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CAP-subgroups and $p\mathfrak{F}$-hypercentrally embedded property

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ABSTRACT: In this paper, we investigate $p\mathfrak{F}$-hypercentrally embedded property of normal subgroups of a finite group and obtain some new results.

KEYWORDS: finite group, hypercentrally embedded, subgroup, Sylow subgroup

MSC2020: 20D10 20D20

INTRODUCTION

All groups in this paper are finite, G is always a finite group. π denotes a set of primes, G_π means a Hall π-subgroup of G, $O_\pi(G)$ is the largest normal π-subgroup of G, and $O^\pi(G)$ is the subgroup generated by all π'-elements of G. We use conventional notions and notations, as in [1, 2].

Recall that a class $\mathfrak{F}$ of groups is called a formation if $\mathfrak{F}$ is closed under taking homomorphic images and $G/(N_1 \cap N_2) \in \mathfrak{F}$ if $G/N_1, G/N_2 \in \mathfrak{F}$ for arbitrary normal subgroups N_1 and N_2 of G. A formation $\mathfrak{F}$ is called saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. We use $\mathfrak{U}$ to denote the formation of all supersolvable groups. It is clear that $\mathfrak{U}$ is a saturated formation. A chief factor H/K of a group G is said to be $\mathfrak{F}$-central in G if $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. A normal subgroup N of G is said to be $p\mathfrak{F}$-hypercentrally embedded (resp., $\mathfrak{F}$-hypercentrally embedded) in G if either $N = 1$ or every p-chief factor of G (resp., chief factor of G) below N is $\mathfrak{F}$-central in G. The product of all normal $p\mathfrak{F}$-hypercentrally embedded subgroups (resp., $\mathfrak{F}$-hypercentrally embedded subgroups) is called the $p\mathfrak{F}$-hypercentre (resp., $\mathfrak{F}$-hypercentre) of G and denoted by $Z_{p\mathfrak{F}}(G)$ (resp., $Z_{\mathfrak{F}}(G)$). Clearly, a normal subgroup N of group G is $p\mathfrak{F}$-hypercentrally embedded in G if and only if $N \leqslant Z_{p\mathfrak{F}}(G)$.

A subgroup H covers A/B if $HA = HB$ and avoids A/B if $H \cap A = H \cap B$, and has the cover-avoiding property in G if H either covers or avoids every chief factor of G (see [3]), in this case we may also say that H is a CAP-subgroup of G. A subgroup H of G is said to be semi cover-avoiding in G if there is a chief series $1 = G_0 < G_1 < \cdots < G_t = G$ of G such that for every $j = 1, 2, \cdots, t$, H either covers G_j/G_{j-1} or avoids G_j/G_{j-1} (see [4]), in this case, H is also called a semi CAP-subgroup of G or partial CAP-subgroup of G in some literatures.

In recent years, many scholars have been interested in the influence of some property of the intersections between some subgroups and the subgroups $O^p(G)$, or $O^p(G_p^*)$, or $G^{\mathfrak{F}}$ on the structure of a group G and give some criteria for p-supersolvability and p-nilpotency. For example, in [5], Guo and Isaacs investigated the supersolvability of a group G by assuming that $H \cap O^p(G) \lhd O^p(G)$ for any normal subgroup H of P with order d, where $P \in Syl_p(G)$ and d is a power of p with $1 \leqslant d < |P|$. In [6], the author proved that G is p-supersolvable if and only if $H \cap O^p(G_p^*)$ is s-permutable in G for all subgroups $H \leqslant P$ with $|H| = p^e$, where e is an integer with $e \geqslant 2$, $P \in Syl_p(G)$ and $|P| \geqslant p^{e+1}$. In addition, there are many literatures that have also investigated the influence of the properties of the intersections mentioned above on a group G (such as [7–11]), the authors obtained many results on p-supersolvability, p-nilpotency and p-supersolvable hypercenter of G. Our motivation is to develop such research by replacing the subgroups $O^p(G)$, or $O^p(G_p^*)$, or $G^{\mathfrak{F}}$ with general normal subgroups and obtain some new results. In [12], Lei, Li and Guo considered that the intersections between some subgroups and a normal subgroup satisfy permutability, their main theorems generalized many known results. In this paper, we continue to study this question and obtain some new characterizations for hypercentrally embedded property of normal subgroups of a finite group G by assuming that the intersections between some subgroups with fixed order and a normal subgroup are CAP-subgroups of G, which generalize the main theorem of [5].

