

Gradient estimate for eigenfunctions of the operator \mathfrak{L} on self-shrinkers

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ABSTRACT: In this paper, we study gradient estimates for eigenfunctions associated to the operator \mathfrak{L} on self-shrinkers. As applications, we obtain a Harnack type inequality concerning those eigenfunctions. Besides, we obtain a gradient estimate of the higher eigenfunctions of the operator \mathfrak{L} on self-shrinkers.

KEYWORDS: eigenfunction, self-shrinker, ∞ -Bakry-Émery Ricci tensor, gradient estimate, Harnack inequality

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INTRODUCTION

Mean curvature flow is an evolution equation where a one-parameter family of $M_t \subset \mathbb{R}^{n+1}$ hypersurfaces flows by mean curvature, that is, it satisfies

$$(\partial_t X)^\perp = -HN, \tag{1}$$

where X is the position vector, H is the mean curvature and N is the outward unit normal. $(\cdot)^\perp$ denotes the projection on the normal bundle of M .

We call a hypersurface $M^n \subset \mathbb{R}^{n+1}$ a self-shrinker, if it satisfies

$$H = \frac{\langle X, N \rangle}{2}. \tag{2}$$

The self-shrinker plays an important role in the study of mean curvature flow. It appears as the rescaling limit of the Type I singularity of the mean curvature flow. For more information on self-shrinkers and singularities of mean curvature flow, we refer the readers to [1–4] and references therein.

In [1], Colding and Minicozzi introduced the following differential operator \mathfrak{L} and used it to study self-shrinkers:

$$\mathfrak{L}(\cdot) = \Delta(\cdot) - \frac{1}{2}\langle X, \nabla(\cdot) \rangle, \tag{3}$$

where Δ, ∇ denote the Laplacian, the gradient operator on the self-shrinker, respectively, $\langle \cdot, \cdot \rangle$ stands for the standard inner product in \mathbb{R}^{n+1} .

In [5], Cheng and Peng investigated the closed eigenvalue problem of the differential operator \mathfrak{L} on an n -dimensional compact self-shrinker, and obtained some universal inequalities for the eigenvalues of the drifting Laplacian. We refer the readers to [6–11] and references therein for more information about the eigenvalues of \mathfrak{L} on self-shrinkers.

In this paper, we will deal with eigenfunctions of the operator \mathfrak{L} on self-shrinkers. Our first result is the next theorem that presents a gradient estimate for eigenfunctions of \mathfrak{L} on a compact self-shrinker with

boundary, under Neumann boundary conditions, as well as on a closed self-shrinker.

Theorem 1 *Let $X: M^n \rightarrow \mathbb{R}^{n+1}$ ($n \geq 2$) be an n -dimensional compact self-shrinker with convex boundary. Suppose $|A| \leq K_1$ and $|X^\top| \leq K_2$, where A and X^\top denote the second fundamental form and the tangential projection of X , respectively, and both $K_1 \geq \sqrt{2}/2$ and K_2 are arbitrary nonnegative constants. Let u be a solution of $\mathfrak{L}u = -\lambda u$, bounded from below, satisfying the Neumann boundary condition $u_\nu = 0$ on ∂M , whenever $\partial M \neq \emptyset$. Then, for any $\alpha > 0$ and $\beta > 0$,*

$$|\nabla u| \leq C(u - \inf_M u), \tag{4}$$

where

$$C = \left\{ \left[\sqrt{\left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right)^2 (1+\alpha)^2 (1+\beta)^2 (n-1)^2 \beta^2 + 4\beta(1+\beta)\lambda^2} + \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) (1+\alpha)(1+\beta)(n-1)\beta \right] \cdot \frac{1}{2\beta} \right\}^{\frac{1}{2}}.$$

Moreover, if $K_1^2 + \frac{K_2^2}{4\alpha(n-1)} = \frac{1}{2}$ and taking the limit as β approaches to infinity, we can assume $C = \sqrt{|\lambda|}$.

Furthermore, we obtain a gradient estimate for eigenfunctions of $\mathfrak{L}u = -\lambda u$ on balls in complete self-shrinkers with $|A| \leq K_3$ ($\geq \sqrt{2}/2$) and $|X^\top| \leq K_4$.

Theorem 2 *Let $X: M^n \rightarrow \mathbb{R}^{n+1}$ ($n \geq 2$) be an n -dimensional complete self-shrinker. Fix a point $x \in M^n$, let $B(x, r)$ be a geodesic ball of radius r and centered at x . And for any $K_3 \geq \sqrt{2}/2$ and $K_4 \geq 0$, we assume $|A| \leq K_3$ and $|X^\top| \leq K_4$ on $B(x, r)$, where A and X^\top denote the second fundamental form and the tangential projection of X , respectively. If u is a positive solution of $\mathfrak{L}u = -\lambda u$ on M , then, for any $\alpha > 0$ and $\beta > 0$,*

$$\sup_{B(x, r/2)} \frac{|\nabla u|}{u} \leq C, \tag{5}$$

where $C = C(\alpha, \beta, n, K_3, K_4, r, \lambda)$ is a positive constant depending on $\alpha, \beta, n, K_3, K_4, r$ and λ , and the supremum is taking over balls $B(x, r/2)$ in M centered at a point x with radius $r/2$.

