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Superstability of a multidimensional pexiderized cosine functional equation

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ABSTRACT: Given an integer $n \geqslant 2$, we will establish the general solution and investigate the superstability of the multidimensional pexiderized cosine functional equation $2^n \prod_{i=1}^{n} f_i(x_i) = \sum_{\substack{\sigma_i=\pm 1 \\ i=1,2,\dots,n}} f_1\left(\sum_{i=1}^{n} \sigma_i x_i\right)$ for complex-valued functions defined on an abelian group.

KEYWORDS: stability, superstability, functional equation, cosine functional equation

MSC2010: 39B82

INTRODUCTION

In 1940, Ulam [1] posed the stability problem for group homomorphisms. Hyers [2] gave the first affirmative answer to Ulam's question for the case of approximate additive mapping on Banach spaces. The stability problem has since become a very active domain of research. Such problem for various types of functional equation has been extentively investigated by a number of mathematicians.

The notion of superstability is about strong stability phenomenon where each approximate homomorphism is actually a true homomorphism, which was probably first observed by Baker et al [3].

In particular, they showed that if a functional f on a real vector space satisfying

$$|f(x+y) - f(x)f(y)| < \delta$$

for some fixed δ and for all x and y in the domain, then f is either bounded or exponential. Baker [4] also proved the superstability of the cosine functional equation, $f(x+y)+f(x-y) = 2f(x)f(y)$, also known as the d' Alembert functional equation, which states that

If $\delta > 0$, G is an abelian group, and f is a complex-valued function defined on G such that

$$|f(x+y)+f(x-y)-2f(x)f(y)| \leqslant \delta$$

for all $x, y \in G$, then either $|f(x)| \leqslant (1+\sqrt{1+4\delta})/2$ or $f(x+y)+f(x-y) = 2f(x)f(y)$ for all $x, y \in G$.

A similar result concerning the superstability of the sine functional equation, $f(x+y)f(x-y) = f(x)^2 - f(y)^2$, was obtained by Cholewa [5].

In 2004, Kim [6] proved a result regarding the superstability of the generalized pexiderized sine functional equation

$$g(x)h(y) = f\left(\tfrac{x+y}{2}\right)^2 - f\left(\tfrac{x-y}{2}\right)^2.$$

In another aspect, the general solution of cosine-type functional equation was investigated by Kannappan [7,8]. In particular, he established the general continuous solution of the functional equation $f(x+y)+f(x-y) = 2f(x)f(y)$ on $\mathbb{R}^n$ and the functional equation $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ on a group G. Kim and Lee [9] studied the generalized cosine functional equation which includes an endomorphism σ of G with $\sigma(\sigma(x)) = x$ for all $x \in G$.

In this paper, we establish the general solution and prove the superstability of the following n-dimensional cosine pexiderized functional equation of the form

$$2^n \prod_{i=1}^{n} f_i(x_i) = \sum_{\substack{\sigma_i=\pm 1 \\ i=1,2,\dots,n}} f_1\left(\sum_{i=1}^{n} \sigma_i x_i\right)$$

for functions $f_1, f_2, \dots, f_n$ defined on an abelian group $(G,+)$. Note that for $n = 2$ and $n = 3$ the equations will take the forms

$$4f_1(x_1)f_2(x_2) = f_1(x_1+x_2) + f_1(x_1-x_2) + f_1(-x_1+x_2) + f_1(-x_1-x_2)$$

and

$$\begin{aligned}8f_1(x_1)f_2(x_2)f_3(x_3) = {} & f_1(x_1+x_2+x_3) + f_1(x_1-x_2+x_3)\\ & + f_1(x_1+x_2-x_3) + f_1(x_1-x_2-x_3)\\ & + f_1(-x_1+x_2+x_3) + f_1(-x_1-x_2+x_3)\\ & + f_1(-x_1+x_2-x_3) + f_1(-x_1-x_2-x_3),\end{aligned}$$

respectively.