PRELIMINARIES

In this section, for the sake of convenience, we present some basic results which will be used in the proofs of the Main Results section in this paper.

Lemma 1 ([13]) *Let S be a CAP-subgroup of G and N a normal subgroup of G. Then*
(1) N is a CAP-subgroup of G;
(2) SN/N is a CAP-subgroup of G/N;
(3) SN is a CAP-subgroup of G;

(4) $S \cap N$ *is a CAP-subgroup of* G.

Lemma 2 ([13]) *Every minimal normal subgroup of* G *is a minimal CAP-subgroup of* G.

Lemma 3 ([14]) *Let* G *be a* p*-supersolvable group. Then the derived subgroup* G' *of* G *is* p*-nilpotent. In particular, if* $O_{p'}(G) = 1$*, then* G *has a unique (normal) Sylow* p*-subgroup.*

Lemma 4 ([15]) *Let* C *be a Thompson critical subgroup of a* p*-group* P*. Then the group* $D := \Omega(C)$ *is of exponent* p *if* p *is an odd prime, or exponent* 4 *if* P *is non-abelian* 2*-group. Moreover, every non-trivial* p'*-automorphism of* P *induces a non-trivial automorphism of* D.

Lemma 5 ([16]) *Let* $\mathfrak{F}$ *be a solvably saturated formation and* P *a normal* p*-subgroup of* G *and* C *is a Thompson critical subgroup of* P*. If either* $\Omega(C) \leqslant Z_{\mathfrak{F}}(G)$ *or* $P/\Phi(P) \leqslant Z_{\mathfrak{F}}(G/\Phi(P))$*, then* $P \leqslant Z_{\mathfrak{F}}(G)$.

Lemma 6 ([12]) *Let* $\mathfrak{F}$ *be a saturated formation,* E *be a normal subgroup of a finite group* G *and* N *a normal subgroup of* G *such that* $N \leqslant \Phi(E)$*. Then* $E \leqslant Z_{p\mathfrak{F}}(G)$ *if and only if* $E/N \leqslant Z_{p\mathfrak{F}}(G/N)$.

MAIN RESULTS

In this section, we present the main results, which give some criterions for $p\mathfrak{F}$-hypercentrally embedded property of subgroups.

Theorem 1 *Let* E *be a normal subgroup of a finite group* G *and* N *a minimal normal subgroup of* G *contained in* E *and* p *be a prime divisor of* $|G|$*. Assume that* $N_p > 1$ *and there is a normal subgroup* P *of* G_p *such that* $1 < N_p \leqslant P$ *and* $H \cap E$ *is a CAP-subgroup of* G *for any subgroup* H *of* P *with order* d*, where* $G_p \in Syl_p(G)$ *and* d *is a power of* p *with* $1 < d < |P|$*. Then* N *is a* p*-group with* $|N| \leqslant d$*. Furthermore, if* $|N| = d$*, or* $|N| < d$ *and* $N \nleq \Phi(P)$*, then* $|N| = p$.

Proof: Since $N_p > 1$, we may let

$$1 < T_0 \leqslant T_1 \leqslant \cdots \leqslant T_n = N_p \leqslant \cdots \leqslant T_k = P \leqslant \cdots \leqslant T_t = G_p$$

be a normal series of G_p such that $|T_i/T_{i-1}| = p$ for any $1 \leqslant i \leqslant t$. Assume that $|T_i| = d$. It is clear that $T_i \cap N = T_i \cap T_n \neq 1$. By the hypothesis and Lemma 1(4), $T_i \cap N = T_i \cap E \cap N$ is a *CAP*-subgroup of G. Then by Lemma 2 and the minimality of N, we have $T_i \cap N = N$. This implies that N is a p-group and $|N| \leqslant |T_i| = d$.