As an application, we have the following Harnack type inequalities:

Corollary 1 *Let $X: M^n \rightarrow \mathbb{R}^{n+1}$ ($n \geq 2$) be an n -dimensional complete self-shrinker. Fix a point $x \in M^n$, let $B(x, r)$ be a geodesic ball of radius r and centered at x . And for any $K_3 \geq \sqrt{2}/2$ and $K_4 \geq 0$, we assume $|A| \leq K_3$ and $|X^\top| \leq K_4$ on $B(x, r)$, where A and X^\top denote the second fundamental form and the tangential projection of X , respectively.*

(i) *If u is a solution of $\mathcal{L}u = -\lambda u$ on a geodesic ball $B(x, r)$, then*

$$\sup_{B(x, r/2)} |\nabla u| \leq 2C \sup_{B(x, r)} |u|.$$

(ii) *If u is a positive solution of $\mathcal{L}u = -\lambda u$ on a geodesic ball $B(x, r)$, then*

$$\sup_{B(x, r/2)} u \leq e^{2Cr} \inf_{B(x, r/2)} u.$$

In both cases $C = C(\alpha, \beta, n, K_3, K_4, r, \lambda)$ is a positive constant depending on $\alpha, \beta, n, K_3, K_4, r$ and λ .

We point out that the above theorems generalize some results due to Zhu and Chen [12] obtained for $\mathcal{L}u = 0$. In the next sections we will present the proofs of them.

In [13], Wang and Zhou showed the lower bound for the higher eigenvalues of the Hodge Laplacian on a Riemannian manifold with Ricci curvature bounded from below. Following the ideas in the paper of Wang and Zhou [13], Dung, Le Hai and Thanh [14] showed a gradient estimate of the higher eigenfunctions of the weighted Laplacian on gradient steady Ricci soliton. Motivated by the above results, we will prove the following theorem.

Theorem 3 *Let $X: M^n \rightarrow \mathbb{R}^{n+1}$ be an n -dimensional compact self-shrinker. Suppose $|A| \leq \sqrt{2}/2$ and $|X^\top| \leq 2a$ for some constant $a > 0$, where A and X^\top denote the second fundamental form and the tangential projection of X , respectively. Then*

- (i) $|\nabla \phi_l| \leq c \lambda_l^{(n+2)/4}$, $|\phi_l| \leq c \lambda_l^{n/4}$;
- (ii) $\lambda_l \geq c^{-1} l^{n/2}$.

Here ϕ_l be an eigenfunction of the \mathcal{L} with respect to the eigenvalue λ_l and $\|\phi_l\|_\varphi^2 := \int_M \phi_l^2 e^{-\varphi} dv = 1$.

GRADIENT ESTIMATE ON COMPACT SELF-SHRINKERS WITH BOUNDARY

Let $X: M^n \rightarrow \mathbb{R}^{n+1}$ be an n -dimensional compact hypersurface with boundary ∂M in the Euclidean space \mathbb{R}^{n+1} . We choose a local orthonormal frame field

$\{e_\alpha\}_{\alpha=1}^{n+1}$ in \mathbb{R}^{n+1} with dual coframe field $\{\omega_\alpha\}_{\alpha=1}^{n+1}$, such that, at any $x \in M^n$, e_1, \dots, e_n are the unit tangent vectors and $e_{n+1} = N$ is the unit normal vector to M^n , and $e_n = \nu$ is the unit normal vector to ∂M . Let $\langle \cdot, \cdot \rangle$ and $\bar{\nabla}$ denote the standard inner product and Levi-Civita connection of \mathbb{R}^{n+1} . The coefficients of second fundamental form A of M^n are defined to be $A_{ij} = -\langle \bar{\nabla}_{e_i} e_j, N \rangle$. The mean curvature of M^n is expressed by $H = \sum_{i=1}^n A_{ii}$.

Let $\varphi = |X|^2/4$, and denote by dV the corresponding weighted volume measure of M^n ,

$$dV = e^{-\varphi} dv,$$

where dv is the volume form on M^n . Let g and ∇ be the Riemannian metric on M^n induced by $\langle \cdot, \cdot \rangle$ and the Levi-Civita connection induced $\bar{\nabla}$, respectively. Then $M^n = (M^n, g, dV)$ is a smooth weighted metric measure space, and the drifting Laplacian operator

$$\mathcal{L}(\cdot) = \Delta(\cdot) - g(\nabla\varphi, \nabla(\cdot)) = \Delta(\cdot) - \frac{1}{2}\langle X, \nabla(\cdot) \rangle$$

is a self-adjoint operator with respect to the weighted measure dV , where ∇ and Δ be the gradient and the Laplacian on M^n , respectively. The ∞ -Bakry-Émery Ricci tensor Ric_φ of M^n is defined by

$$\text{Ric}_\varphi = \text{Ric} + \text{Hess}(\varphi).$$

From [15] (see also [12]), we get the following lower bound for the ∞ -Bakry-Émery Ricci tensor Ric_φ of self-shrinkers,

$$\text{Ric}_\varphi \geq \frac{1}{2} - |A|^2. \tag{6}$$

The next algebraic estimate will be useful: for any a, b real numbers and α strictly positive, we have

$$(a + b)^2 \geq \frac{a^2}{1 + \alpha} - \frac{b^2}{\alpha}, \tag{7}$$

and equality holds if and only if $b = -\frac{\alpha}{1+\alpha}a$. Applying (6) and (7) we first deduce the following proposition.

Proposition 1 *Let $X: M^n \rightarrow \mathbb{R}^{n+1}$ ($n \geq 2$) be an n -dimensional compact self-shrinker with $|A| \leq K_1$ and $|X^\top| \leq K_2$, where A and X^\top denote the second fundamental form and the tangential projection of X , respectively, and both K_1 and K_2 are arbitrary nonnegative constants. Let u be a solution of $\mathcal{L}u = -\lambda u$ with λ constant. Then, for any $\alpha > 0$ and $\beta > 0$,*

$$|\nabla u| \mathcal{L}|\nabla u| \geq \frac{|\nabla(|\nabla u|)|^2}{(1 + \alpha)(1 + \beta)(n - 1)} - \frac{(\lambda u)^2}{(1 + \alpha)\beta(n - 1)} - \left(\frac{K_2^2}{4\alpha(n - 1)} + K_1^2 - \frac{1}{2} + \lambda \right) |\nabla u|^2. \tag{8}$$

