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On magnifying elements in semigroups of transformations restricted by an equivalence relation with a restricted range

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ABSTRACT: An element a of a semigroup S is called a right (left) magnifying element if S contains a proper subset M such that $Ma = S$ ($aM = S$). Let Y be a nonempty subset of a set X and ρ be an equivalence relation on X. Denote the semigroup of transformations restricted by an equivalence relation with a restricted range by $E(X, Y, \rho)$. In this paper, we investigate magnifying elements in $E(X, Y, \rho)$ and give the necessary and sufficient conditions for elements in $E(X, Y, \rho)$ to be a right or a left magnifying element.

KEYWORDS: full transformation semigroups, magnifying elements, equivalence relation, restricted range

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INTRODUCTION

In 1963, the publication of Semigroups by Ljapin [1] in *Translations of Mathematical Monographs Vol. 3* motivated many mathematicians who take an interest in semigroup theory to study the magnifying elements. There are many interesting properties about magnifying elements and their minimum proper subsets in a semigroup. For instance, every magnifying element in a semigroup with unit is regular, moreover, it generates an infinite monogenic semigroup [1]. Every semigroup containing magnifying elements is factorizable [2]. In addition, some minimal subsets relative to a magnifying element in a suitable semigroup S can be a subsemigroup of S [3]. These results persuade us to study magnifying elements in a semigroup. We recall the definition of magnifying elements: An element a of a semigroup S is called a right magnifying element (resp., a left magnifying element) if S contains a proper subset M such that $Ma = S$ (resp., $aM = S$). It was justified by Ljapin [1] that no element of a semigroup is simultaneously a left and a right magnifying element. Furthermore, he proved that finite semigroups, commutative semigroups and semigroups with two-sided cancellation do not contain magnifying elements. Therefore, in this work, we will pay our attention to magnifying elements in an infinite noncommutative semigroup without two-sided cancellation.

In 1974, Migliorini [3] proved the theorem of reduction of a minimal subset M relative to a magnifying element a in a semigroup S by establishing an infinite chain of minimal subsets $M^{(n)}$ of M with $M^{(n+1)} \subset M^{(n)}$ for all $n \in \mathbb{N}$, where $M^{(1)} = M$ and the n-th element $M^{(n)}$ is minimal relative to a magnifying element a^n, i.e., $a^n M^{(n)} = S$. In 1992, Catino and Migliorini [4] investigated the conditions for general semigroups in order that they contain magnifying elements. In 1994, the necessary and sufficient conditions for elements in the full transformation semigroups to be magnifying elements were first published by Magill Jr [5]. Moreover, he applied his result to the semigroup of linear transformations over a vector space and obtained the necessary and sufficient conditions for elements to be magnifying elements. The result exemplifies the fact that no elements will be a left and a right magnifying element concurrently which was claimed earlier by Ljapin. According to Tolo [6], if a proper subset M corresponding to a magnifying element a of a semigroup S is a subsemigroup (not necessarily a minimal subset) of S, then a is called a strong magnifying element. In 1996, Gutan [7] constructed the semigroup containing both left strong and left nonstrong magnifying elements. Later, Gutan [8] gave the characterization of this result. He constructed a general method to obtain a semigroup containing left strong magnifying elements such that the corresponding subsemigroup is a minimal subset. Furthermore, he showed that every such a semigroup can be obtained by this method. In 2003, Gutan and Kisielewicz [9] defined the definition of very good, good and bad magnifying elements. They constructed a semigroup having both good and bad magnifying elements and gave some general properties of semigroups with good magnifying elements.

