Remarks on certain sums involving floor function

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\textit{Received 11 Feb 2021}
\textit{Accepted 9 May 2021}

ABSTRACT: For each $a = 1, 2, 3, \ldots, 7$, there exists an integer $b$ depending on $a$ such that

$$\sum_{k=1}^{n} \left\lfloor \frac{ka}{b} \right\rfloor = \left\lfloor \frac{(2n+b)^2}{8a} \right\rfloor$$

for all $n \in \mathbb{N}$.

In this article, we give some remarks on this identity. In particular, we show that the range of $a$ cannot be extended and the value of $b$ is unique.

KEYWORDS: floor function, summation identity, residue class, fractional part

MSC2010: 11A25 11A07 05A19

INTRODUCTION

Recall that the floor function of a real number $x$, denoted by $\lfloor x \rfloor$, is defined to be the largest integer less than or equal to $x$; and the fractional part of $x$, denoted by $\{x\}$, is defined to be $x - \lfloor x \rfloor$. Sums involving the floor function or the fractional part of real numbers have been a popular area of research. For example, in a proof of the quadratic reciprocity law, Gauss shows that for relatively prime positive integers $a$, $b$,

$$\sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

Dirichlet’s divisor problem is to determine the smallest $\theta > 1/4$ such that

$$\sum_{n \leq x} x \frac{\lfloor x \rfloor}{n} = x \log x + (2\gamma - 1)x + O(x^{\theta+\varepsilon})$$

for any $\varepsilon > 0$. Hermite’s identity states that for $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} x + \frac{k}{n} = \lfloor nx \rfloor.$$


In particular, it is an exercise in Apostol’s book [11] to show that for each $a = 1, 2, 3, \ldots, 7$, there exists an integer $b$ depending on $a$ such that

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor = \left\lfloor \frac{(2n+b)^2}{8a} \right\rfloor$$

for all $n \in \mathbb{N}$. (1)

In this article, we give some remarks on this identity. In particular, we show that a simple formula for $a > 7$ does not exist and the value of $b$ for each $a \leq 7$ is unique.

PRELIMINARIES AND LEMMAS

In this section, we give some results which are useful in proving the main theorems. We also give a proof of (1) for completeness. Recall that for $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have $\lfloor n + x \rfloor = n + \lfloor x \rfloor$ and $0 \leq \{x\} < 1$. These are well-known and are often used without
reference. Next, we prove a lemma that are applied throughout this article.

**Lemma 1** Let \( n \geq 0 \), \( a \geq 1 \), \( 0 \leq r < a \) be integers, and let \( n \equiv r \pmod{a} \). Suppose \( b = 2 - a \). Then

\[
\sum_{k=1}^{n} \lfloor \frac{k}{a} \rfloor = \frac{n(n+b)-r(b+r)}{2a}.
\]

(2)

**Proof:** If \( a = 1 \), then \( r = 0 \), \( b = 1 \), and

\[
\sum_{k=1}^{n} \lfloor \frac{k}{a} \rfloor = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n(n+b)-r(b+r)}{2a}.
\]

If \( n < a \), then \( r = n \) and both sides of (2) are zero. So we assume that \( n \geq a \). Since \( n \equiv r \pmod{a} \) and \( n \geq a > r \), there exists \( q \in \mathbb{Z}^+ \) such that \( n = aq + r \). Therefore the left-hand side of (2) can be written as

\[
\sum_{k=0}^{aq-1} \lfloor \frac{k}{a} \rfloor + \sum_{k=aq}^{aq+r} \lfloor \frac{k}{a} \rfloor.
\]

If \( aq \leq k \leq aq+r \), then \( \lfloor k/a \rfloor = q \), and so the second sum above is equal to \( q(r+1) \). The first sum can be written as

\[
\sum_{0 \leq k < aq} \left( \sum_{a \leq k < a(q+1)} \lfloor \frac{k}{a} \rfloor \right) \sum_{0 \leq k < aq} aq = \frac{aq(q-1)}{2}.
\]

