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Multiplicative Jordan derivations on triangular n-matrix rings

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ABSTRACT: Let $\mathscr{T}$ be a triangular n-matrix ring ($n \geqslant 2$) and $\delta : \mathscr{T} \to \mathscr{T}$ a map. It is shown that δ is a multiplicative Jordan derivation if and only if one of the statements holds: (1) if $\mathscr{T}$ is 2-torsion free, then δ is an additive derivation; (2) if $\mathscr{T}$ is 2-torsion, under some mild assumptions, then $\delta(X) = d(X) + \gamma(X)$ holds for all $X \in \mathscr{T}$, where $d : \mathscr{T} \to \mathscr{T}$ is an additive derivation and γ is a map from $\mathscr{T}$ into its center vanishing on all elements $XY + YX$ for $X, Y \in \mathscr{T}$. This generalizes some related known results.

KEYWORDS: Jordan derivations, derivations, triangular rings, triangular n-matrix rings

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INTRODUCTION

Let $\mathscr{A}$ be an associative ring or an algebra. For any $A, B \in \mathscr{A}$, the Jordan product of A, B is defined by $A \circ B = AB + BA$. Recall that a map $\delta : \mathscr{A} \to \mathscr{A}$ is called a multiplicative (or nonlinear) derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathscr{A}$; is called a multiplicative (or nonlinear) Jordan derivation if $\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$ for all $A, B \in \mathscr{A}$. Particularly, if δ is also assumed to be additive (linear), then δ is called additive (linear) derivation and Jordan derivation, respectively. The questions of characterizing (multiplicative) Jordan derivations and revealing the relationship between Jordan derivations and derivations have received many mathematicians' attention (for example, see [1–7] and the references therein).

In [8], the author introduced a class of rings (or algebras) as follows. Let $\mathscr{A}$ and $\mathscr{B}$ be unital rings (or algebras over a commutative ring $\mathscr{R}$), and $\mathscr{M}$ an $(\mathscr{A}, \mathscr{B})$-bimodule which is faithful as a left $\mathscr{A}$-module as well as a right $\mathscr{B}$-module. The associative ring (or $\mathscr{R}$-algebra)

$$\mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B}) = \left\{ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \middle| A \in \mathscr{A}, M \in \mathscr{M}, B \in \mathscr{B} \right\}$$

under the usual matrix operations is called a triangular ring (or algebra). Zhang and Yu [7] show that every linear Jordan derivation on a triangular algebra $\mathscr{U}$ is a derivation under the assumption that the commutative ring $\mathscr{R}$ is 2-torsion free. Recall that a ring $\mathscr{A}$ is said to be 2-torsion free if $2a = 0$ implies $a = 0$ for any $a \in \mathscr{A}$; otherwise, $\mathscr{A}$ is 2-torsion. Xiao [6] generalizes the result of [7] to multiplicative Jordan derivations and obtains the same result.

Recently, Ferreira [9] defined a class of ring called triangular n-matrix ring as follows.

Definition 1 ([9]) Let $\mathscr{R}_1, \mathscr{R}_2, \dots, \mathscr{R}_n$ be unital rings and $\mathscr{M}_{ij}$ be $(\mathscr{R}_i, \mathscr{R}_j)$-bimodules with $\mathscr{M}_{ii} = \mathscr{R}_i$ for all $1 \leqslant i \leqslant j \leqslant n$. Let $\varphi_{ijk} : \mathscr{M}_{ij} \otimes_{\mathscr{R}_j} \mathscr{M}_{jk} \to \mathscr{M}_{ik}$ be $(\mathscr{R}_i, \mathscr{R}_k)$-bimodules homomorphisms with $\varphi_{iij} : \mathscr{R}_i \otimes_{\mathscr{R}_i} \mathscr{M}_{ij} \to \mathscr{M}_{ij}$ and $\varphi_{ijj} : \mathscr{M}_{ij} \otimes_{\mathscr{R}_j} \mathscr{R}_j \to \mathscr{M}_{ij}$ the canonical multiplication maps for all $1 \leqslant i \leqslant j \leqslant k \leqslant n$. Write $ab = \varphi_{ijk}(a \otimes b)$ for all $a \in \mathscr{M}_{ij}$ and $b \in \mathscr{M}_{jk}$. Assume that $\mathscr{M}_{ij}$ is faithful as a left $\mathscr{R}_i$-module and faithful as a right $\mathscr{R}_j$-module for all $1 \leqslant i < j \leqslant n$. Let $\mathscr{T} = \mathscr{T}_n(\mathscr{R}_i; \mathscr{M}_{ij})$ be the set

$$\mathscr{T} = \left\{ \begin{pmatrix} r_{11} & m_{12} & \cdots & m_{1(n-1)} & m_{1n} \\ 0 & r_{22} & \cdots & m_{2(n-1)} & m_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{(n-1)(n-1)} & m_{(n-1)n} \\ 0 & 0 & \cdots & 0 & r_{nn} \end{pmatrix} \middle| \; r_{ii} \in \mathscr{R}_i,\ m_{ij} \in \mathscr{M}_{ij}, \quad 1 \leqslant i < j \leqslant n \right\}.$$

Furthermore, assume that $a(bc) = (ab)c$ for all $a \in \mathscr{M}_{ik}, b \in \mathscr{M}_{kl}$, and $c \in \mathscr{M}_{lj}$ with $1 \leqslant i \leqslant k \leqslant l \leqslant j \leqslant m$. Then, with the usual matrix operations, $\mathscr{T}$ is called a triangular n-matrix ring.

It is obvious that upper triangular matrix rings $\mathscr{T}_n(\mathscr{R})$ with $n \geqslant 3$ over a unital associative ring $\mathscr{R}$

are triangular n-matrix rings; and that triangular 2-matrix rings are usual triangular rings. However, for $n \geqslant 3$, a triangular ring may not be a triangular n-matrix ring; conversely, a triangular n-matrix ring may not be a triangular ring (see Example 2.1 in [10]). For some results on triangular n-matrix rings, see [9, 11].

The purpose of the present paper is to discuss the structure of multiplicative Jordan derivations on triangular n-matrix rings for any $n \geqslant 2$. Assume that δ is a map on a triangular n-matrix ring $\mathscr{T}$. We show that δ is a multiplicative Jordan derivation if and only if one of the following statements is true: (1) if $\mathscr{T}$ is 2-torsion free, then δ is an additive derivation; (2) if $\mathscr{T}$ is 2-torsion, then δ has the form $\delta(X) = d(X) + \gamma(X)$ for all $X \in \mathscr{T}$ under some mild assumptions on $\mathscr{T}$, where d is an additive derivation on $\mathscr{T}$, and γ is a map from $\mathscr{T}$ into its center vanishing on all elements $XY + YX$ for $X, Y \in \mathscr{T}$.

