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On value distribution of certain delay-differential polynomials

Nan Li[a,*], **Lianzhong Yang**[b]

[a] School of Mathematics, Qilu Normal University, Jinan, Shandong 250200 China
[b] School of Mathematics, Shandong University, Jinan, Shandong 250100 China

*Corresponding author, e-mail: nanli32787310@163.com

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ABSTRACT: Given an entire function f of finite order ρ, let $L(z,f)=\sum_{j=0}^{m} b_j(z)f^{(k_j)}(z+c_j)$ be a linear delay-differential polynomial of f with small coefficients in the sense of $O(r^{\lambda+\varepsilon})+S(r,f)$, $\lambda<\rho$. Provided α and β are similar small functions, we consider the zero distribution of $L(z,f)-\alpha f^n-\beta$ for $n\geqslant 3$ and $n=2$, respectively. Our results are improvements and complements of Chen [Abstract Appl Anal **2011** (2011):ID 239853), and Laine [J Math Anal Appl **469** (2019):808–826].

KEYWORDS: meromorphic functions, delay-differential polynomial, value distribution

MSC2020: 30D35

INTRODUCTION

Let $f(z)$ be a transcendental meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and main results in Nevanlinna theory (see [1–3]). A meromorphic function α is said to be a λ-small function of a meromorphic function f of finite order ρ, if there exists $\lambda<\rho$, such that for any $\varepsilon\in(0,\rho-\lambda)$,

$$T(r,\alpha)=O\left(r^{\lambda+\varepsilon}\right)+S(r,f), \tag{1}$$

outside a possible exceptional set F of finite logarithmic measure (see [4]). Here, $S(r,f)$ is any quantity that satisfies $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside a set F. For the sake of simplicity, the right-hand side in (1) will be denoted by $S_\lambda(r,f)$.

Hayman [5] proved the following theorem.

Theorem 1 *If $f(z)$ is a transcendental entire function, $n\geqslant 3$ is an integer and $a(\neq 0)$ is a constant, then $f'(z)-af(z)^n$ assumes all finite values infinitely often.*

Recently, several articles (see [4, 6–18]) have focused on complex differences, giving many difference analogues in value distribution theory of meromorphic functions.

In 2011, Chen [8] obtained the following Theorem 2, an almost direct difference analogue of Theorem 1, and gave an estimate of numbers of b-points, namely, $\lambda(\Psi_n(z)-b)=\sigma(f)$ for every $b\in\mathbb{C}$.

Theorem 2 *Let $f(z)$ be a transcendental entire function of finite order ρ, and let $\alpha,c\in\mathbb{C}\backslash\{0\}$ be constants, with c such that $f(z+c)\not\equiv f(z)$. Set $\Psi_n(z)=\Delta f(z)-\alpha f(z)^n$, where $\Delta f(z)=f(z+c)-f(z)$ and $n\geqslant 3$ is an integer. Then $\Psi_n(z)$ assumes all finite values infinitely often, and for every $\beta\in\mathbb{C}$, we have $\lambda(\Psi_n(z)-\beta)=\rho$.*

In 2013, Liu and Yi [15] replaced $\Delta f(z)$ in Theorem 2 by a more general linear difference operator $g(f)=\sum_{j=1}^{k} a_j f(z+c_j)$, where $a_j,c_j (j=1,2,\ldots,k)$ are complex constants, and obtained the following result.

Theorem 3 *Let $f(z)$ be a transcendental entire function of finite order ρ, let α,β be complex constants. Set $\Psi_n=g(f)-\alpha f^n(z)$, where $n\geqslant 3$ is an integer. Then Ψ_n have infinitely many zeros and $\lambda(\Psi_n-\beta)=\rho$ provided that $g(f)\not\equiv\beta$.*

In 2019, Laine [12] generalized the coefficients from complex constants to λ-small functions, released the assumption on β that $g(f)\not\equiv\beta$, and obtained the following theorem.

Theorem 4 *Let f be an entire function of finite order ρ, α, β, $b_0,\ldots,b_k$ be λ-small functions of f, $g(f):=\Sigma_{j=1}^{k} b_j(z)f(z+c_j)(\not\equiv 0)$ and $n\geqslant 3$. Then for $\Psi_n:=g(f)-\alpha f^n$, $\Psi_n-\beta$ has sufficiently many zeros to satisfy $\lambda(\Psi_n-\beta)=\rho$.*

But a bit regret, the proof of dealing with $G(z,f)\equiv 0$ in [12, Theorem 5.1] is not complete.

We now introduce the generalized linear delay-differential operator of $f(z)$,

$$L(z,f)=\sum_{j=0}^{m} b_j(z)f^{(k_j)}(z+c_j), \tag{2}$$

where b_j are λ-small functions of f, c_j are distinct complex numbers and k_j are non-negative integers. In view of the above theorems, it is quite natural to study the value distribution of $\Psi_n-\beta$ when the linear difference operator $g(f)$ is changed to the linear delay-differential operator $L(z,f)$ with the restriction on β be omitted.

In this paper, we study the above problem and obtain the following result.