Combining the first and second sums and substituting \( q = \frac{n-r}{a} \), we see that the left-hand side of (2) is equal to

\[
\frac{1}{2} \left( aq^2 - aq + 2ar + 2q ight) = \frac{1}{2a} \left( a^2 \left( \frac{n-r}{a} \right)^2 - a^2 \left( \frac{n-r}{a} \right) + 2(n-r)r + 2(n-r) \right)
\]

\[
= \frac{1}{2a} \left( n^2 - 2nr + r^2 - an + ar + 2nr - 2r^2 + 2n - 2r \right)
\]

\[
= \frac{1}{2a} (n(n+2-a) - r(r+2-a)),
\]

which is equal to the right-hand side of (2). \( \square \)

By applying Lemma 1, we can prove (1) conveniently as shown in the next theorem.

**Theorem 1** If \( a \leq 7 \) is a positive integer, then we can choose \( b = 2 - a \) so that

\[
\sum_{k=1}^{n} \lfloor \frac{k}{a} \rfloor = \left\lceil \frac{(2n+b)^2}{8a} \right\rceil \quad \text{for all } n \in \mathbb{N}.
\]

(3)

**Proof:** We first consider the case \( a = 1 \). Then the left-hand side of (3) is \( n(n+1)/2 \) while the right-hand side of (3) is equal to

\[
\left\lceil \frac{(2n+1)^2}{8} \right\rceil = \lfloor \frac{n(n+1)}{2} + \frac{1}{8} \rfloor = \frac{n(n+1)}{2},
\]

where the last equality is obtained from the fact that \( n(n+1)/2 \) is an integer. The proofs for \( a = 2 \) to \( a = 7 \) are similar, so we show the details only in the cases \( a = 6 \) and \( a = 7 \). So suppose that \( a = 6 \). By Lemma 1, we obtain

\[
\sum_{k=1}^{n} \lfloor \frac{k}{6} \rfloor = \begin{cases}
\frac{n(n-4)}{12}, & \text{if } n \equiv 0, 4 \pmod{6}; \\
\frac{n(n-4)+3}{12}, & \text{if } n \equiv 1, 3 \pmod{6}; \\
\frac{n(n-4)+4}{12}, & \text{if } n \equiv 2 \pmod{6}; \\
\frac{n(n-4)-5}{12}, & \text{if } n \equiv 5 \pmod{6}.
\end{cases}
\]

(4)

The right-hand side of (3) is equal to

\[
\left\lceil \frac{(2n-4)^2}{48} \right\rceil = \left\lceil \frac{4n^2-16n+16}{48} \right\rceil = \frac{n(n-4)+16}{12}.
\]

If \( n \equiv 0, 4, 6, 10 \pmod{12} \), then \( n(n-4) \equiv 0 \pmod{12} \), and so

\[
\frac{n(n-4)+16}{12} = \frac{n(n-4)}{12}.
\]

If \( n \equiv 1, 3, 7, 9 \pmod{12} \), then \( n(n-4) \equiv -3 \pmod{12} \), and therefore

\[
\frac{n(n-4)+16}{12} = \frac{n(n-4)+3}{12}.
\]

If \( n \equiv 2, 8 \pmod{12} \), then \( n(n-4) \equiv -4 \pmod{12} \), and thus

\[
\frac{n(n-4)+16}{12} = \frac{n(n-4)+4}{12}.
\]

If \( n \equiv 5, 11 \pmod{12} \), then \( n(n-4) \equiv 5 \pmod{12} \), and hence

\[
\frac{n(n-4)+16}{12} = \frac{n(n-4)-5}{12}.
\]