MAIN RESULT

For 2-torsion free triangular n-matrix rings, we have the following result.

Theorem 1 *Let $\mathscr{T}$ be a 2-torsion free triangular n-matrix ring ($n \geqslant 2$). Then a map $\delta : \mathscr{T} \to \mathscr{T}$ is a multiplicative Jordan derivation if and only if δ is an additive derivation.*

Note that, if a ring $\mathscr{R}$ is 2-torsion, then for any elements $T, S \in \mathscr{R}$, it is clear that $TS + ST = TS - ST$. In this case, Jordan derivations and Lie derivations are equivalent. Thus, by Theorem 2.3 in [12], we can give the characterization of multiplicative Jordan derivations on 2-torsion triangular n-matrix rings.

Fix any $i \in \{1, 2, \dots, n\}$. Let E_i stand for the nontrivial idempotent in $\mathscr{T}$ whose elements with (i, i) position 1 and the rest 0. Write $P_i = E_1 + E_2 + \cdots + E_i$ and $Q_i = I - P_i$.

Theorem 2 *Let $\mathscr{T}$ be a 2-torsion triangular n-matrix ring ($n \geqslant 2$). Assume that $P_{[\frac{n}{2}]}\mathscr{Z}(\mathscr{T})P_{[\frac{n}{2}]} = \mathscr{Z}(P_{[\frac{n}{2}]}\mathscr{T}P_{[\frac{n}{2}]})$ and $Q_{[\frac{n}{2}]}\mathscr{Z}(\mathscr{T})Q_{[\frac{n}{2}]} = \mathscr{Z}(Q_{[\frac{n}{2}]}\mathscr{T}Q_{[\frac{n}{2}]})$. Then a map $\delta : \mathscr{T} \to \mathscr{T}$ is a multiplicative Jordan derivation if and only if $\delta(X) = d(X) + \gamma(X)$ holds for all $X \in \mathscr{T}$, where $d : \mathscr{T} \to \mathscr{T}$ is an additive derivation and $\gamma : \mathscr{T} \to \mathscr{Z}(\mathscr{T})$ is a map vanishing on any $[X, Y] = XY + YX$. Here, $[s]$ is the integer part of s and $\mathscr{Z}(\mathscr{T})$ is the center of $\mathscr{T}$.*

Particularly, if $n = 2$, we can obtain a complete characterization of multiplicative Jordan derivations on triangular rings, which is a generalization of the related results in [6, 7].

Corollary 1 *Let $\mathscr{A}$ and $\mathscr{B}$ be unital rings, and $\mathscr{M}$ be an $(\mathscr{A}, \mathscr{B})$-bimodule, which is faithful as a left $\mathscr{A}$-module and also as a right $\mathscr{B}$-module. Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ be the triangular ring. Then a map $\delta : \mathscr{U} \to \mathscr{U}$ is a multiplicative Jordan derivation if and only if the following statements hold:*

(i) if $\mathscr{U}$ is 2-torsion free, then δ is an additive derivation;

(ii) if $\mathscr{U}$ is 2-torsion, $P\mathscr{Z}(\mathscr{U})P = \mathscr{Z}(P\mathscr{U}P)$ and $(I-P)\mathscr{Z}(\mathscr{U})(I-P) = \mathscr{Z}((I-P)\mathscr{U}(I-P))$, then $\delta = d + \gamma$, where $d : \mathscr{U} \to \mathscr{U}$ is an additive derivation and $\gamma : \mathscr{U} \to \mathscr{Z}(\mathscr{U})$ is a map vanishing on all commutators.

Set for $1 \leqslant i \leqslant j \leqslant n$,

$$\mathscr{T}_{ij} = \left\{ (m_{kt}) \,\middle|\, m_{kt} = \begin{cases} m_{ij}, & \text{if } (k,t) = (i,j), \\ 0, & \text{if } (k,t) \neq (i,j), \end{cases} \right\} \subset \mathscr{T}.$$

Then we can write $\mathscr{T} = \sum_{1 \leqslant i \leqslant j \leqslant n} \mathscr{T}_{ij}$. Obviously, for any element $a_{ij} \in \mathscr{T}_{ij}$, $a_{ij}a_{kj} = 0$ whenever $j \neq k$.

Denote by $\mathscr{A}_i = P_i\mathscr{T}P_i$, $\mathscr{B}_i = Q_i\mathscr{T}Q_i$ and $\mathscr{M}_i = P_i\mathscr{T}Q_i$ ($i \in \{1, 2, \dots, n\}$). Then $\mathscr{T}$ can also be written as $\mathscr{T} = \mathscr{A}_i + \mathscr{M}_i + \mathscr{B}_i$ for each i. In this paper, if no confusion occurs, for any $A_i \in \mathscr{A}_i$, $M_i \in \mathscr{M}_i$ and $B_i \in \mathscr{B}_i$, we always identify

$$A_i \cong \begin{pmatrix} r_{11} & m_{12} & \cdots & m_{1i} \\ 0 & r_{22} & \cdots & m_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{ii} \end{pmatrix},$$

$$M_i \cong \begin{pmatrix} m_{1,i+1} & m_{1,i+2} & \cdots & m_{1n} \\ m_{2,i+1} & m_{2,i+2} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{i,i+1} & m_{i,i+2} & \cdots & m_{in} \end{pmatrix},$$

and

$$B_i \cong \begin{pmatrix} r_{i+1,i+1} & m_{i+1,i+2} & \cdots & m_{in} \\ 0 & r_{i+2,i+2} & \cdots & m_{i+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{pmatrix}.$$

Proof of Theorem 1

The "if" part is obvious. For the "only if" part, we will prove it by checking a series of claims. Firstly, let $i \in \{1, 2, \dots, n\}$.

Claim 1. *For any $A_i \in \mathscr{A}_i$, $M_i \in \mathscr{M}_i$ and $B_i \in \mathscr{B}_i$, we have* *(i)* $Q_i\delta(A_i)Q_i = 0$;

(ii) $P_i\delta(B_i)P_i = 0$;

(iii) $P_i\delta(M_i)P_i = Q_i\delta(M_i)Q_i = 0$.

