \max_{1 \leq i \leq 3} \{T(r, f_i)\}$ and $r \notin E$.

Lemma 7 ([2]) Let f be a nonconstant meromorphic function, let $n (\geq 2)$ be a positive integer, and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be distinct small functions of f . Then

$$m\left(r, \frac{1}{f - \alpha_1}\right) + \dots + m\left(r, \frac{1}{f - \alpha_n}\right) \leq m\left(r, \frac{1}{f - \alpha_1} + \dots + \frac{1}{f - \alpha_n}\right) + S(r, f).$$

Lemma 8 ([2]) Let n, k be two positive integers with $n > k + 5$, let f and g be two meromorphic functions all whose zeros and poles have multiplicity at least n , and let $f^{(k)} \not\equiv 0$ and $g^{(k)} \not\equiv 0$. If $f^{(k)}$ and $g^{(k)}$ share z CM, then

$$T(r, f) = O(T(r, g)), \quad T(r, g) = O(T(r, f)).$$

Proof: From the condition of Lemma 8, we obtain

$$T(r, f) \geq n \log r + O(1),$$

$$T(r, g) \geq n \log r + O(1). \tag{1}$$

By Lemma 2, Lemma 4, Lemma 7 and Nevanlinna's first fundamental theorem, we have

$$m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f^{(k)} - z}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + m\left(r, \frac{1}{f^{(k+1)} - 1}\right) + S(r, f) \leq m\left(r, \frac{1}{f^{(k+2)}}\right) + S(r, f) \leq T\left(r, f^{(k+2)}\right) - N\left(r, \frac{1}{f^{(k+2)}}\right) + S(r, f) \leq T\left(r, f^{(k)}\right) + 2\bar{N}(r, f) - N\left(r, \frac{1}{f^{(k+2)}}\right) + S(r, f).$$

From (2) and Nevanlinna's first fundamental theorem, we have

$$T(r, f) + T\left(r, f^{(k)} - z\right) \leq T\left(r, f^{(k)}\right) + 2\bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - z}\right) - N\left(r, \frac{1}{f^{(k+2)}}\right) + S(r, f) \leq T\left(r, f^{(k)}\right) + 2\bar{N}(r, f) + (k + 2)\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - z}\right) + S(r, f). \tag{3}$$

It follows from (1) that

$$T\left(r, f^{(k)} - z\right) \geq T\left(r, f^{(k)}\right) - \log r \geq T\left(r, f^{(k)}\right) - \frac{1}{n} T(r, f). \tag{4}$$

Hence, by (3), (4), and $f^{(k)}$ and $g^{(k)}$ share z CM, we have

$$\frac{n-1}{n} T(r, f) \leq 2\bar{N}(r, f) + (k + 2)\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g^{(k)} - z}\right) + S(r, f) \leq \frac{2}{n} N(r, f) + \frac{k+2}{n} N\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{g^{(k)} - z}\right) + S(r, f) \leq \frac{k+4}{n} T(r, f) + T\left(r, \frac{1}{g^{(k)} - z}\right) + S(r, f). \tag{5}$$

From (1), (5), Nevanlinna's first fundamental theorem and Lemma 4, we get

$$\frac{n-k-5}{n} T(r, f) \leq T\left(r, \frac{1}{g^{(k)} - z}\right) + S(r, f) \leq T\left(r, g^{(k)} - z\right) + S(r, f) \leq T\left(r, g^{(k)}\right) + \log r + S(r, f) \leq T(r, g) + k\bar{N}(r, g) + \log r + S(r, f) + S(r, g) \leq T(r, g) + \frac{k}{n} N(r, g) + \frac{1}{n} T(r, g) + S(r, f) + S(r, g) \leq \frac{n+k+1}{n} T(r, g) + S(r, f) + S(r, g).$$

Then

$$T(r, f) \leq \frac{n+k+1}{n-k-5} T(r, g) + S(r, f) + S(r, g).$$

By $n > k + 5$, we get $T(r, f) = O(T(r, g))$. Similarly, we obtain $T(r, g) = O(T(r, f))$. \square

By the proof of Lemma 8, we have the following results.

Lemma 9 Let n, k be two positive integers with $n > k + 2$, let f and g be two nonconstant meromorphic functions all whose zeros and poles have multiplicity at least n , and let $f^{(k+1)} \not\equiv 0$ and $g^{(k+1)} \not\equiv 0$. If $f^{(k)}$ and $g^{(k)}$ share 1 CM. Then

$$T(r, f) = O(T(r, g)), \quad T(r, g) = O(T(r, f)).$$

Lemma 10 ([12]) Let f be a nonconstant entire function, and let $k(\geq 2)$ be a positive integer. If $f f^{(k)} \neq 0$, then $f(z) = e^{az+b}$, where $a(\neq 0)$, b are two constants.

Lemma 11 ([9]) Let f and g be two nonconstant entire functions, and let k be a positive integer. If all the zeros of both f and g are of multiplicity at least $k + 1$, and $f^{(k)}g^{(k)} \equiv 1$, then $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)c^{2k} = 1$.

Lemma 12 ([13]) Let f be a meromorphic function, and let $k(\geq 2)$ be a positive integer. If f and $f^{(k)}$ have finitely many zeros, then $f = Re^P$, where R is a rational function and P is a polynomial.

Lemma 13 Let k be a positive integer, let f and g be two nonconstant entire functions whose zeros are of multiplicity at least $k + 1$. If $f^{(k)}g^{(k)} \equiv z^2$, then

- (1) $k = 1$, either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$ or $f(z) = \frac{Cz^2}{2}$, $g(z) = \frac{z^2}{2C}$, where c_1, c_2, c and C are nonzero constants satisfying $4c_1c_2c^2 = -1$;
- (2) $k \geq 2$, $f(z) = \frac{Cz^{k+1}}{(k+1)!}$, $g(z) = \frac{z^{k+1}}{C(k+1)!}$, where C is a nonzero constant.

