Unicity of meromorphic functions concerning small functions and differences

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ABSTRACT: In this paper, we study the unicity of meromorphic functions concerning their small functions and differences. Our results improve and extend the existed results of Chen-Chen [Bull Malays Math Sci Soc 35 (2012):765–774] and Qi-Li-Yang [Comput Methods Funct Theory 18 (2018):567–582].

KEYWORDS: unicity, meromorphic functions, shifts, derivatives

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INTRODUCTION AND MAIN RESULTS

In this paper, we assume that the readers are familiar with the basic notions of Nevanlinna value distribution theory, see ([1–3]). In the following, a meromorphic function means meromorphic in the whole complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of an exceptional set $E$ with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$. A meromorphic function $a$ is said to be a small function of $f$ if it satisfies $T(r, a) = o(T(r, f))$. We say that two non-constant meromorphic functions $f$ and $g$ share small function $a$ IM(CM) if $f-a$ and $g-a$ have the same zeros ignoring multiplicities (counting multiplicities). Let $f$ be a non-constant meromorphic function. We denote by $N(r, 1/f)$ the counting function of simple zeros of $f$.

Let $f$ be a non-constant meromorphic function. The order of $f$ is defined by

$$\lambda = \lim_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}.$$ 

Let $a$ be a small function of $f$ and $g$ and let $S(f = a = g)$ be the set of all common zeros of $f-a$ and $g-a$ counting multiplicities. We say that two non-constant meromorphic functions $f$ and $g$ share small function $a$ CM almost if

$$N(r, \frac{1}{f-a}) + N(r, \frac{1}{g-a}) - 2N(r, f = a = g) = S(r, f) + S(r, g).$$

Let $c$ be a nonzero finite complex constant, and let $f$ be a meromorphic function, we define its shift by $f(z+c)$ and its difference operator by

$$\Delta_c f(z) = f(z+c) - f(z).$$

Nevanlinna [2] proved the following famous five-value theorem.

Theorem A Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a_j$ $(j = 1, 2, 3, 4, 5)$ be five distinct values in the extended complex plane. If $f(z)$ and $g(z)$ share $a_j$ $(j = 1, 2, 3, 4, 5)$ IM, then $f(z) \equiv g(z)$.

In 2000, Li and Qiao [4] proved that Theorem A is still valid for five small functions, they proved

Theorem B Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a_j(z)$ $(j = 1, 2, 3, 4, 5)$ (one of them can be $\infty$) be five distinct small functions of $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a_j(z)$ $(j = 1, 2, 3, 4, 5)$ IM, then $f(z) \equiv g(z)$.

Recently, value distribution in difference analogue of meromorphic functions has become a subject of some interests, see [5–18].

In 2012, Chen and Chen [5] proved

Theorem C Let $f(z)$ be a non-constant meromorphic function of finite order, let $a, c$ be two nonzero finite values, and let $n \geq 7$ be a positive integer. If $[f(z)]^n$ and $[\Delta f(z)]^n$ share a CM, $f(z)$ and $\Delta f(z)$ share $\infty$ CM, then $f(z) \equiv t \Delta f(z)$, where $t^n = 1, t \neq 1$.

In 2018, Qi, Li and Yang [15] proved

Theorem D Let $f(z)$ be a non-constant meromorphic function of finite order, let $a, c$ be two nonzero finite values, and let $n \geq 9$ be a positive integer. If $[f'(z)]^n$ and $[f(z+c)]^n$ share a CM, $f'(z)$ and $f(z+c)$ share $\infty$ CM, then $f'(z) \equiv tf(z+c)$, where $t^n = 1$.

In 2020, Wang and Fang [16] removed the condition that the function $f(z)$ is of finite order in Theorems D and E, and proved
Theorem F  Let \( f(z) \) be a non-constant meromorphic function, let \( a, c \) be two nonzero finite values, and let \( n \geq 5, k \) be two positive integers. If \( [f^{(i)}(z)]^n \) and \( [f(z + c)]^n \) share a CM, \( f^{(k)}(z) \) and \( f(z + c) \) share \( \infty \) CM, then \( f^{(k)}(z) \equiv tf(z + c) \), where \( t^n = 1 \).