It is clear that $\delta(0) = 0$. For any $A_i \in \mathscr{A}_i$, since $A_i \circ Q_i = 0$, one gets

$$\begin{aligned}0 &= \delta(A_i \circ Q_i) = \delta(A_i) \circ Q_i + A_i \circ \delta(Q_i)\\ &= P_i\delta(A_i)Q_i + 2Q_i\delta(A_i)Q_i + A_i\delta(Q_i) + \delta(Q_i)A_i.\end{aligned}$$

Multiplying by Q_i from the two sides in the above equation, and by the 2-torsion freeness of $\mathscr{T}$, one has $Q_i\delta(A_i)Q_i = 0$.

Similarly, by using the relation $P_i \circ B_i = 0$, one can show $P_i\delta(B_i)P_i = 0$. So, the statements (i)–(ii) are true.

For any $M_i \in \mathscr{M}_i$, we have

$$\begin{aligned}\delta(M_i) &= \delta(P_i \circ M_i) = \delta(P_i) \circ M_i + P_i \circ \delta(M_i)\\ &= \delta(P_i)M_i + M_i\delta(P_i) + 2P_i\delta(M_i)P_i + P_i\delta(M_i)Q_i.\end{aligned}$$

Multiplying by P_i and Q_i from the two sides in the above equation, respectively, one gets $P_i\delta(M_i)P_i = 0$ and $Q_i\delta(M_i)Q_i = 0$. So, the statement (iii) is true, and the claim holds.

Now, define two maps $d_i, \delta_i : \mathscr{T} \to \mathscr{T}$ by

$$d_i(X) = [\delta(P_i), X] \quad\text{and}\quad \delta_i(X) = \delta(X) + d_i(X). \tag{1}$$

It is easy to check that d_i is an additive derivation and δ_i is a multiplicative Jordan derivation. Then, Claim 1 is also true for δ_i, that is,

$$\begin{gathered}Q_i\delta_i(A_i)Q_i = 0, \qquad P_i\delta_i(B_i)P_i = 0,\\ P_i\delta_i(M_i)P_i = Q_i\delta_i(M_i)Q_i = 0.\end{gathered} \tag{2}$$

So

$$\delta_i(P_i)=\delta(P_i)+d_i(P_i)=\delta(P_i)+[\delta(P_i), P_i]=P_i\delta(P_i)P_i.$$

This yields that

$$\delta_i(P_i) = P_i\delta(P_i)P_i = P_i\delta_i(P_i)P_i. \tag{3}$$

We will discuss the properties of δ_i for $i \in \{1, 2, \ldots, n\}$ by Claims 2–5.

Claim 2. *For any $A_i \in \mathscr{A}_i$, $M_i \in \mathscr{M}_i$, and $B_i \in \mathscr{B}_i$, we have*

(i) $\delta_i(M_i) \in \mathscr{M}_i$, $\delta_i(A_i) \in \mathscr{A}_i$, *and* $\delta_i(B_i) \in \mathscr{B}_i$;

(ii) $\delta_i(P_i)M_i = M_i\delta_i(Q_i) = 0$.

By (2), it is true that $\delta_i(M_i) \in \mathscr{M}_i$. By (2)–(3), one has

$$0 = \delta_i(P_i \circ Q_i) = \delta_i(P_i) \circ Q_i + P_i \circ \delta_i(Q_i) = P_i\delta_i(Q_i)Q_i,$$

and so $\delta_i(Q_i) = Q_i\delta_i(Q_i)Q_i$. Thus, for any $A_i \in \mathscr{A}_i$, by (2) again, one obtains

$$0 = \delta_i(A_i \circ Q_i) = \delta_i(A_i) \circ Q_i + A_i \circ \delta_i(Q_i) = P_i\delta_i(A_i)Q_i.$$

It follows that $\delta_i(A_i) = P_i\delta_i(A_i)P_i \in \mathscr{A}_i$.

Similarly, one can show $\delta_i(B_i) = Q_i\delta_i(B_i)Q_i \in \mathscr{B}_i$ for all $B_i \in \mathscr{B}_i$. The statement (i) is true.

Finally, for any $M_i \in \mathscr{M}_i$, by Claim 2(i), one gets

$$\begin{aligned}\delta_i(M_i) = \delta_i(P_i \circ M_i) &= \delta_i(P_i) \circ M_i + P_i \circ \delta_i(M_i)\\ &= \delta_i(P_i)M_i + \delta_i(M_i)\end{aligned}$$

and

$$\begin{aligned}\delta_i(M_i) = \delta_i(M_i \circ Q_i) &= \delta_i(M_i) \circ Q_i + M_i \circ \delta_i(Q_i)\\ &= \delta_i(M_i) + M_i\delta_i(Q_i),\end{aligned}$$

which imply that $\delta_i(P_i)M_i = M_i\delta_i(Q_i) = 0$. So, the statement (ii) is true.

Claim 3. *For any $A_i \in \mathscr{A}_i$, $M_i \in \mathscr{M}_i$, and $B_i \in \mathscr{B}_i$, we have*

(i) $\delta_i(A_i + M_i) \in \mathscr{A}_i + \mathscr{M}_i$ *and* $\delta_i(M_i) = P_i\delta_i(A_i + M_i)Q_i$;

(ii) $\delta_i(M_i + B_i) \in \mathscr{M}_i + \mathscr{B}_i$ *and* $\delta_i(M_i) = P_i\delta_i(M_i + B_i)Q_i$.

We only give the proof of (i). The proof of (ii) is similar and we omit it here. For any $A_i \in \mathscr{A}_i$ and $M_i \in \mathscr{M}_i$, by Claim 2, we get

$$\begin{aligned}\delta_i(M_i) &= \delta_i(M_i \circ Q_i) = \delta_i((A_i + M_i) \circ Q_i)\\ &= \delta_i(A_i + M_i) \circ Q_i + A_i \circ \delta_i(Q_i) + M_i \circ \delta_i(Q_i)\\ &= P_i\delta_i(A_i + M_i)Q_i + 2Q_i\delta_i(A_i + M_i)Q_i.\end{aligned}$$

It follows from Claim 2(i) and 2-torsion freeness of $\mathscr{T}$ that $\delta_i(M_i) = P_i\delta_i(A_i + M_i)Q_i$ and $Q_i\delta_i(A_i + M_i)Q_i = 0$.

Claim 4. *For any $X = A_i + M_i + B_i \in \mathscr{T}$, we have*

(i) $P_i\delta_i(X)Q_i = \delta_i(P_iXQ_i)$;

(ii) $E_t(\delta_i(A_i+M_i+B_i)-\delta_i(A_i)-\delta_i(M_i)-\delta_i(B_i))E_t = 0$, *where* $t = 1, \ldots, n$.