From these, we obtain

\[
\left\lceil \frac{(2n+b)^2}{8a} \right\rceil = \begin{cases}
\frac{n(n-4)+16}{12}, & \text{if } n \equiv 0, 4, 6, 10 \pmod{12}; \\
\frac{n(n-4)+3}{12}, & \text{if } n \equiv 1, 3, 7, 9 \pmod{12}; \\
\frac{n(n-4)+4}{12}, & \text{if } n \equiv 2, 8 \pmod{12}; \\
\frac{n(n-4)-5}{12}, & \text{if } n \equiv 5, 11 \pmod{12}.
\end{cases}
\]

(5)

Observe that \( n \equiv 0, 4 \pmod{6} \) if and only if \( n \equiv 0, 4, 6, 10 \pmod{12} \); \( n \equiv 1, 3 \pmod{6} \) if and only if \( n \equiv 1, 3, 7, 9 \pmod{12} \); \( n \equiv 2 \pmod{6} \) if and only if \( n \equiv 2, 8 \pmod{12} \); \( n \equiv 5 \pmod{6} \) if and only if \( n \equiv 5, 11 \pmod{12} \). Comparing (4) and (5), we see that

\[
\sum_{k=1}^{n} \lfloor \frac{k}{6} \rfloor = \left\lceil \frac{(2n+b)^2}{8a} \right\rceil.
\]
So this theorem is proved for \( a = 6 \). Next, consider \( a = 7 \). By Lemma 1, we have

\[
\sum_{k=1}^{n} \left[ \frac{k}{7} \right] = \begin{cases} \frac{n(n-5)}{14}, & \text{if } n \equiv 0, 5, 7, 12 \pmod{14}; \\ \frac{n(n-5)+4}{14}, & \text{if } n \equiv 1, 4, 8, 11 \pmod{14}; \\ \frac{n(n-5)+6}{14}, & \text{if } n \equiv 2, 3, 9, 10 \pmod{14}; \\ \frac{n(n-5)-6}{14}, & \text{if } n \equiv 6, 13 \pmod{14}. \end{cases} \tag{6}
\]

The right-hand side of (3) is equal to

\[
\sum_{k=1}^{n} \left[ \frac{k}{7} \right] = \begin{cases} \frac{n(n-5)}{14}, & \text{if } n \equiv 0, 5, 7, 12 \pmod{14}; \\ \frac{n(n-5)+4}{14}, & \text{if } n \equiv 1, 4, 8, 11 \pmod{14}; \\ \frac{n(n-5)+6}{14}, & \text{if } n \equiv 2, 3, 9, 10 \pmod{14}; \\ \frac{n(n-5)-6}{14}, & \text{if } n \equiv 6, 13 \pmod{14}. \end{cases} \tag{7}
\]

Comparing (6) and (7), we see that this theorem is verified for \( a = 7 \). Hence the proof is complete. \( \square \)

**MAIN RESULTS**

In this section, we show that \( b \) in Theorem 1, after a reduction, is necessarily equal to \( 2-a \) and the range of \( a \leq 7 \) cannot be extended to any positive integer larger than 7.

**Theorem 2** Let \( a \in \mathbb{N} \), \( b, c, d \in \mathbb{R} \), and \( d \neq 0 \). Suppose that \( A \subseteq \mathbb{N} \) is an infinite set and

\[
\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor = \left\lfloor \frac{(cn+b)^2}{da} \right\rfloor \text{ for all } n \in A. \tag{8}
\]

Then \( \frac{(cn+b)^2}{da} = \frac{(2n+2-a)^2}{8a} \) for all \( n \in A \).