Proof: In the following, we consider two cases.

Case 1. $k = 1$. Then $f'g' \equiv z^2$. Since all zeros of f and g are of multiplicity at least 2, f and g are two entire functions, then we know that $f = z^l e^\alpha$, $g = z^m e^\beta$, where $l, m \in \{0, 2, 3\}$ such that $l + m \leq 4$, and α, β are two entire functions.

Next, we consider five subcases.

Case 1.1. $l = 0, m = 0$. Then $f = e^\alpha, g = e^\beta$. It follows

$$f'g' = \alpha'\beta' e^{\alpha+\beta} \equiv z^2. \tag{6}$$

Now, by (6) we consider three subcases.

Case 1.1.1. $\alpha' = z^2 e^{\gamma_1}, \beta' = e^{\delta_1}$, where γ_1, δ_1 are two entire functions. Obviously, $T(r, \gamma_1') = m(r, \gamma_1') = m\left(r, \frac{e^{\gamma_1}}{e^{\gamma_1}}\right) = S(r, e^{\gamma_1})$. Similarly, $T(r, \delta_1') = S(r, e^{\delta_1})$.

By (6) we have $e^{\alpha+\beta+\gamma_1+\delta_1} \equiv 1$. It follows $\alpha' + \beta' + \gamma_1' + \delta_1' \equiv 0$. Then,

$$z^2 e^{\gamma_1} + e^{\delta_1} + \gamma_1' + \delta_1' \equiv 0. \tag{7}$$

We claim $\gamma_1' + \delta_1' \equiv 0$. Suppose, on the contrary, that $\gamma_1' + \delta_1' \neq 0$. By (6), (7) and Lemma 3 we have

$$\begin{aligned} T(r, e^{\delta_1}) &\leq \bar{N}(r, e^{\delta_1}) + \bar{N}\left(r, \frac{1}{e^{\delta_1}}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{e^{\delta_1+\gamma_1'+\delta_1'}}\right) + S(r, e^{\delta_1}) \\ &\leq \bar{N}\left(r, \frac{1}{z^2 e^{\gamma_1}}\right) + S(r, e^{\delta_1}) \\ &\leq \log r + S(r, e^{\delta_1}). \end{aligned}$$

Hence, e^{δ_1} is a constant, which yields δ_1 is a constant.

If γ_1 is a constant, then $\gamma_1' + \delta_1' \equiv 0$, a contradiction. If γ_1 is a nonconstant, then

$$\begin{aligned} m(r, z^2 e^{\gamma_1}) &\leq m(r, e^{\gamma_1}) + m(r, z^2) \\ &= m(r, e^{\gamma_1}) + S(r, e^{\gamma_1}). \end{aligned} \tag{8}$$

$$\begin{aligned} m(r, e^{\gamma_1}) &\leq m(r, z^2 e^{\gamma_1}) + m\left(r, \frac{1}{z^2}\right) \\ &= m(r, z^2 e^{\gamma_1}) + S(r, e^{\gamma_1}). \end{aligned} \tag{9}$$

By (7)–(9) and $m(r, \gamma_1') = S(r, e^{\gamma_1})$, we have

$$\begin{aligned} T(r, e^{\gamma_1}) &= m(r, e^{\gamma_1}) \\ &= m(r, z^2 e^{\gamma_1}) + S(r, e^{\gamma_1}) \\ &= m(r, e^{\delta_1} + \gamma_1') = S(r, e^{\gamma_1}), \end{aligned}$$

a contradiction.

Hence, $\gamma_1' + \delta_1' \equiv 0$. That is $\gamma_1 + \delta_1 = C$, where C is a constant. Then by (7) we have $z^2 e^{2\gamma_1} + e^C \equiv 0$, a contradiction.

Case 1.1.2. $\alpha' = z e^{\gamma_2}, \beta' = z e^{\delta_2}$, where γ_2, δ_2 are two entire functions. By (6) we have $z e^{\gamma_2} + z e^{\delta_2} + \gamma_2' + \delta_2' \equiv 0$. Using the same argument as used in Case 1.1.1, we know $\gamma_2' + \delta_2' \equiv 0$. That is $z e^{\gamma_2} + z e^{\delta_2} \equiv 0$. Hence, we know that γ_2, δ_2 are two constants. Thus $\alpha' = 2cz, \beta' = -2cz$, where c is a nonzero constant. It follows $\alpha(z) = cz^2 + \log c_1, \beta(z) = -cz^2 + \log c_2$, where c_1, c_2 are two nonzero constants. Therefore, $f(z) = e^{\alpha(z)} = c_1 e^{cz^2}, g(z) = e^{\beta(z)} = c_2 e^{-cz^2}$, where c, c_1, c_2 satisfying $4c_1c_2c^2 = -1$.

Case 1.1.3. $\alpha' = e^{\gamma_3}, \beta' = z^2 e^{\delta_3}$, where γ_3, δ_3 are two entire functions. Using the same argument as used in Case 1.1.1, we get a contradiction.

Case 1.2. $l = 2, m = 0$. Then $f = z^2 e^\alpha, g = e^\beta$. It follows from $f'g' \equiv z^2$ that $z(2 + z\alpha')\beta' e^{\alpha+\beta} \equiv z^2$. Therefore, we know that there exist two entire functions γ_4 and δ_4 such that $2 + z\alpha' = e^{\gamma_4}, \beta' = z e^{\delta_4}$. It follows $e^{\gamma_4} + z^2 e^{\delta_4} + z\gamma_4' + z\delta_4' \equiv 2$. Using the same argument as used in Case 1.1, we obtain a contradiction.