For any $X = A_i + M_i + B_i \in \mathscr{T}$, Claim 2 gives

$$\delta_i(2A_i + M_i) = \delta_i(X \circ P_i) = \delta_i(X) \circ P_i + X \circ \delta_i(P_i)$$
$$= 2P_i\delta_i(X)P_i + P_i\delta_i(X)Q_i + A_i\delta_i(P_i)P_i + P_i\delta_i(P_i)A_i \quad (4)$$

and

$$\delta_i(2A_i + M_i) = \delta_i((A_i + M_i) \circ P_i)$$
$$= \delta_i(A_i + M_i) \circ P_i + (A_i + M_i) \circ \delta_i(P_i)$$
$$= 2P_i\delta_i(A_i+M_i)P_i+P_i\delta_i(A_i+M_i)Q_i+A_i\delta_i(P_i)P_i+P_i\delta_i(P_i)A_i. \quad (5)$$

Comparing (4)–(5) and Claim 3(i) gives

$$P_i\delta_i(X)Q_i = P_i\delta_i(A_i + M_i + B_i)Q_i$$
$$= P_i\delta_i(A_i + M_i)Q_i = \delta_i(M_i) = \delta_i(P_iXQ_i)$$

and

$$2P_i\delta_i(X)P_i = 2P_i\delta_i(A_i + M_i)P_i. \quad (6)$$

Since $\mathscr{T}$ is 2-torsion free, (6) implies

$$P_i\delta_i(X)P_i = P_i\delta_i(A_i + M_i)P_i. \quad (7)$$

Next, take any $M_i' \in \mathscr{M}_i$. Note that

$$\delta_i((A_i+M_i)\circ M_i') = \delta_i(A_i+M_i)\circ M_i'+(A_i+M_i)\circ\delta_i(M_i')$$

and

$$\delta_i((A_i+M_i)\circ M_i')=\delta_i(A_i\circ M_i')=\delta_i(A_i)\circ M_i'+A_i\circ\delta_i(M_i').$$

Combining the above two equations, Claim 2(i) and Claim 3(i) yields

$$(\delta_i(A_i + M_i) - \delta_i(A_i) - \delta_i(M_i))M_i' = 0, \quad \forall M_i' \in \mathscr{M}_i.$$

Moreover, since $\mathscr{M}_{ij}$ is a faithful left $\mathscr{R}_i$-module, one can show that for $t = 1, \dots, i$,

$$E_t(\delta_i(A_i + M_i) - \delta_i(A_i) - \delta_i(M_i))E_t = 0. \quad (8)$$

By the fact $P_i\delta_i(B_i)P_i = 0$ and (7)–(8), one achieves, for $t = 1, 2, \dots, i$,

$$E_t(\delta_i(A_i+M_i+B_i)-\delta_i(A_i)-\delta_i(M_i)-\delta_i(B_i))E_t=0. \quad (9)$$

Finally, by calculating $\delta_i(M_i' \circ (M_i + B_i))$, a similar argument to that of the above gives, for $t = i+1, \dots, n$,

$$E_t(\delta_i(A_i+M_i+B_i)-\delta_i(A_i)-\delta_i(M_i)-\delta_i(B_i))E_t=0. \quad (10)$$

It follows from (9)–(10) that $E_t(\delta_i(A_i + M_i + B_i) - \delta_i(A_i) - \delta_i(M_i) - \delta_i(B_i))E_t = 0$ holds for $t = 1, \dots, n$. The claim holds.

Claim 5. *δ_i is additive on $\mathscr{M}_i$.*

For P_i and any $M_i \in \mathscr{M}_i$, on the one hand, by Claim 4(i), one has

$$\delta_i(2P_i) = \delta_i((P_i + M_i) \circ (P_i - Q_i))$$
$$= \delta_i(P_i + M_i) \circ (P_i - Q_i) + (P_i + M_i) \circ \delta_i(P_i - Q_i)$$
$$= 2P_i\delta_i(P_i + M_i)P_i + 2P_i\delta_i(P_i - Q_i)P_i$$
$$+P_i\delta_i(P_i-Q_i)Q_i+M_i\delta_i(P_i-Q_i)Q_i+P_i\delta_i(P_i-Q_i)M_i.$$

On the other hand, by Claim 2(i) and Claim 4(i), one has

$$\delta_i(2P_i) = \delta_i(P_i \circ (P_i - Q_i))$$
$$= \delta_i(P_i) \circ (P_i - Q_i) + P_i \circ \delta_i(P_i - Q_i)$$
$$= 2P_i\delta_i(P_i)P_i + 2P_i\delta_i(P_i - Q_i)P_i + P_i\delta_i(P_i - Q_i)Q_i$$
$$= 2\delta_i(P_i) + 2P_i\delta_i(P_i - Q_i)P_i.$$

Comparing the above two equations yields

$$P_i\delta_i(P_i + M_i)P_i = \delta_i(P_i)$$

by the 2-torsion freeness of $\mathscr{T}$, which together with Claim 3(i) and Claim 4(i), implies that

$$\delta_i(P_i + M_i) = P_i\delta_i(P_i + M_i)P_i + P_i\delta_i(P_i + M_i)Q_i$$
$$= \delta_i(P_i) + \delta_i(M_i). \quad (11)$$

Symmetrically, one can show that

$$\delta_i(M_i + Q_i) = \delta_i(M_i) + \delta_i(Q_i). \quad (12)$$

Now, for any $M_i, M_i' \in \mathscr{M}_i$, by Claim 2 and (11)–(12), one achieves

$$\delta_i(M_i + M_i') = \delta_i((P_i + M_i) \circ (M_i' + Q_i))$$
$$= \delta_i(P_i+M_i) \circ (M_i'+Q_i)+(P_i+M_i) \circ \delta_i(M_i'+Q_i)$$
$$= (\delta_i(P_i) + \delta_i(M_i)) \circ (M_i' + Q_i)$$
$$+ (P_i + M_i) \circ (\delta_i(M_i') + \delta_i(Q_i))$$
$$= \delta_i(M_i) + \delta_i(M_i'),$$

that is, δ_i is additive on $\mathscr{M}_i$, completing the proof of the claim.

From now on, let $i \in \{2, \dots, n-1\}$. Define another maps $\tau_i : \mathscr{T} \to \mathscr{T}$ by

$$\tau_i(X) = \delta_1(X) + [\delta_1(P_i), X]. \quad (13)$$

By the same arguments as those of Claims 2–5 for δ_i, we can prove that τ_i is also a multiplicative Jordan derivation satisfying $\tau_i(P_i) = P_i\tau_i(P_i)P_i$, and Claims 2–5 still hold for the map τ_i.