Assume $|N| = d$ and let $N \lhd U \leqslant P$. We can pick a subgroup U_1 of U such that $N \nleq U_1$ and $|U_1| = d$. If not, then N is the unique maximal subgroup of U with order d and so $\Phi(U) = N$. This shows that U is cyclic. It follows that N is also cyclic and so $|N| = p$. By the hypothesis and Lemma 1(4), we get that $U_1 \cap N$ is a *CAP*-subgroup of G. By Lemma 2 and the minimality of N, we have $U_1 \cap N = 1$ and so $|N| = |U_1| = p$, as required. Assume that $|N| < d$ and $N \nleq \Phi(P)$. Then $n < i$ and $N \nleq \Phi(T_{i+1})$ by [1, Hilfssatz III.3.3]. Hence we can pick a maximal subgroup K of T_{i+1} such that $N \nleq K$. Clearly, $|K| = d$. It is easy to see that $K \cap N$ is a *CAP*-subgroup of G. Then by Lemma 2 and the minimality of N, we have $K \cap N = 1$ and so $|N| = p$. □

Based on Theorem 1, we can obtain the following Theorem 2.

Theorem 2 *Let* $\mathfrak{F}$ *be a saturated formation containing* $\mathfrak{U}$ *and* E *be a normal subgroup of a finite group* G *such that* $G/E \in \mathfrak{F}$ *and* p *be a prime divisor of* $|G|$*. Let* K *be a normal subgroup of* G *and* $P \in Syl_p(K)$*. Assume that* $H \cap E$ *is a CAP-subgroup of* G *for any normal subgroup* H *of* P *with order* d*, where* d *is a power of* p *with* $1 \leqslant d < |P|$*. Then either* $K \leqslant Z_{p\mathfrak{F}}(G)$ *or else* $|P \cap E| > d$.

Proof: Let (G, E, K) be a counterexample with $|G| + |E| + |K|$ minimal order and let $T = P \cap E$. Then $K \nleq Z_{p\mathfrak{F}}(G)$ and $|T| \leqslant d$. Assume that $O_{p'}(G) > 1$. By Lemma 1(2), it is easy to prove that $(G/O_{p'}(G), EO_{p'}(G)/O_{p'}(G), KO_{p'}(G)/O_{p'}(G)$ satisfies the hypothesis of theorem. By the minimal choice of (G, E, K), we have $KO_{p'}(G)/O_{p'}(G) \leqslant Z_{p\mathfrak{F}}(G/O_{p'}(G))$ or $|PO_{p'}(G)/O_{p'}(G) \cap EO_{p'}(G)/O_{p'}(G)| > d$, which implies that either $K \leqslant Z_{p\mathfrak{F}}(G)$ or $|P \cap E| > d$, a contradiction. Hence $O_{p'}(G) = 1$. Since $G/E \in \mathfrak{F}$, by [2, Proposition IV.1.5], every chief factor of G/E is $\mathfrak{F}$-central. In particular, every chief factor of G/E below KE/E is $\mathfrak{F}$-central. Let A/B be a chief factor of G such that $K \cap E \leqslant B \leqslant A \leqslant K$, then $B \cap E = A \cap E$ and so A/B is G-isomorphic to AE/BE. It is clear that $C_G(A/B) \leqslant C_G(AE/BE)$. Assume that $x \in C_G(AE/BE)$, then $[x, A] \leqslant [x, AE] \leqslant BE$. Since $A \trianglelefteq G$, $[x, A] \leqslant A$. Hence $[x, A] \leqslant A \cap BE = B(A \cap E) = B$ and so $x \in C_G(A/B)$. Thus $C_G(A/B) = C_G(AE/BE)$. Since AE/BE is G-isomorphic to A/B,

$$(A/B) \rtimes (G/C_G(A/B)) \cong (AE/BE) \rtimes (G/C_G(AE/BE)) \in \mathfrak{F}.$$