GENERAL SOLUTION

For the sake of convenience, given a function f, we define the symmetric sum S_f by

$$S_f(x_1,x_2,\dots,x_n) := 2^{-n} \sum_{\substack{\sigma_i=\pm1\\ i=1,2,\dots,n}} f\left(\sum_{i=1}^{n} \sigma_i x_i\right), \quad (1)$$

where $\sum\limits_{\substack{\sigma_i=\pm1\\ i=1,2,\dots,n}} = \sum\limits_{\sigma_1=\pm1}\sum\limits_{\sigma_2=\pm1}\cdots\sum\limits_{\sigma_n=\pm1}$. Note that S_f is invariant under any permutation and a sign switching of any of its arguments.

Lemma 1 *Given a function f and an integer $n \geqslant 3$, we have*

$$2S_f(x_1,x_2,\dots,x_n) = S_f(x_1,\dots,x_{n-2},x_{n-1}+x_n) + S_f(x_1,\dots,x_{n-2},x_{n-1}-x_n).$$

Proof: Observe that

$$\begin{aligned}&\sum_{\substack{\sigma_i=\pm1\\ i=1,2,\dots,n}} f\left(\sum_{i=1}^{n}\sigma_i x_i\right)\\ &= \sum_{\substack{\sigma_i=\pm1\\ i=1,2,\dots,n-2}}\sum_{\sigma_{n-1}=\pm1}\sum_{\sigma_n=\pm1} f\left(\sum_{i=1}^{n-2}\sigma_i x_i + \sigma_{n-1}x_{n-1} + \sigma_n x_n\right).\end{aligned}$$

Upon evaluating σ_{n-1} and σ_n, the result can be written collectively as

$$\begin{aligned}&\sum_{\substack{\sigma_i=\pm1\\ i=1,2,\dots,n}} f\left(\sum_{i=1}^{n}\sigma_i x_i\right)\\ &= \sum_{\substack{\sigma_i=\pm1\\ i=1,2,\dots,n-1}} f\left(\sum_{i=1}^{n-2}\sigma_i x_i + \sigma_{n-1}(x_{n-1}+x_n)\right)\\ &\quad + \sum_{\substack{\sigma_i=\pm1\\ i=1,2,\dots,n-1}} f\left(\sum_{i=1}^{n-2}\sigma_i x_i + \sigma_{n-1}(x_{n-1}-x_n)\right).\end{aligned}$$

By multiplying $2^{-(n-1)}$ to the above equation, the desired result simply follows. □

The following two theorems establish the general solution of the proposed functional equation.

Theorem 1 *Let $n \geqslant 2$ be an integer and let $(G,+)$ be an abelian group. A function $f: G \to \mathbb{C}$ satisfies the functional equation*

$$\prod_{i=1}^{n} f(x_i) = S_f(x_1,x_2,\dots,x_n) \quad (2)$$

for all $x_1,x_2,\dots,x_n \in G$ if any only if $f(0)^n = f(0)$ and there is a function $g: G \to \mathbb{C}$ satisfying

$$2g(x)g(y) = g(x+y) + g(x-y) \quad (3)$$

for all $x, y \in G$ such that $f(x) = f(0)g(x)$ for all $x \in G$.

Proof: To show the necessity, we assume that a function $f: G \to \mathbb{C}$ satisfies (2). By setting $x_1 = x_2 = \cdots = x_n = 0$ in (2), we get

$$f(0)^n = f(0).$$

If $f(0) = 0$, then we set $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = 0$ in (2). Therefore, we will get

$$0 = \frac{f(x)}{2} + \frac{f(-x)}{2}$$

for all $x \in G$. Thus, f is an odd function. Consequently, the symmetric sum, $S_f(x_1,x_2,\dots,x_n)$, vanishes for all $x_1,x_2,\dots,x_n \in G$. If we set $x_1 = x_2 = \cdots = x_n = x$ in (2), then $f(x)^n = 0$ for all $x \in G$. Hence, f is identically zero. Thus, we can choose the trivial solution, $g(x) \equiv 0$, of (3) to satisfy $f(x) = f(0)g(x)$ for all $x \in G$.