By above theorems, we naturally pose following problem:

Problem 1 Are Theorem C, Theorem D, and Theorem F still valid if the constant \( a \) is replaced by a small function \( a(z) \) of \( f(z) \)?

In this paper, we study the problem and obtain the following results.

Theorem 1 Let \( f(z) \) be a non-constant meromorphic function, let \( c \) be a nonzero value and \( n \geq 6 \) a positive integer, and let \( a(z) \neq 0 \) be a small function of \( f(z) \). If \( [f^{(i)}(z)]^n \) and \( [\Delta, f(z)]^n \) share \( a(z) \) CM, \( f(z) \) and \( \Delta, f(z) \) share \( \infty \) CM, then \( f^{(k)}(z) \equiv tf(z + c) \), where \( t^n = 1, t \neq 1 \).

Hence, Theorem C is still valid if the constant \( a \) is replaced by a small function \( a(z) \) of \( f(z) \).

Theorem 2 Let \( f(z) \) be a non-constant meromorphic function, let \( c \) be a nonzero value and \( n \geq 5 \) a positive integer, and let \( a(z) \neq 0 \) be a small function of \( f(z) \). If \( [f^{(i)}(z)]^n \) and \( [f(z + c)]^n \) share \( a(z) \) CM, \( f(z) \) and \( f(z + c) \) share \( \infty \) CM, then either \( f^{(k)}(z) \equiv tf(z + c) \), where \( t^n = 1 \) or \( [f^{(i)}(z)]^n[f(z + c)]^n \equiv a^2(z) \).

Remark 1 The following example shows that Theorem F is not valid if the constant \( a \) is replaced by a small function \( a(z) \) of \( f(z) \). In other words, \( [f^{(i)}(z)]^n[f(z + c)]^n \equiv a^2(z) \) can not be removed in Theorem 2.

Example 1 Let \( f(z) = e^z, c = k \pi i \) and let \( a(z) = e^{i\pi/2} \). Then by calculation we have \( [f'(z)f(z + c)]^n = e^{i\pi/2} = a^2(z) \).

Theorem 3 Let \( f(z) \) be a transcendental meromorphic function of finite order, let \( c \) be a nonzero value and \( n \geq 5 \) a positive integer, and let \( a(z) \neq 0 \) be a small function of \( f(z) \). If \( [f^{(i)}(z)]^n \) and \( [f(z + c)]^n \) share \( a(z) \) CM, \( f^{(k)}(z) \) and \( f(z + c) \) share \( \infty \) CM, then \( f^{(k)}(z) \equiv tf(z + c) \), where \( t^n = 1 \).

Theorem 4 Let \( f(z) \) be a transcendental meromorphic function, let \( c \) be a nonzero value and \( n \geq 5 \) a positive integer, and let \( a(z) \neq 0 \) be a rational function. If \( [f^{(i)}(z)]^n \) and \( [f(z + c)]^n \) share \( a(z) \) CM, \( f^{(k)}(z) \) and \( f(z + c) \) share \( \infty \) CM, then \( f^{(k)}(z) \equiv tf(z + c) \), where \( t^n = 1 \).

LEMNAS

Lemma 1 ([2, 3]) Let \( f(z) \) be a non-constant meromorphic function, and let \( k \) be a positive integer. Then 
\[
m(r, f^{(k)}) = S(r, f).
\]

Lemma 2 ([14]) Let \( f \) be a non-constant meromorphic function, and let \( n \geq 2 \) be a positive integer. If \( f \) and \( f^{(n)} \) have finite many zeros, then \( f \) is of finite order.

Lemma 3 ([2]) Let 
\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right),
\]

where \( F \) and \( G \) are two non-constant meromorphic functions. If \( F \) and \( G \) share 1 CM and \( H \neq 0 \), then 
\[
N_1(r, \frac{1}{H}) \leq N(r, F) + S(r, F) + S(r, G).
\]

Remark 2 We know from the proof in [2] that Lemma 3 is valid when \( F \) and \( G \) share 1 CM almost.