Proof: We start using that $\mathcal{L}|\nabla u|^2 = 2|\nabla u| \mathcal{L}|\nabla u| + 2|\nabla(|\nabla u|)|^2$ and the Bochner formula

$$\frac{1}{2} \mathcal{L}|\nabla u|^2 = |\nabla^2 u|^2 + \text{Ric}_\varphi(\nabla u, \nabla u) + \langle \nabla u, \nabla \mathcal{L}u \rangle, \tag{9}$$

to arrive at the following identity

$$\begin{aligned} |\nabla u|\mathfrak{L}|\nabla u| &= \frac{1}{2}\mathfrak{L}|\nabla u|^2 - |\nabla(|\nabla u|)|^2 \\ &= |\nabla^2 u|^2 + \text{Ric}_\varphi(\nabla u, \nabla u) + \langle \nabla u, \nabla \mathfrak{L}u \rangle - |\nabla(|\nabla u|)|^2 \\ &= |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \text{Ric}_\varphi(\nabla u, \nabla u) - \lambda|\nabla u|^2, \\ &= |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \text{Ric}_\varphi(\nabla u, \nabla u) - \lambda|\nabla u|^2. \end{aligned} \quad (10)$$

Note that $\text{Ric}_\varphi \geq \frac{1}{2} - |A|^2 \geq \frac{1}{2} - K_1^2$, we have

$$|\nabla u|\mathfrak{L}|\nabla u| \geq |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \left(\frac{1}{2} - K_1^2 - \lambda\right)|\nabla u|^2. \quad (11)$$

Proceeding, given $p \in M$ we choose an orthonormal frame $\{e_1, \dots, e_n\}$ around p so that $u_1(p) = |\nabla u|(p)$ and $u_i(p) = 0$, for $2 \leq i \leq n$, where $u_i := e_i(u)$. Thus,

$$|\nabla(|\nabla u|)|^2 = |\nabla u_1|^2 = \sum_{1 \leq j \leq n} u_{1j}^2 \quad (12)$$

and

$$\begin{aligned} -\sum_{2 \leq i \leq n} u_{ii} &= -\Delta u + u_{11} = -\mathfrak{L}u + u_{11} - \langle \nabla \varphi, u_1 e_1 \rangle \\ &= \lambda u + u_{11} - \varphi_1 u_1. \end{aligned} \quad (13)$$

Therefore,

$$\begin{aligned} |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 &= \sum_{1 \leq i, j \leq n} u_{ij}^2 - \sum_{1 \leq j \leq n} u_{1j}^2 \\ &= \sum_{i \neq 1, 1 \leq j \leq n} u_{ij}^2 \\ &\geq \sum_{2 \leq i \leq n} u_{i1}^2 + \sum_{2 \leq i \leq n} u_{ii}^2 \\ &\geq \sum_{2 \leq i \leq n} u_{i1}^2 + \frac{1}{n-1} \left(\sum_{2 \leq i \leq n} u_{ii} \right)^2 \\ &= \sum_{2 \leq i \leq n} u_{i1}^2 + \frac{1}{n-1} (\lambda u + u_{11} - \varphi_1 u_1)^2. \end{aligned}$$

Using twice inequality (7) we obtain, for any α, β , both strictly positive, the following inequality

$$\begin{aligned} (\lambda u + u_{11} - \varphi_1 u_1)^2 &\geq \frac{(\lambda u + u_{11})^2}{1 + \alpha} - \frac{(\varphi_1 u_1)^2}{\alpha} \\ &\geq \frac{1}{1 + \alpha} \left(\frac{u_{11}^2}{1 + \beta} - \frac{(\lambda u)^2}{\beta} \right) - \frac{(\varphi_1 u_1)^2}{\alpha} \\ &= \frac{u_{11}^2}{(1 + \alpha)(1 + \beta)} - \frac{(\lambda u)^2}{(1 + \alpha)\beta} - \frac{(\varphi_1 u_1)^2}{\alpha}. \end{aligned}$$

Hence, for any $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned} |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 &\geq \sum_{2 \leq i \leq n} u_{i1}^2 + \frac{1}{n-1} \left(\frac{u_{11}^2}{(1 + \alpha)(1 + \beta)} - \frac{(\lambda u)^2}{(1 + \alpha)\beta} - \frac{(\varphi_1 u_1)^2}{\alpha} \right) \\ &= \left(\sum_{2 \leq i \leq n} u_{i1}^2 + \frac{u_{11}^2}{(1 + \alpha)(1 + \beta)(n-1)} \right) - \frac{(\lambda u)^2}{(1 + \alpha)\beta(n-1)} - \frac{(\varphi_1 u_1)^2}{\alpha(n-1)} \\ &\geq \frac{1}{(1 + \alpha)(1 + \beta)(n-1)} \sum_{1 \leq i \leq n} u_{i1}^2 - \frac{(\lambda u)^2}{(1 + \alpha)\beta(n-1)} - \frac{(\varphi_1 u_1)^2}{\alpha(n-1)} \\ &= \frac{|\nabla(|\nabla u|)|^2}{(1 + \alpha)(1 + \beta)(n-1)} - \frac{(\lambda u)^2}{(1 + \alpha)\beta(n-1)} - \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\alpha(n-1)} \\ &\geq \frac{|\nabla(|\nabla u|)|^2}{(1 + \alpha)(1 + \beta)(n-1)} - \frac{(\lambda u)^2}{(1 + \alpha)\beta(n-1)} - \frac{|\nabla \varphi|^2 |\nabla u|^2}{\alpha(n-1)}. \end{aligned}$$