Theorem 5 *Let $f(z)$ be an entire function of finite order ρ, $\alpha(\not\equiv 0)$, β be λ-small functions of f, $L(z,f)(\not\equiv 0)$ be linear delay-differential polynomial defined as in (2) and $n \geqslant 3$. Then for $\Phi_n = L(z,f) - \alpha f^n$, $\Phi_n - \beta$ has sufficiently many zeros to satisfy $\lambda(\Phi_n - \beta) = \rho$.*

Remark 1 Omitting the restriction on β is meaningful. In fact, we do not need to worry about that if $L(z,f) \equiv \beta$ and f has a Borel exceptional value 0, then $\lambda(\Phi_n - \beta) = \lambda(-\alpha f^n)$ may be less than ρ. It is because from the proof of Theorem 5, we can get that if 0 is a Borel exceptional value of f, then $L(z,f) \not\equiv \beta$. So $L(z,f) \equiv \beta$ and f has a Borel exceptional value 0 can not hold simultaneously.

Chen [8] also considered the value distribution of Ψ_2 when $n = 2$ and obtained the following Theorem 6 and Theorem 7.

Theorem 6 *Let $f(z)$ be a transcendental entire function of finite order ρ with a Borel exceptional value 0, and let $\alpha, c \in \mathbb{C}\backslash\{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then $\Psi_2(z)$ assumes all finite values infinitely often, and for every $\beta \in \mathbb{C}$ we have $\lambda(\Psi_2 - \beta) = \rho$.*

Theorem 7 *Let $f(z)$ be a transcendental entire function of finite order ρ with a finite nonzero Borel exceptional value d, and let $\alpha, c \in \mathbb{C}\backslash\{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then for every $\beta \in \mathbb{C}$ with $\beta \neq -\alpha d^2$, $\Psi_2(z)$ assumes the value β infinitely often, and $\lambda(\Psi_2 - \beta) = \rho$.*

Liu and Yi [15] replaced $\Delta f(z)$ in Theorem 6 and Theorem 6 to a more general linear difference operator $\sum_{j=1}^{k} a_j(z) f(z+c_j)$ and obtained the following result.

Theorem 8 *Suppose that $f(z)$ be a finite order transcendental entire function with a Borel exceptional value d. Let $\beta(z), \alpha(z)(\not\equiv 0), a_j(z)(j = 1, 2, \ldots, k)$ be polynomials, and let $c_j(j = 1, 2, \ldots, k)$ be complex constants. If either $d = 0$ and $\sum_{j=1}^{k} a_j(z) f(z+c_j) \not\equiv 0$, or $d \neq 0$ and $\sum_{j=1}^{k} d a_j(z) - d^2 \alpha(z) - \beta(z) \not\equiv 0$, then $\Psi_2(z) - \beta(z) = \sum_{j=1}^{k} a_j(z) f(z+c_j) - \alpha(z) f(z)^2 - \beta(z)$ has infinitely many zeros and $\lambda(\Psi_2 - \beta) = \rho(f)$.*

The following Example 1 shows that if the difference operator $\Delta f(z) = f(z+c) - f(z)$ or $\sum_{j=1}^{k} a_j(z) f(z+c_j)$ in Ψ_2 is changed to a linear delay-differential operator $L(z,f)$, the conclusions in Theorem 7 and Theorem 8 may not hold.

Example 1 Let $L(z,f) = f(z+1) - f'(z)$, and $\Phi_2 = L(z,f) - \frac{e-1}{2} f(z)^2$. For $f_1(z) = e^z + 1$, we have $\Phi_2(f_1) = \frac{1-e}{2} e^{2z} + \frac{3-e}{2}$. Here, $d = 1$, $\alpha = \frac{e-1}{2}$, $a_1 = 1$, $a_2 = -1$, and $\beta = \frac{3-e}{2} \neq \sum_{j=1}^{k} d a_j - \alpha d^2 = -\frac{e-1}{2}$, but $\Phi_2 \neq \beta$.

So it is natural to ask: what can we say about $\Phi_2 = L(z,f) - \alpha f^2$? The second aim of this paper is to consider the above problem, and obtain the following results.

Before stating Theorem 9, we recall that the Borel exceptional value for small function β of $f(z)$ satisfies

$$\lambda(f(z) - \beta) < \rho(f),$$

where $\lambda(f - \beta)$ is the exponent of convergence of zeros of $f - \beta$ (see [14]).

Theorem 9 *Let $f(z)$ be a transcendental entire function of finite order ρ with a finite non-zero Borel exceptional value d. Let $\alpha \in \mathbb{C}\backslash\{0\}$ be constant, and β, b_j $(j = 0, 1, \ldots, m)$ be λ-small entire functions of f. Let $L(z,f)(\not\equiv 0)$ be linear delay-differential polynomial defined as in (2). Defining $\Phi_2 = L(z,f) - \alpha f^2$, and $I_1 = \{0 \leqslant j \leqslant m : k_j = 0\}$, we have the following statements:*

(i) If

$$\beta \not\equiv \Big(\sum_{j\in I_1} b_j\Big) d - \alpha d^2,$$

then $\Phi_2(z) - \beta$ has sufficiently many zeros to satisfy $\lambda(\Phi_2 - \beta) = \rho$.