Lemma 4 ([8, 10]) Let \( f \) be a non-constant meromorphic function of finite order, let \( c \) be a nonzero complex number. Then 
\[
m(r, \frac{f(z + c)}{f(z)}) = S(r, f),
\]

for all \( r \) outside of a possible exceptional set \( E \) with finite logarithmic measure.

Lemma 5 ([7, 12]) Let \( f \) be a non-constant meromorphic function of finite order, let \( c \) be a nonzero complex number. Then 
\[
T(r, f(z + c)) = T(r, f) + S(r, f),
\]

\[
N(r, f(z + c)) = N(r, f) + S(r, f),
\]

\[
N(r, \frac{1}{f(z + c)}) = N(r, \frac{1}{f}) + S(r, f).
\]

Lemma 6 ([6, 8]) Let \( f \) be a non-constant meromorphic function of finite order, let \( c \) be a nonzero complex number. If \( f(z + c) \equiv f(z) \), then \( f \) is of order at least 1.

THE PROOF OF Theorem 1

Let 
\[
F = \frac{f^n}{a}, \quad G = \frac{[\Delta, f]^n}{a},
\]

Since \( f^n \) and \([\Delta, f]^n \) share a CM, we know that \( F \) and \( G \) share 1 CM almost. Set 
\[
\phi = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.
\]

We discuss from following two cases.

Case 1: \( \phi \equiv 0 \). By (2) we have 
\[
\frac{F-1}{F} \equiv A \frac{G-1}{G},
\]

where \( A \) is a nonzero value.

If \( A = 1 \), then from (3) we get \( f^n \equiv [\Delta, f]^n \), that is \( f \equiv t \Delta, f \), where \( t \) is a complex number such that \( t^n = 1 \).
If $A \neq 1$, then from (3) we have
\[ \frac{1}{F} - \frac{A}{G} = 1 - A. \] (4)

By (4) we can obtain
\[ T(r, f) = T(r, \Delta f) + S(r, f), \]
\[ S(r, f) = S(r, \Delta f). \] (5)

According to (1), (4), (5) and Nevanlinna’s Second Fundamental Theorem ([2], Page 19, Theorem 1.6) we get
\[ nT(r, f) = T(r, F) + S(r, f) \leq \overline{N}(r, F) \]
\[ + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F-1}) + S(r, f) \]
\[ \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \Delta f) + S(r, f) \]
\[ \leq 3T(r, f) + S(r, f), \] (6)

it follows from (6) and $n \geq 6$ that $T(r, f) = S(r, f)$, a contradiction.

**Case 2:** $\phi \neq 0$. Let $z_0$ be a common pole of $f$ and $\Delta f$ with multiplicity $l$, then by (2) we know that $z_0$ is the zero of $\phi$, and the multiplicity is at least $nl - 1$.

Since $f$ and $\Delta f$ share $\infty$ CM, then
\[ \overline{N}(r, F) = \overline{N}(r, G) \leq \frac{1}{nl-1} \overline{N}(r, \frac{1}{F}) + S(r, f) \]
\[ \leq \frac{1}{nl-1} \overline{T}(r, \phi) + S(r, f) \]
\[ \leq \frac{1}{n-1} \left[ \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F}) \right] + S(r, f). \] (7)

Let $H$ be defined as in Lemma 3. Suppose that $H \neq 0$, by Lemma 3 and Remark 2 we have
\[ N(r, H) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F}) \]
\[ + N_0(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F}) + S(r, f). \] (8)

where $N_0(r, 1/F')$ denotes the counting function corresponding to the zeros of $F'$ which are not the zeros of $F$ and $F - 1$; $N_0(r, 1/G')$ denotes the counting function corresponding to the zeros of $G'$ which are not the zeros of $G$ and $G - 1$. By Nevanlinna’s Second Fundamental Theorem, we get
\[ T(r, F) + T(r, G) \leq \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) \]
\[ + \overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{G-1}) \]
\[ - N_0(r, \frac{1}{F}) - N_0(r, \frac{1}{F}) + S(r, f). \] (9)