Claim 6. *For any $M_i \in \mathscr{M}_i$, we have $\delta_1(M_i) = \tau_i(M_i)$. Therefore, δ_1 is additive on $\mathscr{M}_i$.*

Take any $M_i \in \mathscr{M}_i$. As $\delta_1(M_i) = \tau_i(M_i) - [\delta_1(P_i), M_i] \in \mathscr{M}_i$, it is true that

$$\begin{aligned} P_i\delta_1(M_i)Q_i &= \delta_1(M_i) = \delta_1(P_i \circ M_i) \\ &= \delta_1(P_i) \circ M_i + P_i \circ \delta_1(M_i) \\ &= \delta_1(P_i)M_i + M_i\delta_1(P_i) + P_i\delta_1(M_i)Q_i. \end{aligned}$$

This means

$$\delta_1(P_i)M_i + M_i\delta_1(P_i) = 0.$$

Also note that

$$\begin{aligned} \mathscr{A}_i \ni \tau_i(P_i) &= \delta_1(P_i) + [\delta_1(P_i), P_i] \\ &= P_i\delta_1(P_i)P_i + Q_i\delta_1(P_i)Q_i, \end{aligned}$$

which implies $Q_i\delta_1(P_i)Q_i = 0$. So

$$M_i\delta_1(P_i) = 0 = \delta_1(P_i)M_i, \quad \forall M_i \in \mathscr{M}_i. \tag{14}$$

It follows that $\delta_1(M_i) = \tau_i(M_i)$. Hence δ_1 is additive on $\mathscr{M}_i$.

Claim 7. *For any $X = A_1 + M_1 + B_1 \in \mathscr{T}$, we have $\delta_1(X) = \delta_1(A_1) + \delta_1(M_1) + \delta_1(B_1)$.*

For any $X = A_1 + M_1 + B_1 = A_i + M_i + B_i \in \mathscr{T}$, let

$$\begin{aligned} &H_1 = \delta_1(X) - \delta_1(A_1) - \delta_1(M_1) - \delta_1(B_1) \quad \text{and} \\ &K_i = \tau_i(X) - \tau_i(A_i) - \tau_i(M_i) - \tau_i(B_i). \end{aligned}$$

Our goal is to show that $H_1 = 0$. In fact, by (13), one has

$$\begin{aligned} H_1 - K_i &= \delta_1(X) - \tau_i(X) - \delta_1(A_1) + \tau_i(A_i) \\ &\quad - \delta_1(M_1) + \tau_i(M_i) - \delta_1(B_1) + \tau_i(B_i) \\ &= -[\delta_1(P_i), X] - (\tau_i(A_1) - [\delta_1(P_i), A_1]) + \tau_i(A_i) \\ &\quad -(\tau_i(M_1) - [\delta_1(P_i), M_1]) + (\delta_1(M_i) + [\delta_1(P_i), M_i]) \\ &\quad - (\tau_i(B_1) - [\delta_1(P_i), B_1]) + \tau_i(B_i) \\ &= -\tau_i(A_1) + \tau_i(A_i) - \tau_i(M_1) + \delta_1(M_i) \\ &\quad + [\delta_1(P_i), M_i] - \tau_i(B_1) + \tau_i(B_i). \end{aligned} \tag{15}$$

Observe that, by Claim 2(i) for τ_i, one gets

$$\begin{aligned} \tau_i(A_1) = P_i\tau_i(A_1)P_i, \quad \tau_i(A_i) = P_i\tau_i(A_i)P_i, \\ \tau_i(B_i) = Q_i\tau_i(B_i)Q_i; \end{aligned} \tag{16}$$

by (13) and Claim 2(i) for δ_1, one has

$$\tau_i(M_1) = \delta_1(M_1) + [\delta_1(P_i), M_1] \in \mathscr{M}_1; \tag{17}$$

and by Claim 6 and Claim 2(i) for δ_1, one gets

$$\begin{aligned} \delta_1(M_i) &= \delta_1(P_1M_iP_1) + \delta_1(P_1M_iQ_1) + \delta_1(Q_1M_iQ_1) \\ &= \delta_1(P_1M_iQ_1) + \delta_1(Q_1M_iQ_1) \\ &\in \mathscr{M}_1 + \delta_1(Q_1M_iQ_1). \end{aligned} \tag{18}$$

Now, combining Claim 4(i) and (13)–(18) yields

$$\begin{aligned} E_i(H_1 - K_i)E_j &= E_i\delta_1(M_i)E_j - E_i\tau_i(B_1)E_j \\ &= E_i\delta_1(Q_1M_iQ_1)E_j - E_iP_i\tau_i(B_1)Q_iE_j \\ &= E_i\delta_1(Q_1M_iQ_1)E_j - E_i\tau_i(P_iB_1Q_i)E_j \\ &= E_i(\delta_1(Q_1M_iQ_1) - \tau_i(Q_1M_iQ_1))E_j \\ &= E_i[Q_1M_iQ_1, \delta_1(P_i)]E_j = 0, \end{aligned} \tag{19}$$

where $j = i+1, \dots, n$. Here, the reciprocal 3rd equation is due to $Q_1M_iQ_1 = P_iB_1Q_i \in \sum_{k=2}^{i}\sum_{l=1}^{n-i}\mathscr{T}_{k,i+l}$.

On the other hand, by Claim 4(i) and Claim 2(i) for τ_i, we know that $P_iK_iQ_i = 0$, which implies $E_iK_iE_j = E_iP_iK_iQ_iE_j = 0$ for $j = i+1, \dots, n$. Hence (19) reduces to $E_iH_1E_j = 0$.

Also note that $P_1H_1Q_1 = 0$ and $E_tH_1E_t = 0$, $t = 1, \dots, n$, by Claim 4 and Claim 2(i) for δ_1. Hence $H_1 = 0$, completing the proof of the claim.

Claim 8. *δ_1 is additive on $\mathscr{T}$.*

We will prove the claim by several steps.

Step 8.1. δ_1 is additive on $\mathscr{M}_1$.

By Claim 5, this is true.

Step 8.2. δ_1 is additive on $\mathscr{A}_1$.