(ii) If

$$\beta \equiv \Big(\sum_{j\in I_1} b_j\Big) d - \alpha d^2, \tag{3}$$

then one of the following holds:

(a) β is a Borel exceptional small function of Φ_2, which satisfies

$$\frac{\beta - \Phi_2}{(f-d)^2} = \alpha = \frac{L(z,f) - \alpha d^2 - \beta}{2d(f-d)}. \tag{4}$$

(b) $\Phi_2 - \beta$ has sufficiently many zeros to satisfy

$$N\Big(r, \frac{1}{\Phi_2 - \beta}\Big) = T(r,f) + S_\lambda(r,f).$$

Remark 2 In Example 1, $\sum_{j\in I_1} b_j = 1$, $d = 1$, $\alpha = \frac{e-1}{2}$, and $\beta = \frac{3-e}{2} = \Big(\sum_{j\in I_1} b_j\Big) d - \alpha d^2$ is a Borel exceptional value of Φ_2, which also satisfies (4). Thus Example 1 above illustrates Theorem 9.

Remark 3 Let $L_1(z,f) = f(z+c) - f(z)$, then $\sum_{j\in I_1} b_j = 0$. Let $L_2(z,f) = \sum_{j=1}^{k} a_j(z) f(z+c_j)$, then $\sum_{j=1}^{k} a_j = \sum_{j\in I_1} a_j$. Thus by Theorem 9(i), we can obtain the results in Theorem 7 and Theorem 8 when $d \neq 0$. Therefore Theorem 9 improves Theorem 7 and Theorem 8.

The following theorem deals with the case when $d = 0$.

Theorem 10 *Let $f(z)$ be a transcendental entire function of finite order ρ with a Borel exceptional value 0. Let $\alpha \in \mathbb{C}\backslash\{0\}$ be constant, and β, b_j $(j=0,1,\ldots,m)$ be λ-small entire functions of f. Let $L(z,f)(\not\equiv 0)$ be linear delay-differential polynomial defined as in (2). Defining $\Phi_2 = L(z,f) - \alpha f^2$, then we have $\lambda(\Phi_2 - \beta) = \rho$. Particularly, if $\beta \equiv 0$, then*

$$N\big(r, 1/\Phi_2\big) = T(r,f) + S_\lambda(r,f). \tag{5}$$

Remark 4 The following example shows that when $\beta \equiv 0$, (5) in Theorem 10 occurs.

Example 2 Let $L(z,f) = \mathrm{e}^{-2} f(z+1) - \frac{1}{2} f'(z)$, and $\Phi_2 = L(z,f) - f(z)^2$. For $f_2(z) = \mathrm{e}^{z+1}$, we have $\Phi_2(f_2) = (1 - \mathrm{e}/2)\,\mathrm{e}^z - \mathrm{e}^2\,\mathrm{e}^{2z}$. Here, 0 is a Borel exceptional value of f_2, and $N(r, 1/\Phi_2) = N\big(r, 1/\big(1 - \mathrm{e}/2 - \mathrm{e}^2\,\mathrm{e}^z\big)\big) = T(r, f_2) + S(r, f_2)$.

PRELIMINARY LEMMAS

In this section, we collect the results that are needed for proving the main results.

The following lemma plays an important role in uniqueness problems of meromorphic functions.

Lemma 1 ([3]) *Let $f_j(z)$ $(j=1,\ldots,n)$ $(n \geqslant 2)$ be meromorphic functions, and let $g_j(z)$ $(j=1,\ldots,n)$ be entire functions satisfying*

(i) $\sum_{j=1}^n f_j(z)\,\mathrm{e}^{g_j(z)} \equiv 0$;

(ii) when $1 \leqslant j < k \leqslant n$, then $g_j(z) - g_k(z)$ is not a constant;

(iii) when $1 \leqslant j \leqslant n$, $1 \leqslant h < k \leqslant n$, then

$$T(r, f_j) = o\{T(r, \mathrm{e}^{g_h - g_k})\} \quad (r \to \infty,\ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or logarithmic measure.

Then, $f_j(z) \equiv 0$ $(j=1,\ldots,n)$.

Using the same reasoning as in the proof of [2, Lemma 2.4.2], we easily get the following lemma.

Lemma 2 ([4]) *Let f be a transcendental meromorphic solution of finite order ρ of a differential-difference equation:*

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are delay-differential polynomials in f with λ-small coefficients of f. If the total degree of $Q(z,f)$ is $\leqslant n$, then for each $\varepsilon > 0$,

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r,f).$$

The following lemma, which is a special case of [11, Theorem 3.1], gives a relationship for the Nevanlinna characteristic of a meromorphic function with its shift.

Lemma 3 ([11]) *Let $f(z)$ be a meromorphic function with the hyper-order less than one, and $c \in \mathbb{C}\backslash\{0\}$. Then we have*

$$T(r, f(z+c)) = T(r, f(z)) + S(r,f).$$

Observe that

$$m\left(r, \frac{f^{(k)}(z+c)}{f(z)}\right) \leqslant m\left(r, \frac{f^{(k)}(z+c)}{f(z+c)}\right) + m\left(r, \frac{f(z+c)}{f(z)}\right),$$

by using Logarithmic Derivative Lemma and its difference analogues (see [2, 9–11]), Lemma 3, we obtain the following lemma, see also [14].

Lemma 4 *Let f be a transcendental meromorphic function of finite order. Then*

$$m\left(r, \frac{f^{(k)}(z+c)}{f(z)}\right) = S(r,f), \tag{6}$$

outside a possible exceptional set of finite logarithmic measure.

Applying Lemma 4, we obtain the following lemma.