In 2018, the publications on some particular elements in transformation semigroups involved with preserving or being restricted by an equivalence relation were disseminated by many authors. Sawatraksa et al [10] described an E-inversive element in some full transformation semigroups that preserve an equivalence relation. In addition, Sawatraksa and Namnak [11] generalized some subsemigroups of the full transformation semigroups, the set of all functions α such that for all x, y in a nonemptyset X if $(x, y) \in \sigma$, then $(x\alpha, y\alpha) \in \rho$, where σ and ρ are equivalence relations on X with $\rho \subseteq \sigma$, and gave conditions for these semigroups whose bi-ideals and quasi-ideals coincide. Sawatraksa et al [12] characterized left regular, right regular and completely regular elements in a semigroup of transformations restricted by an equivalence relation. Moreover, they provided the conditions in which this semigroup is left regular, right regular or completely regular. Sawatraksa and Namnak [13] established the necessary and sufficient conditions for two transformation semigroups restricted by an equivalence relation to be isomorphic to each other. Furthermore, many studies of magnifying elements in transformation semigroups are published between 2018 and 2021. For example, Chinram and Baupradist illustrated the necessary and sufficient conditions for elements in a semigroup of transformations with restricted range [14] and in a semigroup of transformations with invariant set [15] to be magnifying elements. In 2019, Prakitsri [16] investigated magnifying elements in some linear transformation semigroups. He showed that linear transformation semigroups with infinite nullity have no right magnifying elements while those with infinite co-rank have no left magnifying elements. Furthermore, he showed that all magnifying elements in these semigroups are strong. In 2020, Kaewnoi et al [17] generalized Chinram and Baupradist's results [14], and hence conditions for elements in a semigroup of transformations with restricted range preserving an equivalence relation to be right magnifying elements are established. Luangchaisri et al [18] gave the necessary and sufficient conditions for elements in a partial transformation semigroup to be magnifying elements. Later, the conditions for elements in some generalized partial transformation semigroups to be magnifying elements were provided by Chinram et al [19]. Recently, Kaewnoi et al [20] published the necessary and sufficient conditions for elements in the semigroups of transformations with a fixed point set restricted by an equivalence relation to be a left or a right magnifying element.

Hereafter, we denote the full transformation semigroup on X and an equivalence relation on X by $T(X)$ and ρ, respectively. All functions will be written from the right to the left, i.e., we write $x\alpha$ instead of $\alpha(x)$ and $x\alpha\beta$ instead of $(\beta\circ\alpha)(x)$ for all functions $\alpha,\beta\in T(X)$. The image of a function $\alpha\in T(X)$ is denoted by ran α. For a nonempty subset Y of X, the restriction of α to Y is denoted by $\alpha_{|Y}$. The equivalence relation ρ is trivial if $\rho = X\times X$ or $\rho=\Delta X$ where ΔX is an identity relation. Let Y be a nonempty subset of a set X. We let $T(X,Y)=\{\alpha\in T(X)\mid \text{ran } \alpha\subseteq Y\}$ and $E(X,\rho)=\{\alpha\in T(X)\mid \forall(x,y)\in X\times X, (x,y)\in\rho \text{ implies } x\alpha = y\alpha\}$. It is obvious that $T(X,Y)$ and $E(X,\rho)$ are subsemigroups of $T(X)$. We then denote the intersection $T(X,Y)\cap E(X,\rho)$ by $E(X,Y,\rho)$. Clearly, if $E(X,Y,\rho)$ is nonempty, then it is a semigroup. We then call $E(X,Y,\rho)$ the semigroup of transformations restricted by an equivalence relation with a restricted range.

Transformation semigroups with a restricted range have been the subject of research for many years. Meanwhile, it is well known that an equivalence relation is interesting and widely studied in transformation semigroups as well. However, no one has yet examined $E(X,Y,\rho)$ and it would be of special interest to determine magnifying elements in this semigroup. In addition, developing an approach to obtain magnifying elements in $E(X,Y,\rho)$ might shed new light on further research of magnifying elements in other semigroups relating to $E(X,Y,\rho)$. In this paper, we show that $E(X,Y,\rho)$ does not contain magnifying elements if $\rho = X\times X$ and establish the conditions for elements in order to be magnifying elements in $E(X,Y,\rho)$ by extending the results in [14] and [17]. Despite the fact that $E(X,Y,\rho)$ is a subsemigroup of $T(X,Y)$ [14] and $T_\rho(X,Y)$ [17], the conditions for the elements in these semigroups to be magnifying elements differ, and some proofs are entirely different. Furthermore, we show how the results obtained here are different from those presented in [14] and [17].