**Proof**: Let \( n \in A \). By Lemma 1, the left-hand side of (8) is equal to

\[
\frac{n(n+2-a)-r(2-a+r)}{2a} = \frac{n^2}{2a} + \frac{n(2-a)}{2a} - \frac{r(2-a+r)}{2a},
\]

where \( 0 \leq r < a \) and \( n \equiv r \pmod{a} \). Recall that \( [x] = x - \{x\} \) and \( 0 \leq \{x\} < 1 \). So the right-hand side of (8) can be written as

\[
\frac{n^2c^2}{da} + \frac{2nbc}{da} + \frac{b^2}{da} - f_1(n, a, b, c, d),
\]

where \( 0 \leq f_1(n, a, b, c, d) < 1 \). Dividing both sides of (8) by \( n^2 \), we obtain

\[
\frac{1}{2a} + \frac{2-a}{2an} - \frac{r(2-a+r)}{2an^2} = \frac{c^2}{da} + \frac{2bc}{dan} + \frac{b^2}{dan^2} - \frac{f_1(n, a, b, c, d)}{n^2}. \tag{9}
\]

Since (9) holds for all \( n \in A \), we can take limit as \( n \to \infty \) on both sides of (9) which leads to

\[
2 - a = \frac{r(2-a+r)}{2an}. \tag{10}
\]

Multiplying both sides of (10) by \( n \) and taking limit as \( n \in A \) and \( n \to \infty \), we obtain \( \frac{2-a}{2} = \frac{2bc}{dan} \) and (9) reduces to

\[
\frac{(2-a)c}{2} = \frac{b^2}{d}. \tag{11}
\]

From these, we obtain

\[
\frac{(cn+b)^2}{da} = \frac{c^2n^2}{da} + \frac{2bcn}{da} + \frac{b^2}{da} = \frac{n^2}{2a} + \frac{(2-a)n}{2a} + \frac{(2-a)^2c^2}{4da} = \frac{4n^2}{8a} + \frac{4(2-a)n}{8a} + \frac{(2-a)^2}{8a} = \frac{(2n+2-a)^2}{8a}. \tag{12}
\]

This completes the proof. \( \square \)

**Theorem 2** immediately implies that it is necessary to choose \( b = 2-a \) in Theorem 1.

**Corollary 1** The value \( b = 2-a \) in Theorem 1 is unique. That is, if \( b \in \mathbb{R}, a \in \mathbb{N}, a \leq 7 \), and

\[
\sum_{k=1}^{n} \frac{k}{a} = \left\lfloor \frac{(2n+b)^2}{8a} \right\rfloor \text{ for infinitely many } n \in \mathbb{N},
\]

then \( b = 2-a \).

**Proof**: By Theorem 2, we have

\[
\frac{(2n+b)^2}{8a} = \frac{(2n+2-a)^2}{8a}. \tag{13}
\]

Since (11) holds for infinitely many \( n \in \mathbb{N} \), we can choose distinct positive integers \( n_0 \) and \( n_1 \) and substitute \( n = n_0 \) and \( n = n_1 \) in (13) to obtain

\[
4n_0b + b^2 = 4n_0(2-a) + (2-a)^2, \tag{12}
\]

\[
4n_1b + b^2 = 4n_1(2-a) + (2-a)^2. \tag{13}
\]
Subtracting (13)–(12), we obtain \( b = 2 - a \), as desired. \( \square \)

Next, we show that the range of \( a \leq 7 \) in Theorem 1 cannot be extended.

**Theorem 3** For each positive integer \( a \geq 8 \) and for any choice of \( b, c, d \in \mathbb{R} \) with \( d \neq 0 \), there are infinitely many \( n \in \mathbb{N} \) such that

\[
\sum_{k=1}^{n} \left[ \frac{k}{a} \right] \neq \left\lfloor \frac{(cn + b)^2}{da} \right\rfloor.
\]

**Proof:** Suppose for a contradiction that there exist \( a \in \mathbb{N} \) and \( b, c, d \in \mathbb{R} \) such that \( a \geq 8 \), \( d \neq 0 \), and

\[
\sum_{k=1}^{n} \left[ \frac{k}{a} \right] = \left\lfloor \frac{(cn + b)^2}{da} \right\rfloor
\]

for only a finite number of \( n \in \mathbb{N} \). Then there exists \( M \in \mathbb{N} \) such that

\[
\sum_{k=1}^{n} \left[ \frac{k}{a} \right] = \left\lfloor \frac{(cn + b)^2}{da} \right\rfloor \text{ for all } n \geq M.
\]