Since $|\nabla \varphi| = |X^\top|/2 \leq K_2/2$, we have

$$\begin{aligned} |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 &\geq \frac{|\nabla(|\nabla u|)|^2}{(1 + \alpha)(1 + \beta)(n-1)} \\ &\quad - \frac{(\lambda u)^2}{(1 + \alpha)\beta(n-1)} - \frac{K_2^2}{4\alpha(n-1)} |\nabla u|^2. \end{aligned} \quad (14)$$

From inequalities (11) and (14) we arrive at

$$\begin{aligned} |\nabla u|\mathfrak{L}|\nabla u| &\geq |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \left(\frac{1}{2} - K_1^2 - \lambda\right)|\nabla u|^2 \\ &\geq \frac{|\nabla(|\nabla u|)|^2}{(1 + \alpha)(1 + \beta)(n-1)} - \frac{(\lambda u)^2}{(1 + \alpha)\beta(n-1)} \\ &\quad - \frac{K_2^2}{4\alpha(n-1)} |\nabla u|^2 + \left(\frac{1}{2} - K_1^2 - \lambda\right)|\nabla u|^2 \\ &= \frac{|\nabla(|\nabla u|)|^2}{(1 + \alpha)(1 + \beta)(n-1)} - \frac{(\lambda u)^2}{(1 + \alpha)\beta(n-1)} \\ &\quad - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} + \lambda \right) |\nabla u|^2. \end{aligned}$$

We complete the proof of Proposition 1. □

PROOF OF THEOREMS 1 AND 2

We will start with the proof of Theorem 1.

Proof: We can suppose u positive, otherwise, we replace u by $u - \inf_M u$. With this choice we can define $\phi := |\nabla u|/u = |\nabla \ln u|$. Then, we infer

$$\nabla \phi = \frac{\nabla|\nabla u|}{u} - \frac{|\nabla u|\nabla u}{u^2}. \quad (15)$$

At any point where $|\nabla u| \neq 0$, we have

$$\begin{aligned} \mathfrak{L}|\nabla u| &= u\mathfrak{L}\phi + \phi\mathfrak{L}u + 2\langle \nabla \phi, \nabla u \rangle \\ &= u\mathfrak{L}\phi - \lambda|\nabla u| + 2\langle \nabla \phi, \nabla u \rangle. \end{aligned}$$

Using Proposition 1, we deduce for any $\alpha > 0$ and $\beta > 0$,

$$\begin{aligned} \mathfrak{L}\phi &= \frac{\mathfrak{L}|\nabla u|}{u} + \frac{\lambda|\nabla u|}{u} - \frac{2\langle \nabla\phi, \nabla u \rangle}{u} \\ &\geq \frac{1}{u|\nabla u|} \left\{ \frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)} \right. \\ &\quad \left. - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} + \lambda \right) |\nabla u|^2 \right\} + \frac{\lambda|\nabla u|}{u} \\ &\quad - \frac{2\langle \nabla\phi, \nabla u \rangle}{u} \\ &= \frac{1}{u|\nabla u|} \left\{ \frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)} \right. \\ &\quad \left. - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) |\nabla u|^2 \right\} - \frac{2\langle \nabla\phi, \nabla u \rangle}{u} \\ &= \frac{1}{(1+\alpha)(1+\beta)(n-1)} \frac{|\nabla(|\nabla u|)|^2}{u|\nabla u|} \\ &\quad - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) \phi - \frac{1}{(1+\alpha)\beta(n-1)} \frac{\lambda^2}{\phi} \\ &\quad - \frac{2\langle \nabla\phi, \nabla u \rangle}{u}. \end{aligned}$$

We have for any $\varepsilon > 0$,

$$\begin{aligned} \frac{2\langle \nabla\phi, \nabla u \rangle}{u} &= (2-\varepsilon) \frac{\langle \nabla\phi, \nabla u \rangle}{u} + \varepsilon \frac{\langle \nabla(|\nabla u|), \nabla u \rangle}{u^2} - \varepsilon \frac{|\nabla u|^3}{u^3} \\ &\leq (2-\varepsilon) \frac{\langle \nabla\phi, \nabla u \rangle}{u} + \varepsilon \frac{|\nabla(|\nabla u|)||\nabla u|}{u^2} - \varepsilon\phi^3 \end{aligned}$$

and

$$\varepsilon \frac{|\nabla(|\nabla u|)||\nabla u|}{u^2} \leq \frac{\varepsilon}{2} \left(\frac{|\nabla(|\nabla u|)|^2}{|\nabla u|u} + \frac{|\nabla u|^3}{u^3} \right).$$

Therefore

$$\begin{aligned} \mathfrak{L}\phi &\geq \frac{1}{(1+\alpha)(1+\beta)(n-1)} \frac{|\nabla(|\nabla u|)|^2}{u|\nabla u|} \\ &\quad - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) \phi - \frac{1}{(1+\alpha)\beta(n-1)} \frac{\lambda^2}{\phi} \\ &\quad - \frac{2\langle \nabla\phi, \nabla u \rangle}{u} \\ &\geq \frac{2}{(1+\alpha)(1+\beta)(n-1)} \frac{|\nabla(|\nabla u|)||\nabla u|}{u^2} \\ &\quad - \frac{1}{(1+\alpha)(1+\beta)(n-1)} \phi^3 - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) \phi \\ &\quad - \frac{1}{(1+\alpha)\beta(n-1)} \frac{\lambda^2}{\phi} - (2-\varepsilon) \frac{\langle \nabla\phi, \nabla u \rangle}{u} \\ &\quad - \varepsilon \frac{|\nabla(|\nabla u|)||\nabla u|}{u^2} + \varepsilon\phi^3. \end{aligned}$$

Taking $\varepsilon = 2/(1+\alpha)(1+\beta)(n-1)$, we conclude that

$$\begin{aligned} \mathfrak{L}\phi &\geq - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) \phi - \frac{1}{(1+\alpha)\beta(n-1)} \frac{\lambda^2}{\phi} \\ &\quad - \left(2 - \frac{2}{(1+\alpha)(1+\beta)(n-1)} \right) \frac{\langle \nabla\phi, \nabla u \rangle}{u} \\ &\quad + \frac{1}{(1+\alpha)(1+\beta)(n-1)} \phi^3. \end{aligned} \tag{16}$$