Hence $K/K \cap E \leqslant Z_{p\mathfrak{F}}(G/K \cap E)$. If $T = 1$, then $K \cap E$ is a p'-group. But $O_{p'}(G) = 1$, so $K \cap E = 1$. It follows that $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. This contradiction shows that $T \neq 1$. Let N be a minimal normal subgroup of G contained in T^G. Clearly, $N \leqslant T^G \leqslant E \cap K$. Since $|T| \leqslant d$ and $T \trianglelefteq P$, there is a normal subgroup H of P such that $T \leqslant H \leqslant P$ and $|H| = d$. Hence $T = P \cap E = H \cap E$ is a *CAP*-subgroup of G. If T avoids $N/1$, then $T \cap N = P \cap N = 1$. It implies that N is a p'-subgroup, which contradicts $O_{p'}(G) = 1$. Hence T must covers $N/1$, that is, $TN = T$, so $N \leqslant P \cap E$. This shows that $|N| \leqslant d$. Assume that $|N| = d$, then $N = P \cap E = T$ is a Sylow p-subgroup of $K \cap E$. It implies that $(K \cap E)/N$ is a p'-group. By the previous proof, we have $K/(K \cap E) \leqslant Z_{p\mathfrak{F}}(G/(K \cap E))$. Hence $K/N \leqslant Z_{p\mathfrak{F}}(G/N)$. Assume that $|N| < d$. By Lemma 1(2), we can get that $(G/N, E/N, K/N)$ satisfies the hypothesis. Hence $K/N \leqslant Z_{p\mathfrak{F}}(G/N)$ or $|P/N \cap E/N| > d/|N|$ by the minimal choice of (G, E, K). If $|P/N \cap E/N| > d/|N|$, then $|T| = |P \cap E| > d$, a contradiction. Hence $K/N \leqslant Z_{p\mathfrak{F}}(G/N)$ whether $|N| < d$ or $|N| = d$. If $N \leqslant \Phi(K)$, then by Lemma 6, we have $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. Hence we assume that $N \nleq \Phi(K)$ and so $N \nleq \Phi(P)$. Let M be any maximal subgroup of P. Since $M \cap E \trianglelefteq P$ and $|M \cap E| \leqslant |P \cap E| \leqslant d$, there is a normal subgroup H of P such that $M \cap E \leqslant H \leqslant M$ and $|H| = d$. Hence $M \cap E = H \cap E$ is a *CAP*-subgroup of G. By Theorem 1, $|N| = p$. Hence $N \leqslant Z_{p\mathfrak{F}}(G)$, which implies that $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. The contradiction completes the proof. □

By Theorem 2, we can easily obtain the following corollaries.

Corollary 1 *Let E be a normal subgroup of a finite group G such that G/E is supersolvable and p be a prime divisor of $|G|$ and let $P \in Syl_p(G)$. Assume that $H \cap E$ is a CAP-subgroup of G for any normal subgroup H of P with order d, where d is a power of p with $1 \leqslant d < |P|$. Then either G is p-supersolvable or else $|P \cap E| > d$.*

Corollary 2 ([5]) *Let p be a prime, let $P \in Syl_p(G)$, where G is a finite group, and let d be a power of p such that $1 \leqslant d < |P|$. Write $U = O^p(G)$, and assume that $H \cap U \lhd U$ for all subgroups $H \lhd P$ with $|H| = d$. Then either G is p-supersolvable or else $|P \cap U| > d$.*

Remark 1 Corollary 2 represents one of the main theorems in [5]. It is easy to observe that this result can be directly derived from Theorem 2. Indeed, since $H \cap O^p(G) \lhd O^p(G)$ and $H \lhd P$, we have $H \cap O^p(G) \lhd G$. Clearly, $H \cap O^p(G)$ is a *CAP*-subgroup of G. Hence this result is a direct corollary of Theorem 2.