Since $F$ and $G$ share 1 almost CM, we have
\[ \overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) \]
\[ \leq N_1(r, \frac{1}{F-1}) + \frac{1}{2} \overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}). \] (10)

By (8)–(10) we have
\[ T(r, F) + T(r, G) \leq \overline{N}(r, F) + 2\overline{N}(r, \frac{1}{F}) + \overline{N}(r, G) + 2\overline{N}(r, \frac{1}{G}) \]
\[ + \frac{1}{2} \overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) \] (11)

By Nevanlinna’s First Fundamental Theorem ([2], Page 12, Theorem 1.2), we have
\[ N(r, \frac{1}{F-1}) + N(r, \frac{1}{G-1}) \leq T(r, F) + T(r, G) + S(r, f). \] (12)

By (7), (11) and (12), we can obtain
\[ T(r, F) + T(r, G) \leq 4\overline{N}(r, \frac{1}{F}) + 4\overline{N}(r, \frac{1}{G}) + 2\overline{N}(r, f) \]
\[ + 2\overline{N}(r, \Delta f) + S(r, f) \]
\[ \leq (4 + \frac{4}{n-1})(T(r, F) + T(r, G)) + S(r, f). \] (13)

Obviously, by (1) we have
\[ \overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G}) + S(r, f), \]
\[ \overline{N}(r, F) = \overline{N}(r, G) + S(r, f), \]
\[ \overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G}) + S(r, f), \]
\[ \overline{N}(r, G) = \overline{N}(r, \Delta f) + S(r, f), \]
\[ T(r, F) = nT(r, f) + S(r, f), \]
\[ T(r, G) = nT(r, \Delta f) + S(r, f). \]

Hence, by above formulas, (13) and Nevanlinna’s First Fundamental Theorem, we get
\[ n(T(r, f) + T(r, \Delta f)) \]
\[ \leq (4 + \frac{4}{n-1})(T(r, F) + T(r, G)) + S(r, f) \]
\[ \leq (4 + \frac{4}{n-1})(T(r, F) + T(r, \Delta f)) + S(r, f), \]

and it follows that
\[ n(1 - \frac{1}{n-1})(T(r, f) + T(r, \Delta f)) \leq S(r, f). \] (14)

Thus by (14) and $n \geq 6$, we get $T(r, f) = S(r, f)$, a contradiction.

Hence, $H \equiv 0$. Thus we have
\[ \frac{F'}{F} - \frac{1}{F-1} = \frac{G'}{G} - \frac{2}{G-1}. \]

Solving above equation, we get
\[ \frac{1}{F-1} = \frac{A}{G-1} + B, \quad \frac{A}{G-1} = \frac{1 + B - BF}{F-1} \] (15)

where $A \neq 0$ and $B$ are constants.

Now we consider three subcases.

**Case 2.1:** $B \neq 0, -1$. It follows from (15) that
\[ T(r, \Delta f) = T(r, f) + S(r, f), \]
\[ \overline{N}(r, \frac{1}{F-1}) = \overline{N}(r, G). \] (16)
So by (15), (16), Nevanlinna’s Second Fundamental Theorem, and the fact that \( f \) and \( \Delta, f \) share \( \infty \) CM, we get

\[
nT(r, f) \leq T(r, f) + S(r, f) \\
\leq \bar{N}(r, F) + \bar{N}(r, \frac{1}{F}) + \frac{1}{r-\frac{1}{F}} + S(r, f) \\
\leq \bar{N}(r, \frac{1}{\Delta}) + \bar{N}(r, F) + \bar{N}(r, G) + S(r, f) \\
\leq \bar{N}(r, \frac{1}{\Delta}) + \bar{N}(r, f) + \bar{N}(r, \Delta, f) + S(r, f) \\
\leq 3T(f, r) + S(r, f). 
\tag{17}
\]

Therefore, by (17) and \( n \geq 6 \), we can get \( T(r, f) = S(r, f) \), a contradiction.