Take any $A_1, A_1' \in \mathscr{A}_1$ and any $M_1 \in \mathscr{M}_1$. By Claim 2(i) and Step 8.1, we have

$$\begin{aligned} &\delta_1((A_1 + A_1') \circ M_1) \\ &\quad = \delta_1(A_1 + A_1') \circ M_1 + (A_1 + A_1') \circ \delta_1(M_1) \\ &\quad = \delta_1(A_1 + A_1')M_1 + (A_1 + A_1')\delta_1(M_1) \end{aligned}$$

and

$$\begin{aligned} \delta_1((A_1 + A_1') \circ M_1) &= \delta_1(A_1M_1 + A_1'M_1) \\ &= \delta_1(A_1 \circ M_1) + \delta_1(A_1' \circ M_1) \\ &= \delta_1(A_1)M_1 + A_1\delta_1(M_1) + \delta_1(A_1')M_1 + A_1'\delta_1(M_1). \end{aligned}$$

Combining the above two equations gives $(\delta_1(A_1 + A_1') - \delta_1(A_1) - \delta_1(A_1'))M_1 = 0$ for all

$M_1 \in \mathscr{M}_1$. Since $\mathscr{M}_{1j}$ is a faithful left $\mathscr{R}_1$-module, we get $\delta_1(A_1+A_1')-\delta_1(A_1)-\delta_1(A_1')=0$.

Step 8.3. δ_1 is additive on $\mathscr{B}_1$.

For any $B_1, B_1' \in \mathscr{B}_1$ and any $M_1 \in \mathscr{M}_1$, by Claim 8.1, one obtains

$$\begin{aligned}\delta_1(M_1 \circ (B_1+B_1'))\\ &= \delta_1(M_1)\circ(B_1+B_1')+M_1\circ\delta_1(B_1+B_1')\\ &= \delta_1(M_1)(B_1+B_1')+M_1\delta_1(B_1+B_1')\end{aligned}$$

and

$$\begin{aligned}\delta_1(M_1\circ(B_1+B_1')) &= \delta_1(M_1B_1+M_1B_1')\\ &= \delta_1(M_1\circ B_1)+\delta_1(M_1\circ B_1')\\ &= \delta_1(M_1)B_1+M_1\delta_1(B_1)+\delta_1(M_1)B_1'+M_1\delta_1(B_1),\end{aligned}$$

which implies $M_1(\delta_1(B_1+B_1')-\delta_1(B_1)-\delta_1(B_1'))=0$. It follows from the fact $\mathscr{M}_{1t}$ is a faithful right $\mathscr{R}_t$-module that, for $t=2,\dots,n$,

$$E_t(\delta_1(B_1+B_1')-\delta_1(B_1)-\delta_1(B_1'))E_t=0. \quad (20)$$

On the other hand, by Claim 4(i) for τ_i, Claim 5 for τ_i, and (13), one obtains

$$\begin{aligned}P_i\delta_1(B_1+B_1')Q_i &= P_i(\tau_i(B_1+B_1')-[\delta_1(P_i),B_1+B_1'])Q_i\\ &= P_i\tau_i(B_1+B_1')Q_i-P_i([\delta_1(P_i),B_1+B_1'])Q_i\\ &= \tau_i(P_i(B_1+B_1')Q_i)-P_i([\delta_1(P_i),B_1])Q_i\\ &\quad -P_i([\delta_1(P_i),B_1'])Q_i\\ &= \tau_i(P_iB_1Q_i)-P_i([\delta_1(P_i),B_1])Q_i+\tau_i(P_iB_1'Q_i)\\ &\quad -P_i([\delta_1(P_i),B_1'])Q_i\\ &= P_i(\tau_i(B_1)-[\delta_1(P_i),B_1])Q_i\\ &\quad +P_i(\tau_i(B_1')-[\delta_1(P_i),B_1'])Q_i\\ &= P_i\delta_1(B_1)Q_i+P_i\delta_1(B_1')Q_i,\end{aligned}$$

which implies that, for $i=2,\dots,n-1$,

$$P_i(\delta_1(B_1+B_1')-\delta_1(B_1)-\delta_1(B_1'))Q_i=0. \quad (21)$$

Combining (20)–(21) yields $\delta_1(B_1+B_1')-\delta_1(B_1)-\delta_1(B_1')=0$.

Step 8.4. δ_1 is additive on $\mathscr{T}$.

For any $X_1=A_1+M_1+B_1\in\mathscr{T}$ and $X_2=A_1'+M_1'+B_1'\in\mathscr{T}$, by Claim 7 and Steps 8.1–8.3, we achieve

$$\begin{aligned}\delta_1(X_1+X_2) &= \delta_1(A_1+M_1+B_1+A_1'+M_1'+B_1')\\ &= \delta_1(A_1+A_1')+\delta_1(M_1+M_1')+\delta_1(B_1+B_1')\\ &= \delta_1(A_1)+\delta_1(M_1)+\delta_1(B_1)+\delta_1(A_1')\\ &\quad +\delta_1(M_1')+\delta_1(B_1')\\ &= \delta_1(X_1)+\delta_1(X_2).\end{aligned}$$

Claim 9. *δ_1 is a derivation on $\mathscr{T}$.*

We will prove the claim by several steps.

Step 9.1. For any $M_1, M_1' \in \mathscr{M}_1$, we have $\delta_1(M_1M_1')=\delta_1(M_1)M_1'+M_1\delta_1(M_1')=0$.

This is obvious by the fact $\delta_1(\mathscr{M}_1)\subseteq\mathscr{M}_1$.

Step 9.2. For any $A_1, A_1' \in \mathscr{A}_1$ and any $M_1 \in \mathscr{M}_1$, we have

$$\delta_1(A_1M_1)=\delta_1(A_1)M_1+A_1\delta_1(M_1) \quad \text{and}$$
$$\delta_1(A_1A_1')=\delta_1(A_1)A_1'+A_1\delta_1(A_1').$$

Take any $A_1, A_1' \in \mathscr{A}_1$ and any $M_1 \in \mathscr{M}_1$. By Claim 2(i), one has

$$\begin{aligned}\delta_1(A_1A_1'M_1) &= \delta_1((A_1A_1')\circ M_1)\\ &= \delta_1(A_1A_1')\circ M_1+(A_1A_1')\circ\delta_1(M_1)\\ &= \delta_1(A_1A_1')M_1+A_1A_1'\delta_1(M_1); \quad (22)\end{aligned}$$

particularly, we have $\delta_1(A_1M_1)=\delta_1(A_1)M_1+A_1\delta_1(M_1)$. On the other hand, one gets

$$\begin{aligned}\delta_1(A_1A_1'M_1) &= \delta_1(A_1\circ A_1'M_1)\\ &= \delta_1(A_1)\circ(A_1'M_1)+A_1\circ\delta_1(A_1'M_1)\\ &= \delta_1(A_1)A_1'M_1+A_1(\delta_1(A_1')M_1+A_1'\delta_1(M_1))\\ &= \delta_1(A_1)A_1'M_1+A_1\delta_1(A_1')M_1+A_1A_1'\delta_1(M_1). \quad (23)\end{aligned}$$

combining (22)–(23) yields $(\delta_1(A_1A_1')-\delta_1(A_1)A_1'-A_1\delta_1(A_1'))M_1=0$ for all $M_1\in\mathscr{M}_1$. Since $\mathscr{M}_{1j}$ is a faithful left $\mathscr{R}_1$-module, we obtain $\delta_1(A_1A_1')=\delta_1(A_1)A_1'+A_1\delta_1(A_1')$.