By Theorem 2, we have

\[
\sum_{k=1}^{n} \left[ \frac{k}{a} \right] = \left\lfloor \frac{(2n + a - 2)^2}{8a} \right\rfloor \text{ for all } n \geq M. \quad (14)
\]

Let \( n \geq M \) and \( n \equiv a - 1 \pmod{2a} \). Then \( n(n + 2 - a) \equiv a - 1 \pmod{2a} \) and \( n \equiv a - 1 \pmod{a} \). By Lemma 1, we obtain

\[
\sum_{k=1}^{n} \left[ \frac{k}{a} \right] = \frac{n(n + 2 - a) - (a - 1)}{2a} \in \mathbb{Z}. \quad (15)
\]

Next, we calculate the right-hand side of (14). We have

\[
\left\lfloor \frac{(2n + 2 - a)^2}{8a} \right\rfloor = \left\lfloor \frac{n(n + 2 - a) - (a - 1)}{2a} + \frac{a - 1}{2a} + \frac{(2 - a)^2}{8a} \right\rfloor
\]

\[
= \frac{n(n + 2 - a) - (a - 1)}{2a} + \frac{a - 1}{2a} + \frac{(2 - a)^2}{8a}.
\]

But \( \frac{a - 1}{2a} + \frac{(2 - a)^2}{8a} = \frac{a}{8} \geq 1 \), and so

\[
\left\lfloor \frac{a - 1}{2a} + \frac{(2 - a)^2}{8a} \right\rfloor \geq 1. \quad (17)
\]

By (15), (16), and (17), we obtain

\[
\sum_{k=1}^{n} \left[ \frac{k}{a} \right] < \left\lfloor \frac{(2n + 2 - a)^2}{8a} \right\rfloor,
\]

which contradicts (14). Hence the proof is complete. \( \square \)

**Remark 1** Obviously, the sum \( \sum_{k=1}^{n} \left[ \frac{k}{a} \right] \) depends on \( a \) and \( n \). If \( a = 1, 2, 3, \ldots, 7 \), then Theorem 1 simply says that a simple formula for this sum exists; but if \( a \) is a positive integer larger than 7, then Theorem 3 states that such a simple formula does not exist. Nevertheless, we can always use Lemma 1 to evaluate this sum though it may lead to many cases of residues modulo \( a \) as shown in the following example.

**Example 1** If \( a = 8 \), we can apply Lemma 1 to obtain

\[
\sum_{k=1}^{n} \left[ \frac{k}{8} \right] = \begin{cases} 
\frac{n(n-8)}{16}, & \text{if } n \equiv 0, 6 \pmod{8}; \\
\frac{n(n-6)+5}{16}, & \text{if } n \equiv 1, 5 \pmod{8}; \\
\frac{n(n-6)+8}{16}, & \text{if } n \equiv 2, 4 \pmod{8}; \\
\frac{n(n-6)+9}{16}, & \text{if } n \equiv 3 \pmod{8}; \\
\frac{n(n-6)+7}{16}, & \text{if } n \equiv 7 \pmod{8}.
\end{cases}
\]

**Questions:** We have obtained the results for all positive integers \( a \). Can we extend them to negative integers? What about nonzero rational numbers? Can we say something nontrivial about the sum \( \sum_{k=1}^{n} \left[ \frac{k}{a} \right] \) when \( a \) is positive irrational? What happen if we replace the floor by the ceiling function? We leave these questions to the interested readers.

**Acknowledgements:** We thank the reviewers for their comments and suggestions which improve the quality of this paper. Prapanpong Pongsriiam is jointly supported by the Faculty of Science Silpakorn University and the National Research Council of Thailand, grant number NRCT5-RSA63021-02. Kittipong Subwattanachai has received a DPST scholarship of IPST, Thailand.

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