If $f(0) \neq 0$, then $f(0)^{n-1} = 1$. Since $S_f(x_1,x_2,\dots,x_n)$ is invariant under a sign switching of any of its arguments, we can see that

$$\begin{aligned}f(x)f(0)\cdots f(0) &= S_f(x,0,\dots,0)\\ &= S_f(-x,0,\dots,0)\\ &= f(-x)f(0)\cdots f(0)\end{aligned}$$

for all $x \in G$. Thus, $f(x) = f(-x)$ for all $x \in G$, and hence f is an even function. By putting $x_1 = x, x_2 = y$, and if $n > 2$, $x_3 = x_4 = \cdots = x_n = 0$ in (2), we are left with

$$f(x)f(y)f(0)^{n-2} = \frac{1}{4}\Big[f(x+y) + f(x-y) + f(-x+y) + f(-x-y)\Big]$$

for all $x, y \in G$. By the evenness of f and recalling that $f(0)^{n-1} = 1$, the above equation reduces to

$$2\left(\frac{f(x)}{f(0)}\frac{f(y)}{f(0)}\right) = \frac{f(x+y)}{f(0)} + \frac{f(x-y)}{f(0)}$$

for all $x, y \in G$. Therefore, if we define a function $g: G \to \mathbb{C}$ by $g(x) = f(x)/f(0)$ for all $x \in G$, then g satisfies the cosine functional equation given by (3) as desired.

To prove the sufficiency, we suppose that there is a function $g: G \to \mathbb{C}$ satisfying (3). By putting $x = y = 0$ in (3), we obtain

$$2g(0)^2 = 2g(0).$$

If $g(0) = 0$, by putting $y = 0$ in (3), then

$$0 = 2g(x)g(0) = g(x) + g(x)$$

for all $x \in G$, which implies that g is identically zero. Therefore, the function $f: G \to \mathbb{C}$ defined by $f(x) = f(0)g(x) = 0$ for all $x \in G$, satisfies (2).

If $g(0) \neq 0$, then $g(0) = 1$. By putting $x = 0$ in (3), we obtain

$$2g(y) = g(y) + g(-y)$$

for all $y \in G$. Thus, $g(y) = g(-y)$ for all $y \in G$, and hence g is an even function. Therefore,

$$\begin{aligned} S_g(x_1, x_2) &= 2^{-2}\big[g(x_1 + x_2) + g(x_1 - x_2) \\ &\quad + g(-x_1 + x_2) + g(-x_1 - x_2)\big] \\ &= 2^{-1}\big[g(x_1 + x_2) + g(x_1 - x_2)\big] \\ &= g(x_1)g(x_2) \end{aligned}$$

for all $x_1, x_2 \in G$. Now for an integer $n \geqslant 2$, we have

$$S_g(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} g(x_i)$$

for all $x_1, x_2, \dots, x_n \in G$, and hence, by Lemma 1,

$$\begin{aligned} 2S_g(x_1, \dots, x_n, x_{n+1}) &= S_g(x_1, \dots, x_{n-1}, x_n + x_{n+1}) \\ &\quad + S_g(x_1, \dots, x_{n-1}, x_n - x_{n+1}) \\ = \left(\prod_{i=1}^{n-1} g(x_i)\right) g(x_n + x_{n+1}) &+ \left(\prod_{i=1}^{n-1} g(x_i)\right) g(x_n - x_{n+1}). \end{aligned}$$

Since g satisfies (3), $g(x_n + x_{n+1}) + g(x_n - x_{n+1}) = 2g(x_n)g(x_{n+1})$. Thus, for all $x_1, x_2, \dots, x_{n+1} \in G$,

$$2S_g(x_1, \dots, x_n, x_{n+1}) = 2\prod_{i=1}^{n+1} g(x_i).$$

By mathematical induction, we conclude that

$$\prod_{i=1}^{n} g(x_i) = S_g(x_1, x_2, \dots, x_n) \tag{4}$$

for all $x_1, x_2, \dots, x_n \in G$ and for all integers $n \geqslant 2$.

Define a function $f: G \to \mathbb{C}$ by $f(0)^n = f(0)$ and $f(x) = f(0)g(x)$ for all $x \in G$. If (4) is multiplied by $f(0)^n = f(0)$, then f certainly satisfies (2) as desired. □

Now, we can generalize Theorem 1 to a pexiderized form of the functional equation.