Step 9.3. For any $B_1, B_1' \in \mathscr{B}_1$ and any $M_1 \in \mathscr{M}_1$, we have

$$\delta_1(M_1B_1)=\delta_1(M_1)B_1+M_1\delta_1(B_1) \quad \text{and}$$
$$\delta_1(B_1B_1')=\delta_1(B_1)B_1'+B_1\delta_1(B_1').$$

Taking any $B_1, B_1' \in \mathscr{B}_1$ and any $M_1 \in \mathscr{M}_1$, by the same arguments as those of Step 9.2, we can prove that

$$\delta_1(M_1B_1)=\delta_1(M_1)B_1+M_1\delta_1(B_1) \quad \text{and}$$
$$M_1(\delta_1(B_1B_1')-\delta_1(B_1)B_1'-B_1\delta_1(B_1'))=0.$$

Since $\mathscr{M}_{ij}$ is a faithful right $\mathscr{R}_j$-module, one gets for $t=2,\dots,n$,

$$E_t(\delta_1(B_1B_1')-\delta_1(B_1)B_1'-B_1\delta_1(B_1'))E_t=0. \quad (24)$$

Now, writing $B_1=(b_{kl})_{n\times n}$ and $B_1'=(b_{st}')_{n\times n}$; then

$$B_1B_1'=\sum_{\substack{2\leqslant k\leqslant l\leqslant n,\\ 2\leqslant s\leqslant t\leqslant n}}B_{kl}B_{st}'=\sum_{2\leqslant k\leqslant l\leqslant t\leqslant n}B_{kl}B_{lt}',$$

where B_{kl} is the element with (k,l) position $b_{k,l}$ and other positions 0. Thus, to show that $\delta_1(B_1B_1') = \delta_1(B_1)B_1' + B_1\delta_1(B_1')$, by the additivity of δ_1, one only needs to check that δ_1 satisfies the following equations:

$$\begin{cases}\delta_1(B_{kl}B_{lt}') = \delta_1(B_{kl})B_{lt}' + B_{kl}\delta_1(B_{lt}') \text{ and}\\ \delta_1(B_{kl})B_{st}' + B_{kl}\delta_1(B_{st}') = 0,\end{cases} \quad (25)$$

for all $2 \leqslant k \leqslant l \leqslant t \leqslant n$ and $2 \leqslant k \leqslant l \leqslant n$, $2 \leqslant s \leqslant t \leqslant n$ with $l \neq s$, respectively.

Observe that, by taking $B_1 = B_{kk}$ and $B_1' = E_k$ with $k \neq t$ in (24), one achieves

$$\begin{aligned}0 &= E_t(\delta_1(B_{kk}E_k) - \delta_1(B_{kk})E_k - B_{kk}\delta_1(E_k))E_t\\ &= E_t\delta_1(B_{kk})E_t,\end{aligned}$$

that is,

$$E_t\delta_1(B_{kk})E_t = 0, \quad t,k \in \{2,\dots,n\},\ t \neq k. \quad (26)$$

Step 9.3.1. For any B_{kk}, B_{ss}' with $k < s$, we have

$$0 = \delta_1(B_{kk})B_{ss}' + B_{kk}\delta_1(B_{ss}') \quad \text{and}$$
$$\delta_1(B_{ss}')B_{kk} = B_{ss}'\delta_1(B_{kk}) = 0.$$

For B_{kk} and B_{ss}' with $k < s$, we have

$$\begin{aligned}0 = \delta_1(B_{kk} \circ B_{ss}') &= \delta_1(B_{kk}) \circ B_{ss}' + B_{kk} \circ \delta_1(B_{ss}')\\ &= \delta_1(B_{kk})B_{ss}' + B_{ss}'\delta_1(B_{kk})\\ &\quad + B_{kk}\delta_1(B_{ss}') + \delta_1(B_{ss}')B_{kk}. \quad (27)\end{aligned}$$

Note that (26) implies

$$\begin{aligned}\delta_1(B_{kk})B_{ss}' &\in \sum_{j=2}^{s-1}\mathscr{T}_{js}, B_{ss}'\delta_1(B_{kk})\\ &\in \sum_{j=s+1}^{n}\mathscr{T}_{sj}, B_{kk}\delta_1(B_{ss}')\\ &\in \sum_{j=k+1}^{n}\mathscr{T}_{kj}, \delta_1(B_{ss}')B_{kk} \in \sum_{j=2}^{k-1}\mathscr{T}_{jk}.\end{aligned}$$

These and (27) mean that $\delta_1(B_{kk})B_{ss}' + B_{kk}\delta_1(B_{ss}') = 0 = B_{ss}'\delta_1(B_{kk}) = \delta_1(B_{ss}')B_{kk}$.

Note that, by Step 9.3.1, one can check that, for $k = 2,\dots,n$,

$$\delta_1(B_{kk}) \in \mathscr{T}_{2k} + \cdots + \mathscr{T}_{kk} + \mathscr{T}_{k(k+1)} + \cdots + \mathscr{T}_{kn}. \quad (28)$$

Step 9.3.2. For any B_{kl}, B_{st}' with $k \leqslant l$ and $s \leqslant t$, we have

(i) $B_{st}'\delta_1(B_{kl}) = \delta_1(B_{st}')B_{kl} = 0$ if $k < s$ or $k = s$, $s < t$;

(ii) $\delta_1(B_{kl})B_{st}' = B_{kl}\delta_1(B_{st}') = 0$ if $k > s$ or $k = s$, $k < l$.

Note that, by Claim 6 (that is, $\delta_1(M_i) = \tau_i(M_i)$ for all M_i), we know that $\delta_1(B_{kl}) \subseteq \mathscr{M}_k \cap \mathscr{B}_1$ holds for all B_{kl} with $k < l$. In addition, by (28), it is true that $B_{st}'\delta_1(B_{kk}) = 0$ for $k < s$. Now, the step is easily checked.

Step 9.3.3. For any B_{kk} and B_{kk}', we have $\delta_1(B_{kk}B_{kk}') = \delta_1(B_{kk})B_{kk}' + B_{kk}\delta_1(B_{kk}')$.