Suppose that ϕ attains its maximum at a point $x_0 \in M$. We claim that x_0 is an interior point of M . Otherwise, by the strong maximum principle, $\phi_\nu(x_0) > 0$. Indeed, suppose that $x_0 \in \partial M$. Proceeding, we choose an orthonormal frame $\{e_1, \dots, e_n = \nu\}$ on TM . Then, at x_0 ,

$$u^2|\nabla u|\phi_\nu = u \left(\sum_{j=1}^{n-1} u_j u_{j\nu} + u_\nu u_{\nu\nu} \right) - |\nabla u|^2 u_\nu.$$

Let us denote by a_{jk} the components of the second fundamental form of ∂M to deduce, from Neumann condition, the following identity

$$u^2|\nabla u|\phi_\nu = u \sum_{j=1}^{n-1} u_j u_{j\nu} = -u \sum_{j,k=1}^{n-1} a_{jk} u_j u_k.$$

From the convexity boundary condition, we obtain $\phi_\nu(x_0) \leq 0$, which is a contradiction. Thus, x_0 lies in the interior of M . Moreover, $\nabla\phi(x_0) = 0$ and $\mathfrak{L}\phi(x_0) \leq 0$. Whence, using inequality (16), we deduce

$$\begin{aligned} 0 &\geq - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) \phi(x_0) \\ &\quad - \frac{1}{(1+\alpha)\beta(n-1)} \frac{\lambda^2}{\phi(x_0)} + \frac{1}{(1+\alpha)(1+\beta)(n-1)} \phi^3(x_0). \end{aligned}$$

That is,

$$\begin{aligned} \beta\phi^4(x_0) - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) (1+\alpha)(1+\beta)(n-1)\beta\phi^2(x_0) \\ - (1+\beta)\lambda^2 \leq 0. \end{aligned} \tag{17}$$

Therefore, there is a constant $C = C(n, K_1, K_2, \lambda) > 0$ such that, $\phi(x_0) \leq C$ and hence, $|\nabla u| \leq Cu$ on M . It is easy to verify that $C = \sqrt{|\lambda|}$, when $K_1^2 + \frac{K_2^2}{4\alpha(n-1)} = \frac{1}{2}$ and taking the limit as β approaches to infinity. On the other hand, if $K_1^2 + \frac{K_2^2}{4\alpha(n-1)} \neq \frac{1}{2}$, we obtain, solving inequality (17),

$$\begin{aligned} C &= \left\{ \left[\left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right)^2 (1+\alpha)^2 (1+\beta)^2 (n-1)^2 \beta^2 + 4\beta(1+\beta)\lambda^2 \right. \right. \\ &\quad \left. \left. + \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} \right) (1+\alpha)(1+\beta)(n-1)\beta \right] \cdot \frac{1}{2\beta} \right\}^{\frac{1}{2}} > 0, \end{aligned}$$

which completes the proof of Theorem 1. \square

Remark 1 If we assume $\lambda = 0$ in Theorem 1, we can take the limit on C when $\beta \rightarrow 0$ to obtain the same estimate of Theorem 1.1 due to Zhu and Chen [12].

In order to present the proof of Theorem 2 we will need a generalized Laplacian comparison theorem obtained by Zhu and Chen [12] for $\mathcal{L}d$, where d is a distance function on self-shrinkers.

Proposition 2 (Zhu and Chen) Let $X: M^n \rightarrow \mathbb{R}^{n+1}$ ($n \geq 2$) be an n -dimensional complete self-shrinker. Fix a point $x \in M^n$, let $B(x, r)$ be a geodesic ball of radius r and centered at x . And for any $K_3 \geq 0$ and $K_4 \geq 0$, we assume $|A| \leq K_3$ and $|X^\top| \leq K_4$ on $B(x, r)$, where A and X^\top denote the second fundamental form and the tangential projection of X , respectively. Let $d(y) = d(y, x)$ be the distance function with respect to the fixed point x , then

$$\mathcal{L}d \leq n \frac{G'(d)}{G(d)} \quad \text{on } B(x, r), \quad (18)$$

where $G: [0, r) \rightarrow R^+$ is the solution of the equation

$$\begin{cases} G''(t) - \frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n} G(t) = 0, \\ G(0) = 0, \quad G(d) = 1. \end{cases} \quad (19)$$

Now we begin the proof of Theorem 2.

Proof: We start using inequality (16) to deduce

$$\begin{aligned} \mathcal{L}\phi &\geq -\left(\frac{K_4^2}{4\alpha(n-1)} + K_3^2 - \frac{1}{2}\right)(n-1)\phi \\ &\quad - \frac{1}{(1+\alpha)\beta(n-1)} \frac{\lambda^2}{\phi} \\ &\quad - \left(2 - \frac{2}{(1+\alpha)(1+\beta)(n-1)}\right) \frac{\langle \nabla\phi, \nabla u \rangle}{u} \\ &\quad + \frac{1}{(1+\alpha)(1+\beta)(n-1)} \phi^3. \end{aligned} \quad (20)$$

Given $r > 0$, let us define a function F as follows

$$F(y) = (r^2 - d^2(x, y))\phi(y), \quad y \in B(x, r).$$

First we notice that

$$\begin{aligned} \nabla F &= -\phi \nabla(d^2) + (r^2 - d^2) \nabla\phi, \\ \mathcal{L}F &= (r^2 - d^2)\mathcal{L}\phi - \phi \mathcal{L}(d^2) - 2\langle \nabla(d^2), \nabla\phi \rangle. \end{aligned}$$

Suppose $|\nabla u| \neq 0$. Since $F = 0$ on $\partial B(x, r)$ and $F > 0$ in $B(x, r)$, F achieves its maximum at some point $x_0 \in B(x, r)$. By Calabi's argument used in [16, p 21], we can suppose that x_0 is not a cut point of x . Therefore, F is smooth near x_0 and $\nabla F = 0$ and $\Delta F \leq 0$ at x_0 .