Lemma 5 *Let f be an entire function of finite order ρ, $\alpha(\not\equiv 0)$, β be λ-small functions of f, $L(z,f)$ be nonvanishing linear delay-differential polynomial defined as in (2) and $n \geqslant 2$. Then $\Phi_n - \beta$ is transcendental and satisfies $\rho(\Phi_n - \beta) = \rho$.*

Proof: We first assume that $\Phi_n - \beta$ is transcendental. Indeed, if not, then $\Phi_n - \beta = R(z)$ is rational, and $f^n = \alpha^{-1}(L(z,f) - \beta - R(z))$. Therefore, by Lemma 4, we obtain

$$\begin{aligned}
nT(r,f) = T(r,f^n) &\leqslant T(r, L(z,f)) + S_\lambda(r,f) \\
&= m(r, L(z,f)) + S_\lambda(r,f) \\
&\leqslant m\left(r, \frac{L(z,f)}{f}\right) + m(r,f) + S_\lambda(r,f) \\
&\leqslant \sum_{j=0}^m m\left(r, \frac{f^{(k_j)}(z+c_j)}{f(z)}\right) + m(r,f) + S_\lambda(r,f) \\
&\leqslant T(r,f) + S_\lambda(r,f),
\end{aligned}$$

a contradiction follows since $n \geqslant 2$.

Next, we prove that $\rho(\Phi_n - \beta) = \rho(f)$. By Lemma 4, we have

$$\begin{aligned}
T(r, \Phi_n - \beta) &= T\left(r, (L(z,f) - \alpha f^n - \beta)\right) \\
&\leqslant T(r, f^n) + T(r, L(z,f)) + S_\lambda(r,f) \\
&= nT(r,f) + m(r, L(z,f)) + S_\lambda(r,f) \\
&\leqslant nT(r,f) + m\left(r, \frac{L(z,f)}{f}\right) + m(r,f) + S_\lambda(r,f) \\
&= (n+1)T(r,f) + S_\lambda(r,f),
\end{aligned} \tag{7}$$

and

$$\begin{aligned}T(r,\Phi_n-\beta) &= T(r,(L(z,f)-\alpha f^n-\beta))\\ &\geqslant T(r,f^n)-T(r,L(z,f))+S_\lambda(r,f)\\ &= nT(r,f)-m(r,L(z,f))+S_\lambda(r,f)\\ &\geqslant nT(r,f)-m(r,f)+S_\lambda(r,f)\\ &= (n-1)T(r,f)+S_\lambda(r,f). \qquad (8)\end{aligned}$$

Therefore, combining with $\lambda<\rho$, from (7) and (8) we have $\rho(\Phi_n-\beta)=\rho$. □

PROOF OF Theorem 5

Firstly, we prove the case $\rho>0$. Suppose now, contrary to the assertion, that $\lambda(\Phi_n-\beta)=\lambda<\rho$. From Lemma 5, we obtain that $\Phi_n-\beta$ is transcendental and (8) holds. By the standard Hadamard representation, we may write

$$\Phi_n-\beta=L(z,f)-\alpha f^n-\beta=\pi\,\mathrm{e}^g, \qquad (9)$$

where $\pi(\not\equiv 0)$ is a λ-small function of f, and g is a polynomial with $\deg g\leqslant\rho$. Actually, $\deg g=\rho$. Otherwise, if $\deg g\leqslant\mu<\rho$, then from (8) and (9), we obtain

$$\begin{aligned}(n-1)T(r,f)-S_\lambda(r,f) &\leqslant T(r,\Phi_n-\beta)\\ &= O(r^{\mu+\varepsilon})+S_\lambda(r,f),\end{aligned}$$

leading to $\rho\leqslant\max\{\mu,\lambda\}<\rho$ by $n\geqslant 3$, a contradiction.

Differentiating (9) and eliminating e^g, we obtain

$$f(z)^{n-1}G(z,f)=H(z,f), \qquad (10)$$

where

$$G(z,f):=\Big(\Big(\frac{\pi'}{\pi}+g'\Big)\alpha-\alpha'\Big)f-n\alpha f'$$

and

$$H(z,f):=\Big(\frac{\pi'}{\pi}+g'\Big)L-L'-\Big(\frac{\pi'}{\pi}+g'\Big)\beta+\beta'.$$

Case 1. $G(z,f)\equiv 0$. Then we have $\alpha f^n=\widetilde{c}\pi\,\mathrm{e}^g$ for some non-zero constant $\widetilde{c}$. By (9), we get

$$L-\beta=\Big(\frac{1}{\widetilde{c}}+1\Big)\alpha f^n. \qquad (11)$$

Subcase 1.1. $\widetilde{c}=-1$. Then we have $f=(-\pi/\alpha)^{1/n}\mathrm{e}^{g/n}$ and $L\equiv\beta$. This gives that

$$\begin{aligned}L(z,f) &= \sum_{j=0}^{m} b_j(z)\bigg(\Big(-\frac{\pi(z+c_j)}{\alpha(z+c_j)}\Big)^{1/n}\mathrm{e}^{\frac{g(z+c_j)}{n}}\bigg)^{(k_j)}\\ &= \sum_{j=0}^{m} b_j(z)\gamma(z+c_j)\,\mathrm{e}^{\frac{g(z+c_j)}{n}}\\ &= \Big(\sum_{j=0}^{m} b_j(z)\gamma(z+c_j)\,\mathrm{e}^{\frac{g(z+c_j)-g(z)}{n}}\Big)\mathrm{e}^{\frac{g(z)}{n}}=\beta, \qquad (12)\end{aligned}$$

where γ is a differential polynomial of $(-\pi/\alpha)^{1/n}$, g and their shifts. Obviously, by Lemma 3, $T(r,\gamma(z+c_j))=T(r,\gamma(z))+S(r,\gamma(z))=S_\lambda(r,f)$.