**Case 2.2:** \( B = 0 \). By (15) we obtain

\[
F = \frac{G + (A-1)}{A}, \quad G = AF - (A-1). \tag{18}
\]

If \( A \neq 1 \), by (18) we get

\[
\bar{N}(r, \frac{1}{F - \frac{1}{A}}) = \bar{N}(r, \frac{1}{\Delta}) = \bar{N}(r, \frac{1}{\Delta}) + S(r, f). \tag{19}
\]

By (16), (19), Nevanlinna’s Second Fundamental Theorem, and the fact that \( f \) and \( \Delta, f \) share \( \infty \) CM, we get

\[
nT(r, f) \leq T(r, f) + S(r, f) \\
\leq \bar{N}(r, F) + \bar{N}(r, \frac{1}{F}) + \frac{1}{r-\frac{1}{F}} + S(r, f) \\
\leq \bar{N}(r, \frac{1}{\Delta}) + \bar{N}(r, F) + \bar{N}(r, \frac{1}{\Delta}) + S(r, f) \\
\leq \bar{N}(r, \frac{1}{\Delta}) + \bar{N}(r, f) + \bar{N}(r, \frac{1}{\Delta}) + S(r, f) \\
\leq 3T(f, r) + S(r, f). \tag{20}
\]

Therefore, by (20) and \( n \geq 6 \), we can get \( T(r, f) = S(r, f) \), a contradiction.

Hence \( A = 1 \). It follows from (18) that \( F = G \). Thus by (1) we deduce that \( f \equiv T \Delta, f \), where \( t^n = 1, t \neq -1 \).

**Case 2.3:** \( B = -1 \), by (15) we have

\[
F = \frac{A}{-G + A + 1}, \quad G = \frac{(A + 1)F - A}{F}. \tag{21}
\]

If \( A \neq -1 \), we get from (19) that \( \bar{N}(r, \frac{1}{F - \frac{1}{A}}) = \bar{N}(r, \frac{1}{\Delta}) \). Using the same argument as in the Case 2.1, we get a contradiction. Thus, \( A = -1 \).

By (21), we get \( FG \equiv 1 \). It follows from \( FG \equiv 1 \) and (1) that

\[
f^n[\Delta, f] = a^2. \tag{22}
\]

Set \( f \Delta, f = b \), then we get \( b^n = a^2 \). It follows that \( T(r, b) = 2^n T(r, a) \). Thus \( b \neq 0 \) is a small function of \( f \). Since \( f \) and \( \Delta, f \) share \( \infty \) CM, we deduce from \( f \Delta, f = b \) that

\[
N(r, \frac{1}{\Delta}) \leq N(r, \frac{1}{\Delta}) \leq T(r, b) + O(1) = S(r, f). \tag{23}
\]

Thus by Nevanlinna’s Second Fundamental Theorem, (23), (24), and Lemma 5, we get

\[
2T(r, f) = T(r, f^2) \leq T(r, \frac{c}{\Delta}) + T(r, b) + O(1) \\
\leq N(r, \frac{c}{\Delta}) + N(r, \frac{b}{\Delta}) + N(r, \frac{1}{\Delta}) + S(r, f) \\
\leq N(r, \frac{b}{\Delta}) + S(r, f) \leq S(r, f), \tag{25}
\]

that is \( T(r, f) = S(r, f) \), a contradiction.

Hence, we prove that \( f \equiv t \Delta, f \), where \( t^n = 1 \).