Case 1.3. $l = 2, m = 2$. Then $f = z^2 e^\alpha, g = z^2 e^\beta$. It follows from $f'g' \equiv z^2$ that

$$z^2(2 + z\alpha')(2 + z\beta') e^{\alpha+\beta} \equiv z^2. \tag{10}$$

By (10) we know that there exist two entire functions γ_5 and δ_5 such that

$$2 + z\alpha' = e^{\gamma_5}, \quad 2 + z\beta' = e^{\delta_5}. \tag{11}$$

By (10) and (11) we deduce

$$\gamma_5' + \delta_5' + \alpha' + \beta' = 0. \tag{12}$$

It follows from (11) and (12) that $e^{\gamma_5} + e^{\delta_5} + z(\gamma_5' + \delta_5') \equiv 4$. Using the same argument as used in Case 1.1, we know that both γ_5 and δ_5 are constants. Thus $\alpha' + \beta' \equiv 0$.

It follows from (10) that $z^2(4 - z^2\alpha'^2)e^{\alpha+\beta} \equiv z^2$. Hence, $\alpha' = 0, \beta' = 0$. By (10) we have $f(z) = \frac{Cz^2}{2}, g(z) = \frac{z^2}{2C}$, where C is a nonzero constant.

Case 1.4. $l = 3, m = 0$. Then $f = z^3 e^\alpha, g = e^\beta$. It follows from $f'g' \equiv z^2$ that $z^2(3 + z\alpha')\beta' e^{\alpha+\beta} \equiv z^2$. Using the same argument as used in Case 1.2, we obtain a contradiction.

Case 1.5. Either $l = 0, m = 2$ or $l = 0, m = 3$. Using the same argument as used in Case 1.2, we obtain a contradiction.

Case 2. $k \geq 2$. Since f and g are two nonconstant entire functions, then by $f^{(k)}g^{(k)} \equiv z^2$ we know that $f^{(k)}$ and $g^{(k)}$ have zeros only at $z = 0$. It follows from the zeros of f and g are of multiplicity at least $k + 1$ that f and g have zeros only at $z = 0$. Thus, by Lemma 12 we deduce that $f = z^l e^P, g = z^m e^Q$, where $l, m \in \{0, k + 1, k + 2\}$, such that $l + m \leq 2k + 2$, and P, Q are polynomials.

By $f^{(k)}g^{(k)} \equiv z^2$ we have

$$(z^l(P')^k + H_k)(z^m(Q')^k + R_k)e^{P+Q} \equiv z^2, \tag{13}$$

where H_k, R_k are two polynomials with $\deg H_k = l + k \deg P' - 1, \deg R_k = l + k \deg Q' - 1$.

In the following, we consider three subcases.

Case 2.1. $P' \equiv 0, Q' \equiv 0$. Then $f(z) = az^l, g(z) = bz^m$, where a, b are two nonzero constants. By $f^{(k)}g^{(k)} \equiv z^2$, we know that $f = az^{k+1}, g = bz^{k+1}$. Thus $f^{(k)}g^{(k)} = ab((k + 1)!)^2 z^2 \equiv z^2$, which yields $ab((k + 1)!)^2 = 1$. Therefore, $f(z) = \frac{Cz^{k+1}}{(k+1)!}, g(z) = \frac{z^{k+1}}{C(k+1)!}$, where C is a nonzero constant.

Case 2.2. Either $P' \equiv 0, Q' \neq 0$ or $P' \neq 0, Q' \equiv 0$. Without loss of generality, we consider the case of $P' \equiv 0, Q' \neq 0$.

By (13) we have $l(l - 1) \cdots (l - k + 1)z^{l-k} e^P (z^m(Q')^k + R_k)e^Q \equiv z^2$. Obviously, $l(l - 1) \cdots (l - k + 1)z^{l-k} e^P (z^m(Q')^k + R_k)$ is a nonzero polynomial be written as U_k . It follows $T(r, e^Q) = T(r, \frac{z^2}{U_k}) = O(\log r) = S(r, e^Q)$, a contradiction.

Case 2.3. $P' \neq 0, Q' \neq 0$. By (13) we deduce

$$l + m + k(\deg P' + \deg Q') = 2. \tag{14}$$

Next, we consider three subcases.

Case 2.3.1. $\deg P' + \deg Q' \geq 2$. By $k \geq 2, l + m \geq 2$ and (14), we obtain a contradiction.

Case 2.3.2. $\deg P' + \deg Q' = 1$. Without loss of generality, we consider the case of $\deg P' = 1, \deg Q' = 0$. Set

$$P = az^2 + bz + d, \quad Q = qz + r, \tag{15}$$

where $a(\neq 0), b, d, q$, and r are constants.

By (13) and (15) we have $f^{(k)}g^{(k)} = T_k e^{az^2+(b+q)z+d+r} \equiv z^2$, where T_k is a polynomial. Using the same argument as used in Case 2.2, we get a contradiction.

Case 2.3.3. $\deg P' = \deg Q' = 0$. By (14) we have $l + m = 2$. From $l, m \in \{0, k + 1, k + 2\}$ and $k \geq 2$, we get a contradiction. \square

PROOF OF THEOREM 1

Set

$$\phi = \frac{f^{(k+1)}}{f^{(k)}(f^{(k)} - 1)} - \frac{g^{(k+1)}}{g^{(k)}(g^{(k)} - 1)}. \tag{16}$$

Now, we consider two cases.