Thus, at x_0 , we have

$$\mathcal{L}F = \Delta F - \langle \nabla\phi, \nabla F \rangle \leq 0,$$

$$\frac{\nabla\phi}{\phi} = \frac{\nabla(d^2)}{r^2 - d^2},$$

hence

$$\begin{aligned} \frac{\mathcal{L}\phi}{\phi} &\geq \frac{\mathcal{L}(d^2)}{r^2 - d^2} + \frac{2\langle \nabla(d^2), \nabla\phi \rangle}{(r^2 - d^2)\phi} \\ &= \frac{\mathcal{L}(d^2)}{r^2 - d^2} + \frac{2|\nabla(d^2)|^2}{(r^2 - d^2)^2}. \end{aligned}$$

Note that $K_3 \geq \sqrt{2}/2$, $K_4 \geq 0$ and $|\nabla d| = 1$, by (18) and (19), we can get

$$\begin{aligned} \mathcal{L}d &\leq n \sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}} \coth\left(\sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}} d\right) \\ &\leq \frac{n}{d} \left(1 + \sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}} d\right) \end{aligned} \quad (21)$$

and

$$\begin{aligned} \mathcal{L}(d^2) &= 2d\mathcal{L}d + 2|\nabla d|^2 \\ &\leq 2n \left(1 + \sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}} d\right) + 2. \end{aligned} \quad (22)$$

Since $|\nabla(d^2)|^2 = 4d^2$, by inequalities (20) and (22), we obtain, at x_0 ,

$$\begin{aligned} 0 &\geq \frac{\mathcal{L}F}{(r^2 - d^2)\phi} = \frac{\mathcal{L}\phi}{\phi} - \frac{\mathcal{L}(d^2)}{r^2 - d^2} - \frac{8d^2}{(r^2 - d^2)^2} \\ &\geq -\left(\frac{K_4^2}{4\alpha(n-1)} + K_3^2 - \frac{1}{2}\right)(n-1) \\ &\quad - \frac{1}{(1+\alpha)\beta(n-1)} \frac{\lambda^2}{\phi^2} \\ &\quad - \left(2 - \frac{2}{(1+\alpha)(1+\beta)(n-1)}\right) \frac{\langle \nabla\phi, \nabla u \rangle}{\phi u} \\ &\quad + \frac{1}{(1+\alpha)(1+\beta)(n-1)} \phi^2 - \frac{8d^2}{(r^2 - d^2)^2} \\ &\quad - \frac{1}{r^2 - d^2} \left[2n \left(1 + \sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}} d\right) + 2\right]. \end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} \frac{\langle \nabla\phi, \nabla u \rangle}{\phi u} &= \frac{\langle \nabla(d^2), \nabla u \rangle}{u(r^2 - d^2)} \\ &= \frac{2d \langle \nabla d, \nabla u \rangle}{u(r^2 - d^2)} \leq \frac{2d\phi}{r^2 - d^2}. \end{aligned}$$

Then,

$$\begin{aligned}
 0 \geq & -\left(\frac{K_4^2}{4\alpha(n-1)} + K_3^2 - \frac{1}{2}\right)(n-1)(r^2 - d^2)^2 \\
 & - \frac{1}{(1+\alpha)\beta(n-1)} \frac{\lambda^2}{F^2} (r^2 - d^2)^4 \\
 & - 4\left(\frac{(1+\alpha)(1+\beta)(n-1)-1}{(1+\alpha)(1+\beta)(n-1)}\right) dF \\
 & + \frac{1}{(1+\alpha)(1+\beta)(n-1)} F^2 - 8d^2 \\
 & - \left[2n\left(1 + \sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}} d\right) + 2\right] (r^2 - d^2).
 \end{aligned}$$

Note that $r^2 - d^2 \leq r^2$ and $d^2 \leq r^2$, we have

$$\begin{aligned}
 0 \geq & \beta F^4 - 4\beta[(1+\alpha)(1+\beta)(n-1)-1]dF^3 \\
 & - (1+\beta)\lambda^2 r^8 - (1+\alpha)(1+\beta)(n-1)\beta \\
 & \times \left\{\left(K_1^2 + \frac{K_2^2}{4\alpha(n-1)} - \frac{1}{2}\right)(n-1)r^4\right. \\
 & \left. + (10+2n)r^2 + 2n\sqrt{\frac{1}{n}\left(K_3^2 + \frac{1}{4}K_4^2 - \frac{1}{2}\right)}r^3\right\} F^2.
 \end{aligned}$$

Proceeding, we define

$$\begin{aligned}
 \rho(y) = & \beta y^4 - 4\beta[(1+\alpha)(1+\beta)(n-1)-1]dy^3 \\
 & - (1+\beta)\lambda^2 r^8 - (1+\alpha)(1+\beta)(n-1)\beta \\
 & \times \left\{\left(K_1^2 + \frac{K_2^2}{4\alpha(n-1)} - \frac{1}{2}\right)(n-1)r^4\right. \\
 & \left. + (10+2n)r^2 + 2n\sqrt{\frac{1}{n}\left(K_3^2 + \frac{1}{4}K_4^2 - \frac{1}{2}\right)}r^3\right\} y^2. \quad (23)
 \end{aligned}$$