If $\sum_{j=0}^m b_j(z)\gamma(z+c_j)\,\mathrm{e}^{\frac{g(z+c_j)-g(z)}{n}}\equiv 0$, then we have $L(z,f)\equiv\beta\equiv 0$, a contradiction with the assumption that $L(z,f)\not\equiv 0$.

If $\sum_{j=0}^m b_j(z)\gamma(z+c_j)\,\mathrm{e}^{\frac{g(z+c_j)-g(z)}{n}}\not\equiv 0$, then we prove

$$T\bigg(r,\frac{\beta}{\sum_{j=0}^m b_j(z)\gamma(z+c_j)\,\mathrm{e}^{\frac{g(z+c_j)-g(z)}{n}}}\bigg)=S(r,\mathrm{e}^g). \quad (13)$$

By applying the exponential polynomial theory (see [19, Lemma 2.6] or [20]), we have

$$T(r,\mathrm{e}^g)=\frac{|\overline{\omega}_0|}{\pi}r^\rho+o(r^\rho), \qquad (14)$$

where $\omega_0(\neq 0)$ is the leading coefficient of g.

From (7), (8) and (9), we have

$$\begin{aligned}(n-1+o(1))T(r,f)+O(r^{\lambda+\varepsilon}) &\leqslant T(r,\mathrm{e}^g)\\ &= T\Big(r,\frac{1}{\pi}(\Phi_n-\beta)\Big)\\ &\leqslant (n+1+o(1))T(r,f)+O(r^{\lambda+\varepsilon}).\end{aligned}$$

This gives that

$$T(r,f)=O(T(r,\mathrm{e}^g)) \text{ and } T(r,\mathrm{e}^g)=O(T(r,f)).$$

Combining these with (14) , we obtain

$$\begin{aligned}\frac{T(r,b_j)}{T(r,\mathrm{e}^g)} &= \frac{S_\lambda(r,f)}{T(r,\mathrm{e}^g)}\\ &= \frac{O(r^{\lambda+\varepsilon})}{\Big(\frac{|\overline{\omega}_0|}{\pi}+o(1)\Big)r^\rho}+\frac{S(r,f)}{T(r,f)}\cdot\frac{T(r,f)}{T(r,\mathrm{e}^g)}\to 0,\end{aligned}$$

as $r\to\infty$, outside a possible exceptional set with finite logarithmic measure. Thus we have

$$T(r,b_j)=S(r,\mathrm{e}^g).$$

Following the same reason, we also have

$$T(r,\beta)=S(r,\mathrm{e}^g), \quad \text{and} \quad T(r,\gamma(z+c_j))=S(r,\mathrm{e}^g).$$

Therefore, (13) holds. Thus by (12), we have

$$\begin{aligned}T\left(r,\mathrm{e}^{g(z)}\right) &= T\bigg(r,\frac{\beta^n}{\Big(\sum_{j=0}^m b_j(z)\gamma(z+c_j)\,\mathrm{e}^{\frac{g(z+c_j)-g(z)}{n}}\Big)^n}\bigg)\\ &= S(r,\mathrm{e}^g),\end{aligned}$$

which yields a contradiction.

Subcase 1.2. $\widetilde{c} \neq -1$. Then from Lemma 4 and (11), we get

$$nT(r,f) = T(r,f^n) = T\left(r, \frac{L-\beta}{\left(\frac{1}{\widetilde{c}}+1\right)\alpha}\right)$$
$$\leqslant T(r,L) + S_\lambda(r,f)$$
$$\leqslant m\left(r, \frac{L}{f}\right) + m(r,f) + S_\lambda(r,f)$$
$$\leqslant T(r,f) + S_\lambda(r,f),$$

which yields a contradiction since $n \geqslant 3$.

Case 2. $G(z,f) \not\equiv 0$. Since $n \geqslant 3$, by applying Lemma 2 to (10), we obtain

$$T(r,G(z,f)) = m(r,G(z,f)) + N(r,G(z,f))$$
$$= O(r^{\rho-1+\varepsilon}) + S_\lambda(r,f),$$

and

$$T(r,fG(z,f)) = m(r,fG(z,f)) + N(r,fG(z,f))$$
$$= O(r^{\rho-1+\varepsilon}) + S_\lambda(r,f).$$

Therefore,

$$T(r,f) = T\left(r, \frac{fG(z,f)}{G(z,f)}\right)$$
$$\leqslant T(r,fG(z,f)) + T(r,G(z,f))$$
$$= O(r^{\rho-1+\varepsilon}) + S_\lambda(r,f),$$

which is a contradiction. Hence $\lambda(\Phi_n - \beta) = \rho$.