Case 1. $\phi \equiv 0$. It follows from (16) that

$$\frac{f^{(k)} - 1}{f^{(k)}} \equiv C \frac{g^{(k)} - 1}{g^{(k)}}, \tag{17}$$

where C is a nonzero constant.

Next, we consider two subcases.

Case 1.1. $C = 1$. By (17) we have $f^{(k)} \equiv g^{(k)}$. Hence $f \equiv g + P$, where P is a polynomial with $\deg P \leq k - 1$.

If $P \equiv 0$, then $f \equiv g$.

If $P \neq 0$, then we obtain

$$\frac{f}{P} - \frac{g}{P} = 1. \tag{18}$$

Since f and g are two nonconstant meromorphic functions whose zeros and poles multiplicities at least n , then we have

$$\begin{aligned} T(r, f) &\geq n \log r + O(1), \\ T(r, g) &\geq n \log r + O(1). \end{aligned} \tag{19}$$

By (19) and Nevanlinna's first fundamental theorem we get

$$\begin{aligned} T\left(r, \frac{f}{P}\right) &\leq T(r, f) + T(r, P) + O(1) \\ &\leq T(r, f) + (k - 1) \log r + O(1) \\ &\leq T(r, f) + (k - 1)T(r, f) + O(1) \\ &\leq kT(r, f) + O(1). \end{aligned}$$

It follows

$$S\left(r, \frac{f}{P}\right) = S(r, f). \tag{20}$$

By (18)–(20) and Nevanlinna's first and second fundamental theorems we have

$$\begin{aligned} T(r, f) &\leq T\left(r, \frac{f}{P}\right) + T(r, P) + O(1) \\ &\leq \bar{N}\left(r, \frac{f}{P}\right) + \bar{N}\left(r, \frac{P}{f}\right) + \bar{N}\left(r, \frac{1}{\frac{f}{P}-1}\right) \\ &\quad + T(r, P) + S(r, f) \\ &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{P}\right) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{P}{g}\right) \\ &\quad + T(r, P) + S(r, f) \\ &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) \\ &\quad + 2T(r, P) + S(r, f) \\ &\leq \frac{1}{n}N(r, f) + \frac{1}{n}N\left(r, \frac{1}{f}\right) + \frac{1}{n}N\left(r, \frac{1}{g}\right) \\ &\quad + \frac{2k-2}{n}T(r, f) + S(r, f) \\ &\leq \frac{2k}{n}T(r, f) + \frac{1}{n}T(r, g) + S(r, f). \end{aligned} \tag{21}$$

Similarly,

$$T(r, g) \leq \frac{2k}{n}T(r, g) + \frac{1}{n}T(r, f) + S(r, g). \quad (22)$$

Since $n > 3k + 6$, then by (21) and (22) we have $T(r, f) + T(r, g) \leq S(r, f) + S(r, g)$, a contradiction.

Case 1.2. $C \neq 1$. By (17) we have

$$\frac{1}{f^{(k)}} - \frac{C}{g^{(k)}} = 1 - C. \quad (23)$$

Since f and g share ∞ IM, then by (23) we know that $f^{(k)} \neq \infty$ and $g^{(k)} \neq \infty$. Thus, $\frac{1}{f^{(k)}} \neq 0$ and $g^{(k)} \neq \frac{C}{C-1}$.

By Lemma 1 and Nevanlinna's first fundamental theorem, we obtain

$$\begin{aligned} T(r, g) &\leq \bar{N}(r, g) + N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^{(k)} - \frac{C}{C-1}}\right) \\ &\quad - N\left(r, \frac{1}{g^{(k+1)}}\right) + S(r, g) \\ &\leq N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g^{(k+1)}}\right) + S(r, g) \\ &\leq (k+1)\bar{N}\left(r, \frac{1}{g}\right) + S(r, g) \\ &\leq \frac{k+1}{n}N\left(r, \frac{1}{g}\right) + S(r, g) \\ &\leq \frac{k+1}{n}T(r, g) + S(r, g). \end{aligned}$$

It follows from $n > 3k + 6$ that $T(r, g) \leq S(r, g)$, a contradiction.

Case 2. $\phi \neq 0$. By Lemma 4, we have

$$\begin{aligned} T(r, f^{(k)}) &\leq T(r, f) + k\bar{N}(r, f) + S(r, f) \\ &\leq (k+1)T(r, f) + S(r, f). \end{aligned}$$

Thus $S(r, f^{(k)}) = S(r, f)$. Similarly, we obtain $S(r, g^{(k)}) = S(r, g)$.

Let z_0 be a pole of f with multiplicity l_1 . By f and g share ∞ IM, we know that z_0 is a pole of g with multiplicity l_2 . Set $l = \min\{l_1, l_2\}$. By (16) we deduce that z_0 is a zero of ϕ with multiplicity $\geq l + k - 1$. It follows from Lemma 2 and Nevanlinna's first fundamental theorem that

$$\begin{aligned} \bar{N}(r, f) = \bar{N}(r, g) &\leq \frac{1}{l+k-1}N\left(r, \frac{1}{\phi}\right) \\ &\leq \frac{1}{l+k-1}T(r, \phi) + O(1) \\ &\leq \frac{1}{l+k-1}N(r, \phi) + \frac{1}{l+k-1}m(r, \phi) + O(1) \\ &\leq \frac{1}{l+k-1}N(r, \phi) + S(r, f^{(k)}) + S(r, g^{(k)}) \\ &\leq \frac{1}{l+k-1}N(r, \phi) + S(r, f) + S(r, g) \\ &\leq \frac{1}{l+k-1}\left[\bar{N}\left(r, \frac{1}{f^{(k)}}\right) + \bar{N}\left(r, \frac{1}{g^{(k)}}\right)\right] \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (24)$$