Theorem 3 *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and E be a normal subgroup of a finite group G such that $G/E \in \mathfrak{F}$ and p be a prime divisor of $|G|$. Let K be a normal subgroup of G and $P \in Syl_p(K)$. Then $K \leqslant Z_{p\mathfrak{F}}(G)$ if one of the following holds:*

(1) $H \cap E$ is a CAP-subgroup of G for any subgroup H of P with order p. If P is a non-abelian 2-group, assume further that $H \cap E$ is a CAP-subgroup of G for any cyclic subgroup H of P with order 4.

(2) $p = 2$, $|P| \geqslant 8$ and $H \cap E$ is a CAP-subgroup of G for any subgroup H of P with order 4.

Proof: Let (G, E, K) be a counterexample with $|G| + |E| + |K|$ minimal order. Similar to the proof of Theorem 2, we may assume that $O_{p'}(G) = 1$. By the hypothesis, $G/E \in \mathfrak{F}$ and $K \trianglelefteq G$. Similar to the proof of Theorem 2, it is easy to prove that $K/(K \cap E) \leqslant Z_{p\mathfrak{F}}(G/(K \cap E))$. If $P \cap E = 1$, then $K \cap E$ is a p'-group and so $K \cap E = 1$. It follows that $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. Hence we may assume that $P \cap E > 1$, so $K \cap E > 1$.

Firstly, we prove (1). Assume that $K \cap E < K$. Let K/T be a chief factor of G such that $K \cap E \leqslant T < K$. It is easy to see that (G, E, T) satisfies the hypothesis of the theorem. Hence $K \cap E \leqslant T \leqslant Z_{p\mathfrak{F}}(G)$ by the minimal choice of (G, E, K). Since $K/(K \cap E) \leqslant Z_{p\mathfrak{F}}(G/(K \cap E))$, $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. Hence $K \cap E = K$ and so $K \leqslant E$. Then every cyclic subgroup of P with order p and 4 (if P is a non-abelian 2-group) is a *CAP*-subgroup of G. Then by [13, Lemma 2.2(1) and (2)], every cyclic subgroup of P with order p and 4 (if P is a non-abelian 2-group) is a semi *CAP*-subgroup of K. By [17, Lemma 1.4], we can get that K is p-supersolvable. Since $O_{p'}(G) = 1$, by Lemma 3, we get $P\ char\ K$ and so $P \trianglelefteq G$. It is clear that (G, E, P) satisfies the hypothesis. If $P < K$, by the minimal choice of (G, E, K), we have $P \leqslant Z_{p\mathfrak{F}}(G)$. Note that P is a Sylow p-subgroup of K, thus K/P is a p'-group. It follows that $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. This contradiction shows that $K = P$ is a p-group.

If K is a minimal normal subgroup of G, then $|K| = p$ and so $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. Hence K is not minimal normal in G. Now, let K/T be a chief factor of G. Then (G, E, T) satisfies the hypothesis of the theorem, and so $T \leqslant Z_{p\mathfrak{F}}(G)$ by the minimal choice of (G, E, K). Since T is a p-group, $T \leqslant Z_{\mathfrak{F}}(G)$. Let L be any normal subgroup of G such that $L < K$. Then we also have $L \leqslant Z_{p\mathfrak{F}}(G)$. If $L \nleq T$, then $K = TL \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. Hence we may assume that T is the unique normal subgroup of G such that K/T is a chief factor of G. If $|K/T| = p$, then $K/T \leqslant Z_{p\mathfrak{F}}(G/T)$ and so $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. Hence $|K/T| > p$. Let C be a Thompson critical subgroup of K. If $\Omega(C) < K$, we have $\Omega(C) \leqslant T \leqslant Z_{\mathfrak{F}}(G)$. Then, by Lemma 5, $K \leqslant Z_{\mathfrak{F}}(G) \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. This implies that $\Omega(C) = K$. Hence $exp(K) = p$ or 4 by Lemma 4. Let x be an element of $K \setminus T$. Then $|\langle x \rangle| = p$ or 4.