Finally, we prove the case $\rho = 0$. By Lemma 5, we have $0 \leqslant \lambda(\Phi_n - \beta) \leqslant \rho(\Phi_n - \beta) = \rho = 0$. Thus, $\lambda(\Phi_n - \beta) = \rho = 0$. Next we prove that $\Phi_n - \beta$ has infinitely many zeros. Suppose, contrary to the assertion, that $\Phi_n - \beta$ has finitely many zeros, then by the standard Hadamard representation and $\rho = 0$, we may write

$$\Phi_n - \beta = L(z,f) - \alpha f^n - \beta = \widetilde{\pi}, \tag{15}$$

where $\widetilde{\pi}(\not\equiv 0)$ is a small function of f. Thus, by Lemma 4 we have

$$nT(r,f) = T(r,f^n) = T\left(r, \frac{L(z,f) - \widetilde{\pi} - \beta}{\alpha}\right)$$
$$\leqslant T(r,L(z,f)) + S(r,f)$$
$$= m(r,L(z,f)) + S(r,f)$$
$$\leqslant \sum_{j=0}^{m} m\left(r, \frac{f^{(k_j)}(z+c_j)}{f(z)}\right) + m(r,f) + S(r,f)$$
$$\leqslant T(r,f) + S(r,f),$$

leading to a contradiction by $n \geqslant 3$. Thus $\Phi_n - \beta$ has infinitely many zeros.

PROOF OF Theorem 9

(i) Suppose that d is a Borel exceptional value of $f(z)$, and

$$\left(\sum_{j\in I_1} b_j\right) d - \alpha d^2 - \beta \not\equiv 0.$$

Then $f(z)$ can be written in the form

$$f(z) = d + h(z)\,\mathrm{e}^{az^\rho}, \tag{16}$$

where $a \neq 0$ is a constant, $\rho\,(\geqslant 1)$ is an integer, and $h(\not\equiv 0)$ is an entire function such that $\rho(h) < \rho$. Thus

$$f(z+c_j) = d + h(z+c_j)\,\mathrm{e}^{a(z+c_j)^\rho}$$
$$= d + \left(h(z+c_j)\,\mathrm{e}^{a(z+c_j)^\rho - az^\rho}\right)\mathrm{e}^{az^\rho}$$
$$= d + h(z+c_j)\widetilde{h}_{c_j}\,\mathrm{e}^{az^\rho}, \tag{17}$$

where $\widetilde{h}_{c_j} = \mathrm{e}^{a(z+c_j)^\rho - az^\rho}$. Combining with Lemma 3, $\rho(h(z+c_j)\widetilde{h}_{c_j}) < \rho$. For $k_j > 0$, differentiating iteratively, we obtain by elementary computation that

$$f^{(k_j)}(z+c_j) = (d + h(z+c_j)\,\mathrm{e}^{a(z+c_j)^\rho})^{(k_j)}$$
$$= d^{(k_j)} + \left(h(z+c_j)\,\mathrm{e}^{a(z+c_j)^\rho}\right)^{(k_j)}$$
$$= h_{c_j,k_j}\,\mathrm{e}^{a(z+c_j)^\rho} = h_{c_j,k_j}\widetilde{h}_{c_j}\,\mathrm{e}^{az^\rho}, \tag{18}$$

where h_{c_j,k_j} are differential polynomials in $h(z+c_j)$ and $a(z+c_j)^\rho$. Obviously, $\rho(h_{c_j,k_j}\widetilde{h}_{c_j}) < \rho$. On the other hand, we may write $L(z,f)$ as

$$L(z,f) = \sum_{j\in I_1} b_j(z) f(z+c_j) + \sum_{j\in I_2} b_j(z) f^{(k_j)}(z+c_j) \tag{19}$$

where $I_1 = \{0 \leqslant j \leqslant m : k_j = 0\}$ and $I_2 = \{0 \leqslant j \leqslant m : k_j > 0\}$. Thus, by substituting (17) and (18) into (19), we obtain

$$L(z,f)$$
$$= \sum_{j\in I_1} b_j(z)\left(d + h(z+c_j)\widetilde{h}_{c_j}\,\mathrm{e}^{az^\rho}\right) + \sum_{j\in I_2} b_j(z) h_{c_j,k_j}\widetilde{h}_{c_j}\,\mathrm{e}^{az^\rho}$$
$$= \left(\sum_{j\in I_1} b_j\right) d + \left(\sum_{j\in I_1} b_j h(z+c_j)\widetilde{h}_{c_j} + \sum_{j\in I_2} b_j h_{c_j,k_j}\widetilde{h}_{c_j}\right)\mathrm{e}^{az^\rho}. \tag{20}$$

By combining with (16) we get

$$\Phi_2 = L(z,f) - \alpha f^2$$
$$= \left(\sum_{j\in I_1} b_j\right) d + \left(\sum_{j\in I_1} b_j h(z+c_j)\widetilde{h}_{c_j} + \sum_{j\in I_2} b_j h_{c_j,k_j}\widetilde{h}_{c_j}\right)\mathrm{e}^{az^\rho}$$
$$- \alpha(d + h\,\mathrm{e}^{az^\rho})^2$$
$$= \widetilde{\gamma}(z)\,\mathrm{e}^{az^\rho} - \alpha h^2\,\mathrm{e}^{2az^\rho} + \left(\sum_{j\in I_1} b_j\right) d - \alpha d^2, \tag{21}$$

where

$$\widetilde{\gamma}(z)=\sum_{j\in I_1} b_j h(z+c_j)\widetilde{h}_{c_j}+\sum_{j\in I_2} b_j h_{c_j,k_j}\widetilde{h}_{c_j}-2\alpha dh.$$