By Lemma 4 we have

$$\begin{aligned} \bar{N}\left(r, \frac{1}{f^{(k)}}\right) &= N\left(r, \frac{1}{f^{(k)}}\right) - \left[N\left(r, \frac{1}{f^{(k)}}\right) - \bar{N}\left(r, \frac{1}{f^{(k)}}\right)\right] \\ &\leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) \\ &\quad - \left[N\left(r, \frac{1}{f^{(k)}}\right) - \bar{N}\left(r, \frac{1}{f^{(k)}}\right)\right] + S(r, f) \\ &\leq (k+1)\bar{N}\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f) \\ &\leq \frac{2k+1}{n}T(r, f) + S(r, f). \end{aligned} \quad (25)$$

Similarly,

$$\bar{N}\left(r, \frac{1}{g^{(k)}}\right) \leq \frac{2k+1}{n}T(r, g) + S(r, g). \quad (26)$$

By (24)–(26), we get

$$\begin{aligned} \bar{N}(r, f) = \bar{N}(r, g) \\ &\leq \frac{2k+1}{n(l+k-1)}[T(r, f) + T(r, g)] + S(r, f) + S(r, g). \end{aligned} \quad (27)$$

Set

$$\varphi = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)} - 1} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2\frac{g^{(k+1)}}{g^{(k)} - 1}. \quad (28)$$

Let z_0 be a common simple zero of $f^{(k)} - 1$ and $g^{(k)} - 1$. By computation we have $\varphi(z_0) = 0$. Suppose that $\varphi \not\equiv 0$. It follows from Lemma 2 and Nevanlinna's first fundamental theorem that

$$\begin{aligned} N_{11}\left(r, \frac{1}{f^{(k)} - 1}\right) &= N_{11}\left(r, \frac{1}{g^{(k)} - 1}\right) \leq N\left(r, \frac{1}{\varphi}\right) \\ &\leq T(r, \varphi) + O(1) \leq N(r, \varphi) + S(r, f) + S(r, g), \end{aligned} \quad (29)$$

where $N_{11}\left(r, \frac{1}{f^{(k)} - 1}\right)$ is the counting function of simple zeros of $f^{(k)} - 1$. Similarly, we have the notation $N_{11}\left(r, \frac{1}{g^{(k)} - 1}\right)$. Since $f^{(k)}$ and $g^{(k)}$ share 1 CM, and f and g share ∞ IM, then by (28) we have

$$\begin{aligned} N(r, \varphi) &\leq \frac{1}{2}\bar{N}(r, f) + \frac{1}{2}\bar{N}(r, g) + \bar{N}\left(r, \frac{1}{f}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) + N_0\left(r, \frac{1}{g^{(k+1)}}\right), \end{aligned} \quad (30)$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function for which $f^{(k+1)} = 0$ and $f(f^{(k)} - 1) \neq 0$. Similarly, we have the notation $N_0\left(r, \frac{1}{g^{(k+1)}}\right)$.

By Lemma 1, we obtain

$$\begin{aligned} T(r, f) &\leq \bar{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - 1}\right) \\ &\quad - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f), \end{aligned} \quad (31)$$

$$\begin{aligned} T(r, g) &\leq \bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^{(k)} - 1}\right) \\ &\quad - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r, g). \end{aligned} \quad (32)$$

From $f^{(k)}$ and $g^{(k)}$ share 1 CM, we have

$$\begin{aligned} \bar{N}\left(r, \frac{1}{f^{(k)-1}}\right) + \bar{N}\left(r, \frac{1}{g^{(k)-1}}\right) &= N_1\left(r, \frac{1}{f^{(k)-1}}\right) \\ &+ \frac{1}{2}\left[N\left(r, \frac{1}{f^{(k)-1}}\right) + N\left(r, \frac{1}{g^{(k)-1}}\right)\right]. \end{aligned} \quad (33)$$

It follows from (31)–(33) that

$$\begin{aligned} T(r, f) + T(r, g) &\leq \bar{N}(r, f) + \bar{N}(r, g) \\ &+ \frac{k+1}{n}\left[N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right] \\ &+ \frac{1}{2}\left[N\left(r, \frac{1}{f^{(k)-1}}\right) + N\left(r, \frac{1}{g^{(k)-1}}\right)\right] \\ &- \left[N_0\left(r, \frac{1}{f^{(k+1)}}\right) + N_0\left(r, \frac{1}{g^{(k+1)}}\right)\right] \\ &+ N_1\left(r, \frac{1}{f^{(k)-1}}\right) + S(r, f) + S(r, g). \end{aligned} \quad (34)$$

By Lemma 4 and Nevanlinna’s first fundamental theorem we have

$$\begin{aligned} N\left(r, \frac{1}{f^{(k)-1}}\right) + N\left(r, \frac{1}{g^{(k)-1}}\right) &\leq T(r, f^{(k)}) + T(r, g^{(k)}) + O(1) \\ &\leq T(r, f) + T(r, g) + k\bar{N}(r, f) + k\bar{N}(r, g) \\ &+ S(r, f) + S(r, g) \\ &\leq T(r, f) + T(r, g) + \frac{k}{n}N(r, f) + \frac{k}{n}N(r, g) \\ &+ S(r, f) + S(r, g). \end{aligned} \quad (35)$$

Since f and g share ∞ IM, then by (27), (29), (30), (34) and (35), we obtain

$$\begin{aligned} T(r, f) + T(r, g) &\leq \left[\frac{3k+4}{n} + \frac{12k+6}{n(l+k-1)}\right][T(r, f) + T(r, g)] \\ &+ S(r, f) + S(r, g). \end{aligned} \quad (36)$$

By (36) and $l \geq n > 3k+6$ we obtain $T(r, f) + T(r, g) \leq S(r, f) + S(r, g)$, a contradiction. Thus $\varphi \equiv 0$. That is

$$\frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)} - 1} \equiv \frac{g^{(k+2)}}{g^{(k+1)}} - 2\frac{g^{(k+1)}}{g^{(k)} - 1}.$$

It follows

$$\frac{1}{f^{(k)} - 1} = \frac{a}{g^{(k)} - 1} + b, \quad (37)$$

where $a (\neq 0)$, b are two finite complex numbers.