By hypothesis, $\langle x \rangle \cap E$ covers or avoids K/T. In the former case, we have $\langle x \rangle K = \langle x \rangle T$, that is, $K = \langle x \rangle T$. It follows that K/T is cyclic and so $K/T \leqslant Z_{p\mathfrak{F}}(G/T)$, which implies that $K \leqslant Z_{p\mathfrak{F}}(G)$, a contradiction. In the latter case, $\langle x \rangle = \langle x \rangle \cap K = \langle x \rangle \cap T$ implies $\langle x \rangle = 1$ or $\Phi(\langle x \rangle)$, a contradiction. Thus, (1) holds.

Now, we prove (2). Let N be a minimal normal subgroup of G contained in $K \cap E$. By Theorem 1, N is a 2-group and $|N| \leqslant 4$. If $|N| = 4$, again by Theorem 1, $|N| = 2$, a contradiction. Hence $|N| = 2$ and so $N \leqslant Z(G)$. Let x be any element of $P \setminus N$ with order 2. Then $\langle x \rangle N = \langle x \rangle \ltimes N$ is a subgroup of P and $|\langle x \rangle N| = 4$. If $x \notin E$, then $\langle x \rangle \cap E = 1$ is a *CAP*-subgroup of G. If $x \in E$, then $\langle x \rangle N = \langle x \rangle N \cap E$ is a *CAP*-subgroup of G. Let A/B be any chief factor of G. Then $(\langle x \rangle N)A = (\langle x \rangle N)B$ or $\langle x \rangle N \cap A = \langle x \rangle N \cap B$.

Since N is a minimal normal subgroup of G, we distinguish three cases to complete the proof.

Case 1: $N \leqslant B \leqslant A$.

If $(\langle x \rangle N)A = (\langle x \rangle N)B$, then $\langle x \rangle A = \langle x \rangle B$; If $\langle x \rangle N \cap A = \langle x \rangle N \cap B$, then $(\langle x \rangle \cap A)N = (\langle x \rangle \cap B)N$. Note that $\langle x \rangle \cap N = 1$, hence $\langle x \rangle \cap A = \langle x \rangle \cap B$.

Case 2: $N \nleq B$ and $N \leqslant A$.

Obviously, $N \times B = A$ and so $|A/B| = |N| = 2$.

If $(\langle x\rangle N)A=(\langle x\rangle N)B$, then we claim that $\langle x\rangle\cap A=\langle x\rangle\cap B$. If not, then $\langle x\rangle\cap B=1$ and $\langle x\rangle\leqslant A$. It implies that $A=\langle x\rangle NB$ and so $|A/B|=4$, a contradiction; If $\langle x\rangle N\cap A=\langle x\rangle N\cap B$, then $(\langle x\rangle\cap A)N=\langle x\rangle N\cap B$. If $\langle x\rangle\cap A\neq\langle x\rangle\cap B$, then $\langle x\rangle\cap B=1$ and $\langle x\rangle\leqslant A$. Hence $\langle x\rangle N=\langle x\rangle N\cap B$ and so $\langle x\rangle N\leqslant B$. This shows that $\langle x\rangle\leqslant B$, a contradiction. Thus, $\langle x\rangle\cap A=\langle x\rangle\cap B$.

Case 3: $N\nleq B$ and $N\nleq A$.