By Lemma 5, $\rho(\Phi_2-\beta)=\rho$. If $\lambda(\Phi_2-\beta)<\rho=\rho(\Phi_2-\beta)$, then β is a Borel exceptional small function of Φ_2, and we can rewrite Φ_2 as follow:

$$\Phi_2=\beta+h^*(z)\,\mathrm{e}^{bz^\rho}, \tag{22}$$

where $b(\neq 0)$ is a constant, and $h^*(\not\equiv 0)$ is an entire function with $\rho(h^*)<\rho$. By (21) and (22) we have

$$h^*(z)\,\mathrm{e}^{bz^\rho}=\widetilde{\gamma}(z)\,\mathrm{e}^{az^\rho}-\alpha h^2\,\mathrm{e}^{2az^\rho}+\Bigg(\sum_{j\in I_1} b_j\Bigg)d-\alpha d^2-\beta. \tag{23}$$

In (23), there are three cases for b: Case 1. $b\neq a$ and $b\neq 2a$; Case 2. $b=a$; Case 3. $b=2a$.

Applying Lemma 1 to (23) for all these three cases, we obtain

$$\Bigg(\sum_{j\in I_1} b_j\Bigg)d-\alpha d^2-\beta\equiv 0,$$

which contradicts our assumption that

$$\beta\not\equiv\Bigg(\sum_{j\in I_1} b_j\Bigg)d-\alpha d^2.$$

Hence, $\lambda(\Phi_2(z)-\beta)=\rho$.

(ii) Suppose that d is a Borel exceptional value of f, and

$$\Bigg(\sum_{j\in I_1} b_j\Bigg)d-\alpha d^2-\beta\equiv 0. \tag{24}$$

Using the same method as before, we can obtain (16), (20) and (21). By combining (21) with (24), we have

$$\Phi_2-\beta=\widetilde{\gamma}(z)\,\mathrm{e}^{az^\rho}-\alpha h^2\,\mathrm{e}^{2az^\rho}. \tag{25}$$

Next, we discuss the following two cases:

Case 1. $\widetilde{\gamma}(z)\equiv 0$. Then

$$\sum_{j\in I_1} b_j h(z+c_j)\widetilde{h}_{c_j}+\sum_{j\in I_2} b_j h_{c_j,k_j}\widetilde{h}_{c_j}\equiv 2\alpha dh, \tag{26}$$

and (25) can be reduced to

$$\Phi_2-\beta=-\alpha h^2\,\mathrm{e}^{2az^\rho}. \tag{27}$$

By Lemma 5, $\rho(\Phi_2-\beta)=\rho$. Combining with $\rho(h)<\rho$, we obtain $\lambda(\Phi_2-\beta)=\lambda(h^2)\leqslant\rho(h)<\rho$. Thus, β is a Borel exceptional small function of Φ_2.

From (16) and (27), we have

$$\Phi_2=\beta-\alpha\left(h\,\mathrm{e}^{az^\rho}\right)^2=\beta-\alpha(f-d)^2. \tag{28}$$

Hence

$$\frac{\beta-\Phi_2}{(f-d)^2}=\alpha.$$

Combining with (16), (20), (24) and (26), we obtain

$$\begin{aligned}&L(z,f)\\&=\Bigg(\sum_{j\in I_1} b_j\Bigg)d+\Bigg(\sum_{j\in I_1} b_j h(z+c_j)\widetilde{h}_{c_j}+\sum_{j\in I_2} b_j h_{c_j,k_j}\widetilde{h}_{c_j}\Bigg)\mathrm{e}^{az^\rho}\\&=\alpha d^2+\beta+2\alpha dh\,\mathrm{e}^{az^\rho}=\alpha d^2+\beta+2\alpha d(f-d).\end{aligned}$$

Therefore,

$$\frac{L(z,f)-\alpha d^2-\beta}{2d(f-d)}=\alpha.$$

Case 2. $\widetilde{\gamma}(z)\not\equiv 0$. We rewrite (25) as follow:

$$\Phi_2-\beta=\widetilde{\gamma}\,\mathrm{e}^{az^\rho}-\alpha h^2\,\mathrm{e}^{2az^\rho}=\alpha h^2\,\mathrm{e}^{az^\rho}\left(\frac{\widetilde{\gamma}}{\alpha h^2}-\mathrm{e}^{az^\rho}\right). \tag{29}$$