Next, we consider two subcases.

Case 2.1. $b \neq 0$. Since f and g share ∞ IM, we know that $f^{(k)}$ and $g^{(k)}$ share ∞ IM. It follows from (37) that $f^{(k)} \neq \infty$, $g^{(k)} \neq \infty$. Hence $\frac{1}{f^{(k)-1}} \neq 0$. By (37) we have $g^{(k)} \neq \frac{b-a}{b}$.

Now, we consider two subcases.

Case 2.1.1. $b \neq a$. By Lemma 1, we have

$$\begin{aligned} T(r, g) &\leq \bar{N}(r, g) + N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^{(k)-1} - \frac{b-a}{b}}\right) \\ &- N\left(r, \frac{1}{g^{(k+1)}}\right) + S(r, g) \leq (k+1)\bar{N}\left(r, \frac{1}{g}\right) + S(r, g) \\ &\leq \frac{k+1}{n}N\left(r, \frac{1}{g}\right) + S(r, g) \leq \frac{k+1}{n}T(r, g) + S(r, g). \end{aligned} \quad (38)$$

It follows from $n > 3k + 6$ and (38) that $T(r, g) \leq S(r, g)$, a contradiction.

Case 2.1.2. $b = a$. If $b \neq -1$, then using the same argument as used in Case 2.1.1, we get a contradiction. If $b = -1$, then by (37) we have $f^{(k)}g^{(k)} = 1$. It follows from Lemma 11 that $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)c^{2k} = -1$.

Case 2.2. $b = 0$. We consider two subcases.

Case 2.2.1. $a = 1$. By (37) we have $f^{(k)} \equiv g^{(k)}$. Using the same argument as used in Case 1.1, we get $f \equiv g$.

Case 2.2.2. $a \neq 1$. By (37) we obtain $(af - g)^{(k)} \equiv a - 1$. It follows $af - g \equiv Q$, where Q is polynomial with $\deg Q = k$. Using the same argument as used in Case 1.1, we have a contradiction.

This completes the proof of Theorem 1.

PROOF OF THEOREM 2

Imitating the proof of Theorem 1, we can prove Theorem 2 only by replacing (30) with the following inequality.

$$N(r, \varphi) \leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) + N_0\left(r, \frac{1}{g^{(k+1)}}\right).$$

Hence, we omit the details.

PROOF OF THEOREM 3

It follows from $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ IM that

$$H = \frac{f^{(k)} - z}{g^{(k)} - z}, \quad (39)$$

where $H (\neq 0, \infty)$ is a meromorphic function.

By (39) we have

$$\bar{N}(r, H) \leq \bar{N}_L(r, f), \quad \bar{N}\left(r, \frac{1}{H}\right) \leq \bar{N}_L(r, g). \quad (40)$$

Set

$$f_1 = \frac{f^{(k)}}{z}, \quad f_2 = H, \quad f_3 = -\frac{Hg^{(k)}}{z}. \quad (41)$$

Obviously, $f_1 + f_2 + f_3 \equiv 1$.

By (39)–(41) and Lemma 4 we have

$$T(r, f_1) + T(r, f_2) + T(r, f_3) \leq O(T(r, f) + T(r, g)). \quad (42)$$

We suppose that f_2 and f_3 are not constants. If f_1, f_2, f_3 are linearly independent, then by (40)–(42) and Lemma 6 we obtain

$$\begin{aligned} T(r, f_1) &\leq N_2\left(r, \frac{1}{f_1}\right) + N_2\left(r, \frac{1}{f_2}\right) + N_2\left(r, \frac{1}{f_3}\right) \\ &+ \bar{N}(r, f_1) + \bar{N}(r, f_2) + \bar{N}(r, f_3) + o(T(r)) \\ &\leq N_2\left(r, \frac{1}{f^{(k)}}\right) + 2\bar{N}_L(r, g) + N_2\left(r, \frac{1}{g^{(k)}}\right) + 2\bar{N}(r, f) \\ &+ \bar{N}_L(r, f) + 2\log r + S(r, f) + S(r, g). \end{aligned} \quad (43)$$

By (41) and (43) we have

$$\begin{aligned} T(r, f^{(k)}) &\leq T(r, f_1) + \log r \\ &\leq N_2\left(r, \frac{1}{f^{(k)}}\right) + 2\bar{N}_L(r, g) + N_2\left(r, \frac{1}{g^{(k)}}\right) \\ &\quad + 2\bar{N}(r, f) + \bar{N}_L(r, f) + 3\log r + S(r, f) + S(r, g) \\ &= N\left(r, \frac{1}{f^{(k)}}\right) - \left[N_{(3)}\left(r, \frac{1}{f^{(k)}}\right) - 2\bar{N}_{(3)}\left(r, \frac{1}{f^{(k)}}\right)\right] + N\left(r, \frac{1}{g^{(k)}}\right) \\ &\quad - \left[N_{(3)}\left(r, \frac{1}{g^{(k)}}\right) - 2\bar{N}_{(3)}\left(r, \frac{1}{g^{(k)}}\right)\right] + 2\bar{N}_L(r, g) + 2\bar{N}(r, f) \\ &\quad + \bar{N}_L(r, f) + 3\log r + S(r, f) + S(r, g). \end{aligned} \quad (44)$$