Note that $\rho(0) = -(1+\beta)\lambda^2 r^8 < 0$ and hence the polynomial ρ just has two roots, with different signs. Thus, there is a positive constant C , depending on $\alpha, \beta, n, K_3, K_4, r$ and λ , such that $\rho \leq C$, when $\rho(y) \leq 0$. Then, we have $F \leq C$ on $B(x, r)$, and the following estimate holds

$$\frac{3}{4}r^2 \sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leq \sup_{B(x,r/2)} F \leq C,$$

that is,

$$\sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leq \frac{4}{3}Cr^{-2}. \quad (24)$$

Therefore, we obtain the desired estimate and this finishes the proof of Theorem 2. \square

PROOF OF COROLLARY 1

This section is devoted to the proof of Corollary 1. *Proof:* To prove the first assertion we consider $\mathcal{U} = \sup_{B(x,r)} |u|$. For any $\epsilon > 0$, we set $v := u + \mathcal{U} + \epsilon > 0$

on $B(x, r)$. Using Theorem 2 we infer

$$\begin{aligned}
 \sup_{B(x,r/2)} |\nabla u| = \sup_{B(x,r/2)} |\nabla v| & \leq C \sup_{B(x,r/2)} (u + \mathcal{U} + \epsilon) \\
 & \leq C \left(2 \sup_{B(x,r)} |u| + \epsilon\right).
 \end{aligned}$$

Now making $\epsilon \rightarrow 0$ we conclude the claim of the first assertion.

Finally, we choose x_1, x_2 in $B(x, r/2)$ satisfying $u(x_1) = \sup_{B(x,r/2)} u$ and $u(x_2) = \inf_{B(x,r/2)} u$. Let $\gamma \subset B(x, r)$ be a minimal geodesic connecting x_1 to x_2 . Since γ is contained in $B(x, r)$, we obtain from Theorem 2 and triangle inequality,

$$\log \frac{u(x_1)}{u(x_2)} = \left| \int_{\gamma} \frac{d \log u}{ds} ds \right| \leq \int_{\gamma} \frac{|\nabla u|}{u} ds \leq \int_{\gamma} C ds \leq 2Cr.$$

Therefore, $u(x_1) \leq e^{2Cr} u(x_2)$, which ends the proof of Corollary 1. \square

PROOF OF THEOREM 3

In this section, we will give a gradient estimate of the higher eigenfunctions of the \mathcal{L} on compact self-shrinkers. Let $X: M^n \rightarrow \mathbb{R}^{n+1}$ be an n -dimensional compact self-shrinkers. Suppose $|A| \leq \sqrt{2}/2$ and $|X^T| \leq a$ for some constant $a > 0$, where A and X^T denote the second fundamental form and the tangential projection of X , respectively.

First, we consider the eigenfunctions ϕ_i ($i = 0, 1, 2, \dots$) of the \mathcal{L} . Since the differential operator \mathcal{L} is self-adjoint with respect to volume measure $dV = e^{-\varphi} dv$, then the closed eigenvalue problem:

$$\mathcal{L}\phi_i = -\lambda_i \phi_i, \quad \int_M \phi_i \phi_j dV = \delta_{ij}$$

for the differential operator \mathcal{L} on compact self-shrinkers M has a real and discrete spectrum:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_l \leq \dots \rightarrow \infty,$$

where each eigenvalue is repeated according to its multiplicity. For a given constant c , consider the function

$$P(x) = |\nabla \phi|^2 + c\phi^2,$$

where $\phi = \sum_{i=1}^l b_i \phi_i$ with $b_i \in \mathbb{R}$ and $\sum_{i=1}^l b_i^2 = 1$. Let

$$\psi(b_1, \dots, b_l) := \max_{x \in M} P(x).$$

Assume that $\psi(b_1, \dots, b_l)$ achieves its maximum at some point a_1, \dots, a_l .

Lemma 1 Let $u = \sum_{i=1}^l a_i \phi_i$, then

$$|\nabla u|^2 + Lu^2 \leq L \max_M u^2,$$

where $L = (2\lambda_l + a^2 + a\sqrt{4\lambda_l + a^2})/2$.

Proof: We follow the arguments in [14]. Define

$$F(b_1, \dots, b_l, x, \theta) = P(x) - \theta \left(\sum_{i=1}^l b_i^2 - 1 \right).$$

Then, subject to the constrain $\sum_{i=1}^l b_i^2 = 1$, F achieves its maximum value at some point $(a_1, \dots, a_k, x_0, \alpha)$. We now show

$$|\nabla u|^2(x_0) + cu^2(x_0) \leq c \max_M u^2,$$

for $c > (2\lambda_l + a^2 + a\sqrt{4\lambda_l + a^2})/2$.

At the point $(a_1, \dots, a_k, x_0, \alpha)$, F satisfies

$$\begin{cases} \nabla F(a_1, \dots, a_k, x_0, \alpha) = 0 \\ \Delta F(a_1, \dots, a_k, x_0, \alpha) \leq 0 \\ \frac{\partial F}{\partial b_i} = 0 \\ \sum_{i=1}^l a_i^2 = 1. \end{cases} \quad (25)$$

From the third equation of (25), we have

$$\sum_{j=1}^l (2a_j \langle \nabla \phi_i, \nabla \phi_j \rangle + 2ca_j \phi_i \phi_j) - 2\alpha a_i = 0.$$