Next, we prove that $T(r,\widetilde{\gamma}/(\alpha h^2))=S(r,\mathrm{e}^{az^\rho})$. By $\rho(h)<\rho$, we have $T(r,h)=S(r,\mathrm{e}^{az^\rho})$. Combining with Lemma 3, we have $T(r,h(z+c_j))=S(r,\mathrm{e}^{az^\rho})$ and $T(r,h_{c_j,k_j})=S(r,\mathrm{e}^{az^\rho})$. We assert that $T(r,b_j)=S(r,\mathrm{e}^{az^\rho})$. From (16), we have

$$T(r,f)=T(r,\mathrm{e}^{az^\rho})+S(r,\mathrm{e}^{az^\rho}). \tag{30}$$

Thus,

$$\begin{aligned}\frac{T(r,b_j)}{T(r,\mathrm{e}^{az^\rho})}&=\frac{O(r^{\lambda+\varepsilon})}{T(r,\mathrm{e}^{az^\rho})}+\frac{S(r,f)}{T(r,\mathrm{e}^{az^\rho})}\\&=\frac{O(r^{\lambda+\varepsilon})}{\left(\frac{|a|}{\pi}+o(1)\right)r^\rho}+\frac{S(r,f)}{T(r,f)}\to 0,\end{aligned}$$

as $r\to\infty$, outside a possible exceptional set with finite logarithmic measure. So we have $T(r,b_j)=S(r,\mathrm{e}^{az^\rho})$. Thus, $T(r,\widetilde{\gamma}/(\alpha h^2))=S(r,\mathrm{e}^{az^\rho})$.

By the first and second main theorems of Nevanlinna theory, we have

$$\begin{aligned}T\left(r,\mathrm{e}^{az^\rho}\right)&\leqslant N\left(r,\frac{1}{\mathrm{e}^{az^\rho}}\right)+N\left(r,\frac{1}{\mathrm{e}^{az^\rho}-\frac{\widetilde{\gamma}}{\alpha h^2}}\right)\\&\quad+N(r,\mathrm{e}^{az^\rho})+S(r,\mathrm{e}^{az^\rho})\\&=N\left(r,\frac{1}{\mathrm{e}^{az^\rho}-\frac{\widetilde{\gamma}}{\alpha h^2}}\right)+S(r,\mathrm{e}^{az^\rho})\\&\leqslant T\left(r,\mathrm{e}^{az^\rho}\right)+S(r,\mathrm{e}^{az^\rho}).\end{aligned}$$

So

$$\begin{aligned}N\left(r,\frac{1}{\Phi_2-\beta}\right)&=N\left(r,\frac{1}{\mathrm{e}^{az^\rho}-\frac{\widetilde{\gamma}}{\alpha h^2}}\right)+S(r,\mathrm{e}^{az^\rho})\\&=T\left(r,\mathrm{e}^{az^\rho}\right)+S(r,\mathrm{e}^{az^\rho}).\end{aligned} \tag{31}$$

Thus, combining with (30) and (31), we obtain

$$N\left(r,\frac{1}{\Phi_2-\beta}\right)=T(r,f)+S_\lambda(r,f).$$

PROOF OF Theorem 10

Suppose that $d=0$ is the Borel exceptional value of f. Using the same method as before (the proof of Theorem 9), we can obtain (20) and (21) with $d=0$, i.e.,

$$L(z,f)=\widetilde{\gamma}(z)\,e^{az^{\rho}}. \tag{32}$$

and

$$\Phi_2=\widetilde{\gamma}(z)\,e^{az^{\rho}}-\alpha h^2\,e^{2az^{\rho}}, \tag{33}$$

where

$$\widetilde{\gamma}(z)=\sum_{j\in I_1} b_j h(z+c_j)\widetilde{h}_{c_j}+\sum_{j\in I_2} b_j h_{c_j,k_j}\widetilde{h}_{c_j}.$$

Next, we discuss the following two cases:

Case 1. $\beta\not\equiv 0$. By Lemma 5, $\rho(\Phi_2-\beta)=\rho$. If $\lambda(\Phi_2(z)-\beta)<\rho$, then we can rewrite Φ_2 as follow:

$$\Phi_2=\beta+h^*\,e^{bz^{\rho}}, \tag{34}$$

where $b(\neq 0)$ is a constant, and $h^*(\not\equiv 0)$ is an entire function with $\rho(h^*)<\rho$. By (33) and (34) we have

$$\beta+h^*\,e^{bz^{\rho}}=\widetilde{\gamma}(z)\,e^{az^{\rho}}-\alpha h^2\,e^{2az^{\rho}}. \tag{35}$$

In (35), there are three subcases for b: Subcase 1. $b\neq a$ and $b\neq 2a$; Subcase 2. $b=a$; Subcase 3. $b=2a$. Applying Lemma 1 to (35) for these three cases, we obtain $\beta\equiv 0$, which contradicts our assumption that $\beta\not\equiv 0$. Hence $\lambda(\Phi_2(z)-\beta)=\rho$.

Case 2. $\beta\equiv 0$. Obviously, $\widetilde{\gamma}(z)\not\equiv 0$. Otherwise, by (32) we obtain $L(z,f)\equiv 0$, a contradiction. We rewrite (33) as follow:

$$\Phi_2=\alpha h^2\,e^{az^{\rho}}\left(\frac{\widetilde{\gamma}}{\alpha h^2}-e^{az^{\rho}}\right),$$

following the same method as in the proof of case 2 in Theorem 9(ii), we obtain

$$\begin{aligned} N\Big(r,\frac{1}{\Phi_2}\Big)&=N\Bigg(r,\frac{1}{e^{az^{\rho}}-\frac{\widetilde{\gamma}}{\alpha h^2}}\Bigg)+S(r,e^{az^{\rho}})\\ &=T\left(r,e^{az^{\rho}}\right)+S(r,e^{az^{\rho}})\\ &=T(r,f)+S_{\lambda}(r,f). \end{aligned}$$

Hence, we have $\lambda(\Phi_2)=\rho$.

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