Theorem 2 *Let $n \geqslant 2$ be an integer and let $(G, +)$ be an abelian group. Functions $f_1, f_2, \dots, f_n: G \to \mathbb{C}$, none of which is identically zero, satisfy the functional equation*

$$\prod_{i=1}^{n} f_i(x_i) = S_{f_1}(x_1, x_2, \dots, x_n) \tag{5}$$

for all $x_1, x_2, \dots, x_n \in G$ if any only if there exist complex numbers $\lambda_1, \lambda_2, \dots, \lambda_n \neq 0$ with $\lambda_2\lambda_3 \cdots \lambda_n = 1$ such that

$$f_i(x) = \lambda_i g(x)$$

for all $x \in G$ and for $i = 1, 2, \dots, n$, where $g: G \to \mathbb{C}$ is a nontrivial solution of the cosine functional equation

$$2g(x)g(y) = g(x+y) + g(x-y).$$

Proof: To prove the necessity, we suppose that functions $f_1, f_2, \dots, f_n: G \to \mathbb{C}$, none of which is identically zero, satisfy (5). Certainly, there exist $y_1, y_2, \dots, y_n \in G$ such that $f_i(y_i) \neq 0$ for $i = 1, 2, \dots, n$. We have, for each $i = 2, 3, \dots, n$ and for any $x \in G$,

$$\begin{aligned} f_1(y_1)f_2(y_2)\dots f_{i-1}(y_{i-1})f_i(x)f_{i+1}(y_{i+1})\dots f_n(y_n) \\ = S_{f_1}(y_1, y_2, \dots, y_{i-1}, x, y_{i+1}, \dots, y_n), \end{aligned}$$

and by switching y_1 and x, we get

$$\begin{aligned} f_1(x)f_2(y_2)\dots f_{i-1}(y_{i-1})f_i(y_1)f_{i+1}(y_{i+1})\dots f_n(y_n) \\ = S_{f_1}(x, y_2, \dots, y_{i-1}, y_1, y_{i+1}, \dots, y_n). \end{aligned}$$

Since S_{f_1} is invariant under any permutation of the arguments, and $f_i(y_i) \neq 0$ for all $i = 1, 2, \dots, n$, we have

$$f_1(y_1)f_i(x) = f_1(x)f_i(y_1)$$

for all $x \in G$. As $f_1(y_1) \neq 0$, we get

$$f_i(x) = \left(\frac{f_i(y_1)}{f_1(y_1)}\right) f_1(x)$$

for all $x \in G$. If we let $\alpha_i = f_i(y_1)/f_1(y_1)$ for each $i = 2, 3 \ldots, n$, then

$$f_i(x) = \alpha_i f_1(x)$$

for all $x \in G$. Since f_i is not identically zero, we have $\alpha_i \neq 0$ for all i. Now (5) becomes

$$(\alpha_2\alpha_3 \cdots \alpha_n) \prod_{i=1}^{n} f_1(x_i) = S_{f_1}(x_1, x_2, \ldots, x_n).$$

Let ω be a complex number with $\omega^{n-1} = \alpha_2\alpha_3 \cdots \alpha_n$. Then, for all $x_1, x_2, \ldots, x_n \in G$,

$$\prod_{i=1}^{n} \omega f_1(x_i) = S_{\omega f_1}(x_1, x_2, \ldots, x_n).$$

By Theorem 1, there is a solution $g : G \to \mathbb{C}$ of cosine functional equation

$$2g(x)g(y) = g(x + y) + g(x - y)$$

with $\omega f_1(x) = \omega f_1(0)g(x)$ for all $x \in G$ and $(\omega f_1(0))^n = \omega f_1(0)$. We note that $f_1(0) \neq 0$; otherwise by setting $x_1 = x_2 = \cdots = x_{n-1} = 0$ and $x_n = x$ in (5) yields

$$0 = \frac{f_1(x)}{2} + \frac{f_1(-x)}{2}$$

for all $x \in G$, which implies the oddness of f_1. Consequently, $S_{f_1}(x_1, x_2, \ldots, x_n)$ identically vanishes in (5), and

$$\prod_{i=1}^{n} f_i(x_i) = 0$$

for all $x_1, x_2, \ldots, x_n \in G$. If we set $x_i = y_i$ for all $i = 1, 2, \ldots, n$, then $\prod_{i=1}^{n} f_i(y_i) = 0$, which contradicts the fact that $f_i(y_i) \neq 0$ for all $i = 1, 2, \ldots, n$.