If $(\langle x\rangle N)A=(\langle x\rangle N)B$, it is easy to prove that $\langle x\rangle\cap A=\langle x\rangle\cap B$. If not, we have $\langle x\rangle\cap B=1$ and $\langle x\rangle\leqslant A$. Then $|AN|=|\langle x\rangle NB|$ and so $|A||N|=|\langle x\rangle||N||B|$. Hence $|A|=|\langle x\rangle||B|$. This induce that $\langle x\rangle A=A=\langle x\rangle B$; If $\langle x\rangle N\cap A=\langle x\rangle N\cap B$, we also get $\langle x\rangle\cap A=\langle x\rangle\cap B$. If not, $\langle x\rangle\cap B=1$ and $\langle x\rangle\leqslant A$. Thus, $\langle x\rangle=\langle x\rangle(N\cap A)=\langle x\rangle N\cap B\leqslant B$, a contradiction.

The above three cases show that $\langle x\rangle\cap E=\langle x\rangle$ is a *CAP*-subgroup of G. Then by (1), $K\leqslant Z_{p\mathfrak{F}}(G)$. The proof is complete. □

Based on the preceding theorems, we obtain the following Theorem.

Theorem 4 *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, E be a normal subgroup of a finite group G such that $G/E\in\mathfrak{F}$ and p be a prime divisor of $|G|$. Let K be a normal subgroup of G, $P\in Syl_p(K)$ and d a power of p with $1<d<|P|$. Assume that $H\cap E$ is a CAP-subgroup of G for any subgroup H of P with order d. If P is a non-abelian 2-group, assume further that $H\cap E$ is a CAP-subgroup of G for any cyclic subgroup H of P with order 4. Then $K\leqslant Z_{p\mathfrak{F}}(G)$.*

Proof: By Theorem 3, we may assume that $d>p$. Similar to the proof of Theorem 2, we assume further that $O_{p'}(G)=1$ and $K\cap E>1$. Let N be a minimal normal subgroup of G contained in $K\cap E$. Clearly, $1<N_p\leqslant P\trianglelefteq G_p$ for some Sylow p-subgroup G_p of G. By Theorem 1, N is p-group and $|N|\leqslant d$. If $|N|=d$, again by Theorem 1, $d=|N|=p$, a contradiction. Hence $|N|<d$.

Now, we claim that $K/N\leqslant Z_{p\mathfrak{F}}(G/N)$. Actually, if $d/|N|\neq 2$, then $(G/N,E/N,K/N)$ satisfies the hypothesis of the theorem, and so we can get $K/N\leqslant Z_{p\mathfrak{F}}(G/N)$ by induction. Assume $d/|N|=2$. If P/N has a cyclic subgroup X/N of order 4 with $N\leqslant\Phi(X)$, then X is cyclic and therefore N is cyclic. Hence $|N|=2$ and $d=4$. By Theorem 3, $K\leqslant Z_{p\mathfrak{F}}(G)$. Hence, we may assume that $N\nleq\Phi(X)$ for any cyclic subgroup X/N of order 4 in P/N. Then there is a maximal subgroup X_1 of X such that $X=X_1N$. Note that $|X_1|=d$, then $X/N\cap E/N=(X_1N\cap E)/N=(X_1\cap E)N/N$ is a *CAP*-subgroup of G/N by Lemma 1 (2). This shows that K/N satisfies the hypothesis of Theorem 3, hence $K/N\leqslant Z_{p\mathfrak{F}}(G/N)$.

If $N\leqslant\Phi(K)$, then $K\leqslant Z_{p\mathfrak{F}}(G)$ by Lemma 6. If $N\nleq\Phi(K)$, then $|N|=p$ by Theorem 1. Hence $K\leqslant Z_{p\mathfrak{F}}(G)$. □

Corollary 3 *Let G be a finite group, p be an odd prime divisor of $|G|$ and $P\in Syl_p(G)$ and let d be a power of p with $1<d<|P|$. Assume that $H\cap O^p(G)$ is a CAP-subgroup of G for any subgroup H of P with order d. Then G is p-supersolvable.*

CONCLUSION

In this paper, by investigating the intersections of some subgroups with fixed order and a normal subgroup satisfying the *CAP*-property, we obtain some new characterizations for the hypercentrally embedded property of normal subgroups of a finite group (i.e., Theorems 2–4). In particular, Theorem 2 generalizes one of the main theorems of [5].

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