After multiplying by a_i and summing over i , one sees that

$$\alpha = P(u, x_0) = |\nabla u|^2(x_0) + cu^2(x_0).$$

Suppose now that

$$|\nabla u|^2(x_0) + cu^2(x_0) > c \max_M u^2.$$

Then $\nabla u(x_0) \neq 0$ and one can choose an orthonormal frame $\{e_1, \dots, e_n\}$ at x_0 so that

$$\nabla u(x_0) = u_1(x_0)e_1.$$

Now the first equation of (25) becomes

$$2u_1 u_{1i} + 2cuu_i = 0$$

for $i = 1, \dots, n$. This in particular implies

$$|\nabla^2 u|^2 \geq u_{11}^2 = c^2 u^2. \quad (26)$$

On the other hand, at the maximum point $(a_1, \dots, a_k, x_0, \alpha)$,

$$\Delta F(a_1, \dots, a_k, x_0, \alpha) \leq 0$$

or equivalently,

$$\Delta |\nabla u|^2 + c\Delta u^2 \leq 0. \quad (27)$$

Note that $\mathcal{L}(\cdot) = \Delta(\cdot) - g(\nabla \varphi, \nabla(\cdot)) = \Delta(\cdot) - \frac{1}{2} \langle X, \nabla(\cdot) \rangle$. By the Bochner formula, we have

$$\frac{1}{2} \mathcal{L} |\nabla u|^2 = |\nabla^2 u|^2 + \text{Ric}_\varphi(\nabla u, \nabla u) + \langle \nabla u, \nabla \mathcal{L} u \rangle. \quad (28)$$

From (27) and (28), we obtain

$$|\nabla^2 u|^2 + \text{Ric}_\varphi(\nabla u, \nabla u) + \langle \nabla u, \nabla \mathcal{L} u \rangle + \frac{1}{2} \langle \nabla \varphi, \nabla |\nabla u|^2 \rangle + cu\mathcal{L}u + c|\nabla u|^2 + \frac{c}{2} \langle \nabla \varphi, \nabla u^2 \rangle \leq 0. \quad (29)$$

Since the Cauchy-Schwarz inequality, the Kato inequality ($|\nabla |\nabla u|| \leq |\nabla^2 u|$) and $|\nabla \varphi| = |X^\top|/2 \leq a$, we have

$$|\langle \nabla \varphi, \nabla |\nabla u|^2 \rangle| \leq 2|\nabla u| |\nabla \varphi| |\nabla |\nabla u|| \leq 2a|\nabla u| |\nabla^2 u|$$

and

$$|\langle \nabla \varphi, \nabla u^2 \rangle| \leq 2|u| |\nabla u| |\nabla \varphi| \leq 2a|u| |\nabla u|.$$

Since $\text{Ric}_\varphi \geq \frac{1}{2} - |A|^2 \geq 0$, from (26) and (29) we have

$$|\nabla^2 u|^2 + \langle \nabla u, \nabla \mathcal{L} u \rangle + cu\mathcal{L}u + c|\nabla u|^2 - a|\nabla u| |\nabla^2 u| - ca|u| |\nabla u| \leq 0.$$

Using the inequality $xy \leq \frac{x^2}{4\epsilon} + \epsilon y^2$ for any $\epsilon > 0$, the above inequality implies

$$(1 - a\beta) |\nabla^2 u|^2 + \langle \nabla u, \nabla \mathcal{L} u \rangle + cu\mathcal{L}u + c|\nabla u|^2 - \frac{a}{4\beta} |\nabla u|^2 - ca\gamma u^2 - \frac{ca}{4\gamma} \leq 0$$

for any $\beta, \gamma > 0$. Since $\mathcal{L}u = -\sum_{i=1}^l \lambda_i a_i \phi_i$, we can compute

$$\begin{aligned} & \langle \nabla u, \nabla \mathcal{L} u \rangle + cu\mathcal{L}u \\ &= - \sum_{i,j=1}^l \lambda_i a_i a_j \langle \nabla \phi_i, \nabla \phi_j \rangle - c \sum_{i,j=1}^l \lambda_i a_i a_j \phi_i \phi_j \\ &= - \sum_{i=1}^l \lambda_i a_i \sum_{j=1}^l (a_j \langle \nabla \phi_i, \nabla \phi_j \rangle + ca_j \phi_i \phi_j) \\ &= -\alpha \sum_{i=1}^l \lambda_i a_i^2. \end{aligned}$$

Hence, in the view of the inequality (26), if β is small, we have

$$(1 - a\beta)c^2 u^2 - \alpha \sum_{i=1}^l \lambda_i a_i^2 - ca\gamma u^2 + \left(c - \frac{a}{4\beta} - \frac{ca}{4\gamma} \right) |\nabla u|^2 \leq 0.$$

By Dung, Le Hai and Thanh's arguments used to prove Lemma 2.1 in [14], the inequality reduces to

$$\left(c - \lambda_l - \frac{a^2 c}{c - \lambda_l} \right) |\nabla u|^2(x_0) \leq 0.$$

This is impossible if $c > (2\lambda_l + a^2 + a\sqrt{4\lambda_l + a^2})/2$.

The proof is complete by letting c approach $(2\lambda_l + a^2 + a\sqrt{4\lambda_l + a^2})/2$. \square

To prove Theorem 3, we need the following volume comparison theorem for compact self-shrinkers already proved in [17].

Lemma 2 Let $X: M^n \rightarrow \overline{\mathcal{B}}_k^{n+1}(0) \subset \mathbb{R}^{n+1}$ be an n -dimensional compact self-shrinkers with $|A| \leq \sqrt{3}/3$, and $\overline{\mathcal{B}}_k^{n+1}(0)$ denotes the Euclidean closed ball with center 0 and radius k . Then for any $p \in M^n$, $0 < R_1 \leq R_2$, we have

$$\frac{\text{Vol}(B(p, R_2))}{\text{Vol}(B(p, R_1))} \leq e^{3k^2/4} \frac{V(R_2)}{V(R_1)},$$

where $B(p, R)$ is a geodesic ball of M^n with radius R centered at p , and $V(r)$ is the volume of the ball with radius r in Euclidean space \mathbb{R}^n .

Proof of Theorem 3: The proof is similar to the proof of Theorem 2.2 in [14] with note that the Bishop volume comparison theorem in [14] is now replaced by the volume comparison in Lemma 2. Since the proof is essentially the same as in Theorem 2.2 in [14], we omit it here.

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