Let z_0 be a zero of f with multiplicity l . Then z_0 is a zero of $f^{(k)}$ with multiplicity $l - k \geq 3$. Thus

$$N_{(3)}\left(r, \frac{1}{f^{(k)}}\right) - 2\bar{N}_{(3)}\left(r, \frac{1}{f^{(k)}}\right) \geq (l - k - 2)\bar{N}\left(r, \frac{1}{f}\right). \quad (45)$$

Similarly,

$$N_{(3)}\left(r, \frac{1}{g^{(k)}}\right) - 2\bar{N}_{(3)}\left(r, \frac{1}{g^{(k)}}\right) \geq (l - k - 2)\bar{N}\left(r, \frac{1}{g}\right). \quad (46)$$

It follows from Lemma 2 and Nevanlinna's first fundamental theorem that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ &= T(r, f^{(k)}) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f), \\ T(r, f) &\leq T(r, f^{(k)}) - N\left(r, \frac{1}{f^{(k)}}\right) \\ &\quad + N\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (47)$$

By (44)–(47), Lemma 4 and f and g share ∞ IM we obtain

$$\begin{aligned} T(r, f) &\leq \frac{k+2}{n}N\left(r, \frac{1}{f}\right) + \frac{k+2}{n}N\left(r, \frac{1}{g}\right) + 2\bar{N}_L(r, g) \\ &\quad + (k+2)\bar{N}(r, f) + \bar{N}_L(r, f) + 3\log r \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (48)$$

Similarly,

$$\begin{aligned} T(r, g) &\leq \frac{k+2}{n}N\left(r, \frac{1}{g}\right) + \frac{k+2}{n}N\left(r, \frac{1}{f}\right) \\ &\quad + 2\bar{N}_L(r, f) + (k+2)\bar{N}(r, g) \\ &\quad + \bar{N}_L(r, g) + 3\log r + S(r, f) + S(r, g). \end{aligned} \quad (49)$$

Noting that

$$\bar{N}_L(r, f) + \bar{N}_L(r, g) \leq \bar{N}(r, f) = \bar{N}(r, g). \quad (50)$$

By (48)–(50) we have

$$\begin{aligned} (n-2k-4)[T(r, f) + T(r, g)] &\leq (k+\frac{7}{2})[\bar{N}(r, f) + \bar{N}(r, g)] \\ &\quad + 6\log r + S(r, f) + S(r, g). \end{aligned} \quad (51)$$

In the following, we consider two cases.

Case 1. f and g have poles. Since f and g share ∞ IM, then we have

$$\bar{N}(r, f) = \bar{N}(r, g) \geq \log r. \quad (52)$$

Set

$$F = \frac{f^{(k)}}{z}, \quad G = \frac{g^{(k)}}{z}. \quad (53)$$

From $f^{(k)}$ and $g^{(k)}$ share z CM, we know that F and G share 1 CM almost.

By (52), (53) and Lemma 4 we obtain

$$\begin{aligned} T(r, F) &\leq T(r, f^{(k)}) + \log r \\ &\leq T(r, f) + k\bar{N}(r, f) + \log r + S(r, f) \\ &\leq (k+2)T(r, f) + S(r, f). \end{aligned} \quad (54)$$

It follows $S(r, F) = S(r, f)$. Similarly, $S(r, G) = S(r, g)$.

Set

$$\phi = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \quad (55)$$

Now, we consider two subcases.

Case 1.1. $\phi \equiv 0$. By (55) we obtain

$$\frac{F-1}{F} \equiv C \frac{G-1}{G}, \quad (56)$$

where C is a nonzero constant.

In the following, we consider two subcase.

Case 1.1.1. $C = 1$. By (56) we get $F \equiv G$. That is $f^{(k)} \equiv g^{(k)}$. Using the same argument as used in proof of Theorem 1, we obtain $f \equiv g$.

Case 1.1.2. $C \neq 1$. By (56) we have

$$\frac{1}{F} - \frac{C}{G} = 1 - C. \quad (57)$$

Since f and g share ∞ IM, we know that F and G share ∞ IM, then by (57) we obtain a contradiction.

Case 1.2. $\phi \not\equiv 0$. Let z_0 be a pole of f with multiplicity l_1 . By f and g share ∞ IM, we know that z_0 is a pole of g with multiplicity l_2 . Set $l = \min\{l_1, l_2\}$. By (55) we deduce that z_0 is a zero of ϕ with multiplicity $\geq l + k - 1$. From Lemma 2 and Nevanlinna's first fundamental theorem, we get

$$\begin{aligned} \bar{N}(r, f) = \bar{N}(r, g) &\leq \frac{1}{l+k-1}N\left(r, \frac{1}{\phi}\right) \\ &\leq \frac{1}{l+k-1}N(r, \phi) + \frac{1}{n+k-1}m(r, \phi) + O(1) \\ &\leq \frac{1}{l+k-1}[\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right)] \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (58)$$

By Lemma 4 we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f}\right) &= \overline{N}\left(r, \frac{z}{f^{(k)}}\right) = \overline{N}\left(r, \frac{1}{f^{(k)}}\right) \\ &= N\left(r, \frac{1}{f^{(k)}}\right) - \left[N\left(r, \frac{1}{f^{(k)}}\right) - \overline{N}\left(r, \frac{1}{f^{(k)}}\right)\right] \\ &\leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) \\ &\quad - \left[N\left(r, \frac{1}{f^{(k)}}\right) - \overline{N}\left(r, \frac{1}{f^{(k)}}\right)\right] + S(r, f) \\ &\leq (k+1)\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f) \\ &\leq \frac{2k+1}{n}T(r, f) + S(r, f). \end{aligned} \tag{59}$$