THE SEMIGROUP $E(X,Y,\rho)$

Some facts about $E(X,Y,\rho)$ are provided in this section.

Lemma 1 *$E(X,Y,\rho)=T(X,Y)$ if and only if $\rho=\Delta X$ or $|Y|=1$.*

Proof: Suppose that $\rho\neq\Delta X$ and $|Y|\neq 1$. Then there are distinct elements $y_1,y_2\in Y$ and there are distinct elements $a,b\in X$ such that $(a,b)\in\rho$. Define a function $\alpha: X\to Y$ by, for all $x\in X$,

$$x\alpha=\begin{cases} y_1, & \text{if } x=a,\\ y_2, & \text{otherwise.}\end{cases}$$

Clearly, $\alpha\in T(X,Y)$. Since $(a,b)\in\rho$ but $a\alpha=y_1\neq y_2=b\alpha$, we have that $\alpha\notin E(X,Y,\rho)$. Therefore, $E(X,Y,\rho)\neq T(X,Y)$. Conversely, if $|Y|=1$, then $E(X,Y,\rho)=T(X,Y)$. Assume that $\rho=\Delta X$. Clearly, $E(X,Y,\rho)\subseteq T(X,Y)$. Let $\alpha\in T(X,Y)$ and $x,y\in X$ be such that $(x,y)\in\rho$. By assumption, $x=y$ and hence $x\alpha=y\alpha$. Therefore, $\alpha\in E(X,Y,\rho)$. □

According to Ljapin [1], if $E(X,Y,\rho)$ is finite, then it does not contain a magnifying element. So, we may assume that X is infinite. By [21], $|Y|=1$, then $E(X,Y,\rho)$ is a singleton set which consists of only one constant function. Therefore, $E(X,Y,\rho)$ contains no magnifying elements since $E(X,Y,\rho)$ has no nonempty proper subset. Hence, if $E(X,Y,\rho)$ contains a magnifying element, then $|Y|\geqslant 2$. So, in this paper, we set that Y is a proper subset of an infinite set X and has at least two elements.

By [14], if $\rho=\Delta X$, then we obtain the following theorems.

Theorem 1 ([14]) *If $\rho = \Delta X$, then a function α in $E(X,Y,\rho)$ is a right magnifying element if and only if α is onto but not one-to-one and is such that $y\alpha^{-1}\cap Y \neq \varnothing$ for all $y \in Y$ and $|y\alpha^{-1}\cap Y| > 1$ for some $y \in Y$.*

Theorem 2 ([14]) *If $\rho = \Delta X$, $|X| = |Y|$ and $Y \neq X$, then a function α in $E(X,Y,\rho)$ is a left magnifying element if and only if α is one-to-one.*

Next, we will investigate the magnifying elements in $E(X,Y,\rho)$ when $\rho = X\times X$.

Lemma 2 *If $\rho = X\times X$, then all functions in $E(X,Y,\rho)$ are constant and $|E(X,Y,\rho)| = |Y|$.*

Proof: Assume that $\rho = X\times X$. Let $\alpha \in E(X,Y,\rho)$. For each $x \in X$, $x\alpha = y$ for some $y \in Y$. Then, by assumption, each element in $E(X,Y,\rho)$ is a constant function and can be written as $\alpha_y : X \to Y$ defined by $x\alpha_y = y$ for all $x \in X$. Conversely, for each $y \in Y$, the constant function α_y belongs to $E(X,Y,\rho)$. It follows that $f : Y \to E(X,Y,\rho)$ defined by $yf = \alpha_y$ is a bijection from Y to $E(X,Y,\rho)$. Hence, $|E(X,Y,\rho)| = |Y|$. □

Lemma 3 *If $\rho = X\times X$, then $E(X,Y,\rho)$ does not contain any right magnifying element.*