For any B_{kk}, B_{kk}', by Claim 8, one gets

$$\begin{aligned}2\delta_1(B_{kk}) &= \delta_1(B_{kk} \circ E_k)\\ &= \delta_1(B_{kk}) \circ E_k + B_{kk} \circ \delta_1(E_k)\\ &= \delta_1(B_{kk})E_k + E_k\delta_1(B_{kk}) + B_{kk}\delta_1(E_k)\\ &\quad + \delta_1(E_k)B_{kk}. \quad (29)\end{aligned}$$

Multiplying respectively by E_t, $t = 2,\dots,k-1$, and E_k from the left and the right in (29), one obtains

$$E_t\delta_1(B_{kk})E_k = E_t\delta_1(E_k)B_{kk},\ t = 2,\dots,k-1; \quad (30)$$

multiplying respectively by E_k and E_s, $s = k+1,\dots,n$, from the left and the right in (29), we get

$$E_k\delta_1(B_{kk})E_s = B_{kk}\delta_1(E_k)E_s,\ s = k+1,\dots,n. \quad (31)$$

Thus, (30)–(31) imply

$$\begin{aligned}&E_t(\delta_1(B_{kk}B_{kk}') - \delta_1(B_{kk})B_{kk}' - B_{kk}\delta_1(B_{kk}'))E_k\\ &\quad = E_t\delta_1(B_{kk}B_{kk}')E_k - E_t\delta_1(B_{kk})B_{kk}'\\ &\quad = E_t\delta_1(E_k)B_{kk}B_{kk}' - E_t\delta_1(E_k)B_{kk}B_{kk}' = 0 \quad (32)\end{aligned}$$

and

$$E_k(\delta_1(B_{kk}B_{kk}') - \delta_1(B_{kk})B_{kk}' - B_{kk}\delta_1(B_{kk}'))E_s = 0 \quad (33)$$

for $t = 2,\dots,k-1$ and $s = k+1,\dots,n$. In addition, by (28), it is clear that

$$\begin{aligned}&\delta_1(B_{kk}B_{kk}') - \delta_1(B_{kk})B_{kk}' - B_{kk}\delta_1(B_{kk}')\\ &\qquad \in \mathscr{T}_{2k} + \cdots + \mathscr{T}_{kk} + \mathscr{T}_{k(k+1)} + \cdots + \mathscr{T}_{kn}. \quad (34)\end{aligned}$$

Now, combining (32)–(34) and (24) gives $\delta_1(B_{kk}B_{kk}') = \delta_1(B_{kk})B_{kk}' + B_{kk}\delta_1(B_{kk}')$.

Step 9.3.4. For any B_{kk}, B_{kt}' with $k < t$, we have $\delta_1(B_{kk}B_{kt}') = \delta_1(B_{kk})B_{kt}' + B_{kk}\delta_1(B_{kt}')$.

By (28) and the fact $\delta_1(B_{kt}) \subseteq \mathscr{M}_k \cap \mathscr{B}_1$, we have

$$\begin{aligned}\delta_1(B_{kk}B_{kt}') &= \delta_1(B_{kk} \circ B_{kt}') = \delta_1(B_{kk}) \circ B_{kt}' + B_{kk} \circ \delta_1(B_{kt}')\\ &= \delta_1(B_{kk})B_{kt}' + B_{kt}'\delta_1(B_{kk}) + B_{kk}\delta_1(B_{kt}') + \delta_1(B_{kt}')B_{kk}\\ &= \delta_1(B_{kk})B_{kt}' + B_{kk}\delta_1(B_{kt}').\end{aligned}$$

A similar argument to that of Step 9.3.4 can give the following two steps.

Step 9.3.5. For any B_{kl}, B'_{ll} with $k < l$, we have $\delta_1(B_{kl}B'_{ll}) = \delta_1(B_{kl})B'_{ll} + B_{kl}\delta_1(B'_{ll})$.

Step 9.3.6. For any B_{kl}, B'_{lt} with $k < l < t$, we have $\delta_1(B_{kl}B'_{lt}) = \delta_1(B_{kl})B'_{lt} + B_{kl}\delta_1(B'_{lt})$.

Step 9.3.7. For any B_{kl}, B'_{st} with $k \leqslant l$, $s \leqslant t$ and $l \neq s$, we have $\delta_1(B_{kl})B'_{st} + B_{kl}\delta_1(B'_{st}) = 0$.

In fact, if $k \leqslant l < s \leqslant t$, by Step 9.3.2(i), one has

$$\begin{aligned} 0 &= \delta_1(B_{kl} \circ B'_{st}) = \delta_1(B_{kl}) \circ B'_{st} + B_{kl} \circ \delta_1(B'_{st}) \\ &= \delta_1(B_{kl})B'_{st} + B'_{st}\delta_1(B_{kl}) + B_{kl}\delta_1(B'_{st}) + \delta_1(B'_{st})B_{kl} \\ &= \delta_1(B_{kl})B'_{st} + B_{kl}\delta_1(B'_{st}). \end{aligned} \tag{35}$$

Similarly, if $k \leqslant s < l \leqslant t$, $k \leqslant s < t \leqslant l$ or $k < s \leqslant t < l$, one can check that $\delta_1(B_{kl})B'_{st} + B_{kl}\delta_1(B'_{st}) = 0$.

Next, if $k = s = t < l$, we have

$$\begin{aligned} &\delta_1(B'_{kk}B_{kl}) = \delta_1(B_{kl} \circ B'_{kk}) \\ &= \delta_1(B_{kl})B'_{kk} + B'_{kk}\delta_1(B_{kl}) + B_{kl}\delta_1(B'_{kk}) + \delta_1(B'_{kk})B_{kl}, \end{aligned}$$

and Step 9.3.4 gives $\delta_1(B_{kl})B'_{kk} + B_{kl}\delta_1(B'_{kk}) = 0$. Similarly, if $s \leqslant k$ and $l \neq s$, by considering subcases $s \leqslant k \leqslant t \leqslant l$, $s \leqslant k \leqslant l \leqslant t$, and $s \leqslant t \leqslant k \leqslant l$, respectively, one can show that $\delta_1(B_{kl})B'_{st} + B_{kl}\delta_1(B'_{st}) = 0$. The substep is true.

Now, combining Steps 9.3.1–9.3.7, and by a bald calculation, one can show that (25) holds, completing the proof of Step 9.3. It follows from Claim 8 and Steps 9.1–9.3 that δ_1 is a derivation.

Claim 10. *δ is an additive derivation on $\mathscr{T}$.*

By (1), $\delta = \delta_1 - d_1$ with d_1 an additive derivation. Now, it follows from Claims 8–9 that δ is an additive derivation on $\mathscr{T}$.

The proof of Theorem 1 is finished. □

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