Similarly,

$$\overline{N}\left(r, \frac{1}{g}\right) \leq \frac{2k+1}{n}T(r, g) + S(r, g). \tag{60}$$

By (58)–(60) we obtain

$$\begin{aligned} \overline{N}(r, f) &= \overline{N}(r, g) \leq \frac{2k+1}{n(1+k-1)}[T(r, f) + T(r, g)] \\ &\quad + S(r, f) + S(r, g). \end{aligned} \tag{61}$$

It follows from (51), (52) and (61) that

$$\begin{aligned} (n-2k-4)[T(r, f) + T(r, g)] \\ \leq \frac{(2k+13)(2k+1)}{n+k-1}[T(r, f) + T(r, g)] \\ + S(r, f) + S(r, g). \end{aligned} \tag{62}$$

Since $n > 3k + 8$, then by (62) we obtain $T(r, f) + T(r, g) \leq S(r, f) + S(r, g)$, a contradiction.

Therefore, we deduce that f_1, f_2, f_3 are linearly dependent. Hence, there exist three constants c_1, c_2, c_3 with $(c_1, c_2, c_3) \neq (0, 0, 0)$ such that

$$c_1f_1 + c_2f_2 + c_3f_3 \equiv 0. \tag{63}$$

If $c_1 = 0$, then by (63), we have $c_2f_2 + c_3f_3 \equiv 0$ and $c_3 \neq 0$. That is

$$g^{(k)}(z) \equiv \frac{c_2}{c_3}z.$$

Thus g is a polynomial, a contradiction. Hence, $c_1 \neq 0$.

If $c_2 \neq 0$, from (63) and $f_1 + f_2 + f_3 \equiv 1$, we obtain

$$\left(1 - \frac{c_2}{c_1}\right)f_2 + \left(1 - \frac{c_3}{c_1}\right)f_3 \equiv 1, \tag{64}$$

and $c_1 \neq c_2, c_1 \neq c_3$.

By (39), (41), and (64) we get

$$\left(1 - \frac{c_3}{c_1}\right)\frac{g^{(k)}}{z} + \frac{g^{(k)}-z}{f^{(k)}-z} \equiv 1 - \frac{c_2}{c_1}. \tag{65}$$

Since f and g share ∞ IM, and f and g have poles, then by (65), we get a contradiction.

Therefore, $c_2 = 0, c_3 \neq 0$. By (63) and $f_1 + f_2 + f_3 \equiv 1$ we have

$$\left(1 - \frac{c_1}{c_3}\right)f_1 + f_2 \equiv 1.$$

Similar to above discussion, we get a contradiction.

Hence we deduce that either f_2 or f_3 is a constant.

Next, we consider two subcases.

Case 1.2.1 $f_2 = C$, where C is a constant. It follows from (41) and $f_1 + f_2 + f_3 \equiv 1$ that

$$\frac{f^{(k)}}{z} - C\frac{g^{(k)}}{z} \equiv 1 - C. \tag{66}$$

If $C \neq 1$, then by (66), we obtain $(f - Cg)^{(k)} \equiv (1 - C)z$. Hence, $f - Cg \equiv P$, where P is a polynomial with $\deg P = k + 1$. Using the same argument as used in proof of Theorem 1, we get a contradiction.

Therefore $f^{(k)} \equiv g^{(k)}$. Using the same argument as used in proof of Theorem 1, we have $f \equiv g$.

Case 1.2.2. $f_3 = C$, where C is a constant. By (39), (41) and $f_1 + f_2 + f_3 \equiv 1$, we obtain

$$\frac{f^{(k)}}{z} + \frac{f^{(k)}-z}{g^{(k)}-z} = 1 - C. \tag{67}$$

Similar to above discussion, we get a contradiction.

Case 2. f and g are two entire functions. By Lemma 8 we deduce that either f and g are two transcendental entire functions or f and g are two polynomials.

In the following, we consider two subcases.

Case 2.1. f and g are two transcendental entire functions. By the arguments similar to the proof of Case 1, we get either $f^{(k)} \equiv g^{(k)}$ or $f^{(k)}g^{(k)} \equiv z^2$.

If $f^{(k)} \equiv g^{(k)}$, then using the same argument as used in the proof of Theorem 1, we have $f \equiv g$.

If $f^{(k)}g^{(k)} \equiv z^2$, then from $n > k + 3$, we obtain $f \neq 0, g \neq 0$. By Lemma 13, we get

- (1) $k = 1, f(z) = c_1 e^{cz^2}, g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are nonzero constants satisfying $4c_1c_2c^2 = -1$, or $f \equiv g$;
- (2) $k \geq 2, f \equiv g$.

Case 2.2. f and g are two polynomials. Since $f^{(k)}$ and $g^{(k)}$ share z CM, then

$$f^{(k)} - z \equiv c(g^{(k)} - z), \tag{68}$$

where c is a nonzero constant.

If $c \neq 1$, then by (68) we have $(f - cg)^{(k)} \equiv (1 - c)z$. Hence, $f - cg \equiv P$, where P is a polynomial with $\deg P = k + 1$. Using the same argument as used in proof of Theorem 1, we obtain a contradiction. Thus, $f^{(k)} \equiv g^{(k)}$. Using the same argument as used in proof of Theorem 1, we have $f \equiv g$.

This completes the proof of Theorem 3.

PROOF OF THEOREM 4

From $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ CM, we obtain

$$e^h = \frac{f^{(k)} - z}{g^{(k)} - z},$$

where h is a nonconstant entire function.

Using the same argument as used in the proof of Theorem 3, we can prove Theorem 4.

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