Since $f_1(0) \neq 0$, we now have $(\omega f_1(0))^{n-1} = 1$. If we let

$$\lambda_1 = f_1(0) \text{ and } \lambda_i = \alpha_i \lambda_1 \text{ for } i = 2, 3, \ldots, n,$$

then $f_i(x) = \lambda_i g(x)$ for all $i = 1, 2, \ldots, n$, and

$$\lambda_2\lambda_3 \cdots \lambda_n = (\alpha_2\alpha_3 \cdots \alpha_n)\lambda_1^{n-1} = \omega^{n-1} f_1(0)^{n-1} = 1.$$

To prove the sufficiency, we suppose that a nontrivial function $g : G \to \mathbb{C}$ satisfies the cosine functional equation. For any complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \neq 0$ with $\lambda_2\lambda_3 \cdots \lambda_n = 1$, we define $f_i(x) = \lambda_i g(x)$ for all $x \in G$, for all $i = 1, 2, \ldots, n$. Again, by Theorem 1,

$$\prod_{i=1}^{n} g(x_i) = S_g(x_1, x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in G$. Therefore,

$$\begin{aligned}\prod_{i=1}^{n} f_i(x_i) &= \prod_{i=1}^{n} \lambda_i g(x_i) \\ &= (\lambda_2\lambda_3 \cdots \lambda_n)\lambda_1 S_g(x_1, x_2, \ldots, x_n) \\ &= S_f(x_1, x_2, \ldots, x_n)\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in G$ as desired. □

STABILITY

In order to investigate the stability of the proposed functional equation, we need a further property of symmetric sum, S_f, of a function f in the following lemma.

Lemma 2 *Given a function f. If $x_1 = x_1'$, then*

$$\sum_{\substack{\sigma_i=\pm1 \\ i=1,2,\ldots,n}} S_f\left(\sum_{i=1}^{n} \sigma_i x_i, x_2', \ldots, x_n'\right) = \sum_{\substack{\sigma_i=\pm1 \\ i=1,2,\ldots,n}} S_f\left(\sum_{i=1}^{n} \sigma_i x_i', x_2, \ldots, x_n\right).$$

Proof: By the definition of S_f given in (1), we have

$$\begin{aligned}A := &\sum_{\substack{\sigma_i=\pm1 \\ i=1,2,\ldots,n}} S_f\left(\sum_{i=1}^{n} \sigma_i x_i, x_2', \ldots, x_n'\right) \\ &= 2^{-n} \sum_{\substack{\sigma_i=\pm1 \\ i=1,2,\ldots,n}} \sum_{\substack{\sigma_i'=\pm1 \\ i=1,2,\ldots,n}} f\left(\sigma_1' \sum_{i=1}^{n} \sigma_i x_i + \sum_{i=2}^{n} \sigma_i' x_i'\right).\end{aligned}$$

Evaluating the sum on σ_1', we have

$$\begin{aligned}A = 2^{-n} &\sum_{\substack{\sigma_i=\pm1 \\ i=1,2,\ldots,n}} \sum_{\substack{\sigma_i'=\pm1 \\ i=2,3,\ldots,n}} f\left(\sum_{i=1}^{n} \sigma_i x_i + \sum_{i=2}^{n} \sigma_i' x_i'\right) \\ &+ 2^{-n} \sum_{\substack{\sigma_i=\pm1 \\ i=1,2,\ldots,n}} \sum_{\substack{\sigma_i'=\pm1 \\ i=2,3,\ldots,n}} f\left(\sum_{i=1}^{n} (-\sigma_i) x_i + \sum_{i=2}^{n} \sigma_i' x_i'\right).\end{aligned}$$

Since

$$\begin{aligned}&\left\{(\sigma_1, \ldots, \sigma_n) \mid \sigma_i = \pm1, i = 1, \ldots, n\right\} \\ &\qquad = \left\{(-\sigma_1, \ldots, -\sigma_n) | \sigma_i = \pm1, i = 1, \ldots, n\right\},\end{aligned}$$

we have

$$A = 2^{-n+1} \sum_{\substack{\sigma_i=\pm1 \\ i=1,2,\ldots,n}} \sum_{\substack{\sigma_i'=\pm1 \\ i=2,3,\ldots,n}} f\left(\sum_{i=1}^{n} \sigma_i x_i + \sum_{i=2}^{n} \sigma_i' x_i'\right)$$