**THE PROOF OF Theorem 2**

Let

\[
F = \frac{(f^{(k)})^n}{a}, \quad G = \frac{(f_{a})^n}{a}. \tag{26}
\]

Since \( f_{a} \) and \( f^{(k)} \) share \( \infty \) CM, \( f^{(k)} \) has no pole with multiplicity 1. Then we use the same argument as in the proof of Theorem 1 and note that (26) is replaced by the following formula:

\[
\bar{N}(r, \frac{1}{\Delta}) = \bar{N}(r, \frac{1}{\Delta}) \\
\leq \frac{1}{(n-1)} N(r, \frac{1}{\Delta}) + S(r, f) \\
\leq \frac{1}{(n-1)} T(r, \phi) + S(r, f) \\
\leq \frac{1}{(n-1)} [\bar{N}(r, \frac{1}{\Delta}) + \bar{N}(r, \frac{1}{\Delta})] + S(r, f), \tag{27}
\]

and we prove either \( f^{(k)} \equiv tf_{a} \), with \( t^n = 1 \), or \( (f^{(k)})_{a} \equiv a^2 \).

**THE PROOF OF Theorem 3**

By Theorem 2, we obtain either \( f^{(k)} \equiv tf_{a} \), with \( t^n = 1 \), or \( (f^{(k)})_{a} \equiv a^2 \). We claim that \( f^{(k)} \equiv tf_{a} \), with \( t^n = 1 \). Otherwise, we suppose

\[
[f^{(k)}]^n f_{a} \equiv a^2. \tag{28}
\]

Since \( f \) is a meromorphic function of finite order, it follows from (28) that \( N(r, 1/f_{a}) = S(r, f) \). Thus by Lemma 1, Lemma 4, Lemma 5, and Nevanlinna’s First Fundamental Theorem, we have

\[
2nT(r, f) = T(r, tf a f^{2n}) + S(r, f) \\
= m(r, \frac{1}{\Delta}) + N(r, \frac{1}{\Delta}) + S(r, f) = m(r, \frac{1}{\Delta}) + S(r, f) \\
\leq m(r, \frac{(f^{(k)})_{a}}{f^{2n}}) + 2T(r, a) + O(1) \\
\leq m(r, \frac{(f^{(k)})_{a}}{f^{2n}}) + m(r, \frac{1}{\Delta}) + 2T(r, a) + S(r, f) \\
= S(r, f). \tag{29}
\]

which is \( T(r, f) = S(r, f) \), a contradiction.
THE PROOF OF Theorem 4

By Theorem 2, we know that either \( f^{(k)}(z) \equiv t f(z) \), where \( t^n = 1 \) or \( \left[ f^{(k)}(z) \right]^n f^n(z) = a^2(z) \). We claim that

\[
f^{(k)}(z) \equiv t f(z),
\]

(30)

where \( t^n = 1 \). Otherwise, we have

\[
[f^{(k)}(z)]^n f^n(z) \equiv a^2(z).
\]

(31)

Since \( a(z) \) is a rational function, it follows from (31) that both \( f(z) \) and \( f^{(k)}(z) \) have finite many zeros and poles. If \( k \geq 2 \), by Lemma 2 we know that \( f(z) \) is a transcendental meromorphic function of finite order. Thus by Theorem 3 we get a contradiction.

Next, we consider the case of \( k = 1 \). Since \( f(z) \) has finite many zeros and poles, we assume that

\[
f(z) = b(z) e^{\alpha(z)},
\]

(32)

where \( b(z) \) is a rational function and \( \alpha(z) \) is an entire function. By (32) we get

\[
f(z) = b(z) e^{\alpha(z)},
\]

(33)

\[
f'(z) = (b'(z) + \alpha'(z)b(z)) e^{\alpha(z)},
\]

(34)

It follows from (31)–(34) that

\[
\left[ (b'(z) + \alpha'(z)b(z)) b(z) \right] e^{\alpha(z)} \equiv a^2(z).
\]

(35)

Thus, we have

\[
(b'(z) + \alpha'(z)b(z)) b(z) = d(z) e^{\beta(z)},
\]

(36)

where \( d(z) \) is a rational function, and \( \beta(z) \) is an entire function.