If we single out the sum on σ_1, then we can write

$$\sum_{\substack{\sigma_i=\pm 1\\ i=1,2,\dots,n}} S_f\Big(\sum_{i=1}^{n}\sigma_i x_i, x_2', \dots, x_n'\Big)$$
$$= 2^{-n+1}\sum_{\sigma_1=\pm 1}\sum_{\substack{\sigma_i=\pm 1\\ i=2,3,\dots,n}}\sum_{\substack{\sigma_i'=\pm 1\\ i=2,3,\dots,n}} f\Big(\sigma_1 x_1+\sum_{i=2}^{n}\sigma_i x_i+\sum_{i=2}^{n}\sigma_i' x_i'\Big).$$

Similarly, we can show that

$$\sum_{\substack{\sigma_i=\pm 1\\ i=1,2,\dots,n}} S_f\Big(\sum_{i=1}^{n}\sigma_i x_i', x_2, \dots, x_n\Big)$$
$$= 2^{-n+1}\sum_{\sigma_1=\pm 1}\sum_{\substack{\sigma_i=\pm 1\\ i=2,3,\dots,n}}\sum_{\substack{\sigma_i'=\pm 1\\ i=2,3,\dots,n}} f\Big(\sigma_1 x_1'+\sum_{i=2}^{n}\sigma_i x_i'+\sum_{i=2}^{n}\sigma_i' x_i\Big).$$

If $x_1 = x_1'$, then the desired result simply follows from the above two equations. □

The following theorem gives the superstability of the proposed functional equation.

Theorem 3 *Let $n \geqslant 2$ be an integer and let $(G,+)$ be an abelian group. If functions $f_1, f_2, \dots, f_n : G \to \mathbb{C}$, none of which is identically zero, satisfy the inequality*

$$\Big|\prod_{i=1}^{n} f_i(x_i) - S_{f_1}(x_1, x_2, \dots, x_n)\Big| \leqslant \varepsilon \qquad (6)$$

for all $x_1, x_2, \dots, x_n \in G$, for some $\varepsilon > 0$, then either they satisfy

$$\prod_{i=1}^{n} f_i(x_i) = S_{f_1}(x_1, x_2, \dots, x_n) \qquad (7)$$

for all $x_1, x_2, \dots, x_n \in G$ or $f_2, f_3, \dots, f_n$ are bounded.

Proof: If functions $f_1, f_2, \dots, f_n : G \to \mathbb{C}$, none of which is identically zero, satisfy inequality (6), then there exist $y_1, y_2, \dots, y_n$ such that $f_i(y_i) \neq 0$ for all $i = 1, 2, \dots, n$. Suppose that one of the functions, $f_2, f_3, \dots, f_n$, is unbounded. Without loss of generality, we may assume that f_n is unbounded. Hence, there exists a sequence $\{z_k\}$ in G such that

$$0 \neq |f_n(z_k)| \to \infty \text{ as } n \to \infty. \qquad (8)$$

By putting $(x_1, x_2, \dots, x_n) = (x, y_2, \dots, y_{n-1}, z_k)$ in inequality (6), and dividing the result by $|f_n(z_k)|$, we obtain

$$\Big|f_1(x) f_2(y_2) \cdots f_{n-1}(y_{n-1}) - \frac{S_{f_1}(x, y_2, \dots, y_{n-1}, z_k)}{f_n(z_k)}\Big| \leqslant \frac{\varepsilon}{|f_n(z_k)|}$$

for all $x \in G$. If we take the limit as $k \to \infty$, then

$$f_1(x) f_2(y_2) \cdots f_{n-1}(y_{n-1}) = \lim_{k\to\infty} \frac{S_{f_1}(x, y_2, \dots, y_{n-1}, z_k)}{f_n(z_k)} \qquad (9)$$

for all $x \in G$.