By (35) and (36), we get

\[
d^n(z) e^{\beta(z)} e^{\alpha(z) + \alpha(z)} \equiv a^2(z).
\]

(37)

It follows from (37) that

\[
\beta(z) + \alpha(z) + \alpha(z) \equiv A,
\]

(38)

where \( A \) is a finite complex number. Differential both sides of (38) we obtain

\[
\beta'(z) + \alpha'(z) + \alpha'(z) \equiv 0.
\]

(39)

By (36) we have

\[
\alpha'(z) \equiv \frac{c}{b(z)b(z)} \frac{b'(z)}{b(z)}.
\]

(40)

Therefore,

\[
\frac{d(z)}{b(z)b(z)} e^{\beta(z)} + \frac{d(z)}{b(z)b(z)} e^{\beta(z)} = \frac{b'(z)}{b(z)} + \frac{b'(z)}{b(z)} - \beta'(z).
\]

(41)

Next, we consider two cases.

Case 1:

\[
\frac{b'(z)}{b(z)} + \frac{b'(z)}{b(z)} - \beta'(z) \equiv 0.
\]

(42)

Then, we claim that \( \beta'(z) \equiv 0 \). Otherwise, by (42) we get

\[
b(z)b(z) \equiv B e^{\beta(z)},
\]

(43)

where \( B \) is a nonzero constant. Since \( b(z) \) is a rational function, so it is impossible. Thus, \( \beta'(z) \equiv 0 \), that is \( \beta(z) \) is a constant. And then we deduce that

\[
\frac{d(z)}{b(z)b(z)} + \frac{d(z)}{b(z)b(z)} \equiv 0.
\]

(44)

Let \( A(z) = \frac{d(z)}{b(z)b(z)} \). Then \( A(z) + A(z) \equiv 0 \), \( A(z) + A(z) \equiv 0 \), \( A(z) \equiv A(z) \). Then by Lemma 6, \( A(z) \) is a meromorphic function of order \( \geq 1 \), but \( A(z) \) is a rational function, it is impossible.

Case 2:

\[
\frac{b'(z)}{b(z)} + \frac{b'(z)}{b(z)} - \beta'(z) \not\equiv 0.
\]

(45)

We claim that \( \beta'(z) \not\equiv 0 \). Otherwise, \( \beta(z) = D \) is a constant. It follows (41) that

\[
\left[ \frac{d(z)}{b(z)b(z)} + \frac{d(z)}{b(z)b(z)} \right] e^{\beta(z)} \equiv \frac{b'(z)}{b(z)} + \frac{b'(z)}{b(z)}.
\]

(46)

We can rewrite above as

\[
\frac{d(z)}{b(z)b(z)} e^{\beta(z)} - \frac{b'(z)}{b(z)} \equiv \frac{d(z)}{b(z)b(z)} e^{\beta(z)} - \frac{b'(z)}{b(z)}.
\]

(47)

Let \( H(z) \equiv \frac{d(z)}{b(z)b(z)} e^{\beta(z)} - \frac{b'(z)}{b(z)} \). Then (46) implies that \( H(z) + H(z) \equiv 0 \), \( H(z) + H(z) \equiv 0 \), and \( H(z) \equiv H(z) \). It follows from Lemma 6 that \( H(z) \) is a meromorphic function of order \( \geq 1 \), a contradiction. Hence \( \beta'(z) \not\equiv 0 \), and \( e^{\beta(z)} \) is a transcendental entire function. By (41) we have

\[
A_1(z) e^{\beta(z)} + A_2(z) e^{\beta(z)} \equiv 1,
\]

(48)

where

\[
A_1(z) = \frac{d(z)}{b(z)b(z)(\frac{b'(z)}{b(z)} + \frac{b'(z)}{b(z)})},
\]

(49)

and

\[
A_2(z) = \frac{d(z)}{b(z)b(z)(\frac{b'(z)}{b(z)} - \beta'(z))}.
\]

(50)

Since \( T(r, e^{\beta}) = m(r, e^{\beta})/\rho(e^{\beta}) = S(r, e^{\beta}) \), it follows from (48) and (49) that \( A_1(z) \) and \( A_2(z) \) are small functions of \( e^{\beta(z)} \). It follows from above and the Nevanlinna's Second Fundamental Theorem, we can get \( T(r, e^{\beta}) = S(r, e^{\beta}) \), a contradiction.
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