Let $(x_1', x_2', \dots, x_n') = (x, y_2, y_3, \dots, y_{n-1}, z_k)$. By putting $x_1 = \sum_{i=1}^{n} \sigma_i x_i'$ in (6), we get

$$\Big|f_1\Big(\sum_{i=1}^{n}\sigma_i x_i'\Big)\prod_{i=2}^{n} f_i(x_i) - S_{f_1}\Big(\sum_{i=1}^{n}\sigma_i x_i', x_2, \dots, x_n\Big)\Big| \leqslant \varepsilon.$$

Taking the sum over all $\sigma_1, \sigma_2, \dots, \sigma_n = \pm 1$, and multiplying by 2^{-n}, we obtain that

$$2^{-n}\Big|\sum_{\substack{\sigma_i=\pm 1\\ i=1,2,\dots,n}} f_1\Big(\sum_{i=1}^{n}\sigma_i x_i'\Big)\prod_{i=2}^{n} f_i(x_i) - \sum_{\substack{\sigma_i=\pm 1\\ i=1,2,\dots,n}} S_{f_1}\Big(\sum_{i=1}^{n}\sigma_i x_i', x_2, \dots, x_n\Big)\Big|$$
$$\leqslant 2^{-n}\sum_{\substack{\sigma_i=\pm 1\\ i=1,2,\dots,n}}\Big|f_1\Big(\sum_{i=1}^{n}\sigma_i x_i'\Big)\prod_{i=2}^{n} f_i(x_i) - S_{f_1}\Big(\sum_{i=1}^{n}\sigma_i x_i', x_2, \dots, x_n\Big)\Big| \leqslant \varepsilon.$$

By the definition of S_f in (1), and Lemma 2, we obtain

$$\Big|S_{f_1}(x_1', x_2', \dots, x_n')\prod_{i=2}^{n} f_i(x_i) - 2^{-n}\sum_{\substack{\sigma_i=\pm 1\\ i=1,2,\dots,n}} S_{f_1}\Big(\sum_{i=1}^{n}\sigma_i x_i, x_2', \dots, x_n'\Big)\Big| \leqslant \varepsilon,$$

where we have redefined $x_1 = x_1'$ in accordance to Lemma 2. Dividing the above equation by $|f_n(z_k)|$ and substituting $x_2', \dots, x_n'$ by their original values, we get

$$\Big|\frac{S_{f_1}(x_1, y_2, \dots, y_{n-1}, z_k)}{f_n(z_k)}\prod_{i=2}^{n} f_i(x_i) - 2^{-n}\sum_{\substack{\sigma_i=\pm 1\\ i=1,2,\dots,n}}\frac{S_{f_1}\big(\sum_{i=1}^{n}\sigma_i x_i, y_2, \dots, y_{n-1}, z_k\big)}{f_n(z_k)}\Big| \leqslant \frac{\varepsilon}{|f_n(z_k)|}$$

for all $x_1 \in G$. Taking the limit as $k \to \infty$, and applying (9), we have

$$f_1(x_1)f_2(y_2)\cdots f_{n-1}(y_{n-1})\prod_{i=2}^{n} f_i(x_i) = 2^{-n}\sum_{\substack{\sigma_i=\pm 1\\ i=1,2,\ldots,n}} f_1\left(\sum_{i=1}^{n}\sigma_i x_i\right) f_2(y_2)\cdots f_{n-1}(y_{n-1}).$$

By the definition of S_f and that $f_2(y_2), f_3(y_3), \ldots, f_{n-1}(y_{n-1}) \neq 0$, we finally conclude that

$$\prod_{i=1}^{n} f_i(x_i) = S_{f_1}(x_1, x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in G$. This completes the proof. □

Corollary 1 *Let $n \geqslant 2$ be an integer and let $(G,+)$ be an abelian group. If a nontrivial function $f: G \to \mathbb{C}$ satisfies the inequality*

$$\left|\prod_{i=1}^{n} f(x_i) - S_f(x_1, x_2, \ldots, x_n)\right| \leqslant \varepsilon \tag{10}$$

for all $x_1, x_2, \ldots, x_n \in G$ and for some $\varepsilon > 0$, then either f is bounded or f satisfies

$$\prod_{i=1}^{n} f(x_i) = S_f(x_1, x_2, \ldots, x_n) \tag{11}$$

for all $x_1, x_2, \ldots, x_n \in G$.

Proof: By letting $f_1 = f_2 = \cdots = f_n = f$ in Theorem 3, we immediately get the desired result. □

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