Proof: Suppose that there exists a right magnifying element $\alpha \in E(X,Y,\rho)$. Then there is a proper subset M of $E(X,Y,\rho)$ such that $M\alpha = E(X,Y,\rho)$. By the proof of Lemma 2, $\alpha = \alpha_y$ for some $y \in Y$. Since $|Y| > 1$, there exists an element $y' \in Y$ such that $y' \neq y$. Define a function $\alpha_{y'} : X \to Y$ by $x\alpha_{y'} = y'$ for all $x \in X$. Clearly, $\alpha_{y'}$ belongs to $E(X,Y,\rho)$. Then $\beta\alpha_y = \alpha_{y'}$ for some $\beta \in M$. Let $x \in X$. So $y = x\beta\alpha_y = x\alpha_{y'} = y'$, a contradiction. □

Lemma 4 *If $\rho = X\times X$, then $E(X,Y,\rho)$ does not contain any left magnifying element.*

Proof: Suppose that there exists a left magnifying element $\alpha \in E(X,Y,\rho)$. Then there is a proper subset M of $E(X,Y,\rho)$ such that $\alpha M = E(X,Y,\rho)$. Clearly, $M \subseteq E(X,Y,\rho)$. Let $\gamma \in E(X,Y,\rho)$. By the proof of Lemma 2, $\alpha = \alpha_y$ and $\gamma = \alpha_{y'}$ for some $y, y' \in Y$. Then $\alpha_{y'} = \alpha_y\beta$ for some $\beta \in M$. Since β is a constant map and ran $\alpha_y\beta =$ ran $\alpha_{y'} = \{y'\}$, we have $\beta = \alpha_{y'}$. Then $\gamma = \alpha_{y'} = \beta \in M$. This shows that $E(X,Y,\rho) \subseteq M$. Hence, $M = E(X,Y,\rho)$, which is a contradiction. □

Therefore, we will omit the proof for the case of trivial equivalence relations. So in the next section we may assume $\rho \neq X\times X$ and $\rho \neq \Delta X$. Throughout the rest of this paper, the set of all equivalence classes in X with respect to an equivalence relation ρ is denoted by $X/\rho = \{[x_i]_\rho \mid i \in \Lambda\}$ where $[x]_\rho$ is the equivalence class of ρ containing x. Let A, B be any subsets of X and let $a, b \in Y$. We will adopt the following notation to represent a function $\alpha \in E(X,Y,\rho)$:

$$\alpha = \begin{pmatrix} A & B \\ a & b \end{pmatrix}$$

which means that α maps all elements in A and B to a and b, respectively. If $A = \{x\}$, then we can write only x instead of A as follows:

$$\alpha = \begin{pmatrix} x & B \\ a & b \end{pmatrix}.$$

MAGNIFYING ELEMENTS IN $E(X,Y,\rho)$

Right magnifying elements

As in [14] and [17], the authors published the necessary and sufficient conditions for elements in the semigroups $T(X,Y)$ and $T_\rho(X,Y) = \{\alpha \in T(X,Y) \mid \forall (x,y) \in \rho, (x\alpha, y\alpha) \in \rho\}$ to be a right magnifying element. It is clear that $E(X,Y,\rho)$ is a subsemigroup of $T(X,Y)$ and $T_\rho(X,Y)$. However, the proofs of conditions for elements to be a right magnifying element in $E(X,Y,\rho)$ are completely different from those in $T(X,Y)$ and $T_\rho(X,Y)$. We will later explain how the results in this work relate to those in [14] and [17].

Lemma 5 *Let α be a function in $E(X,Y,\rho)$. If α is a right magnifying element, then $\alpha_{|Y}$ is onto Y.*

Proof: Let α be a right magnifying element in $E(X,Y,\rho)$. Then there exists a proper subset M of $E(X,Y,\rho)$ such that $M\alpha = E(X,Y,\rho)$. Suppose that $\alpha_{|Y}$ is not onto Y. Let $y \in Y\backslash \operatorname{ran} \alpha_{|Y}$. Define a function $\gamma : X \to Y$ by $x\gamma = y$ for all $x \in X$. Clearly, $\gamma \in E(X,Y,\rho)$. Then $\beta\alpha = \gamma$ for some $\beta \in M$. For all $x \in X$, $y = x\gamma = x\beta\alpha = (x\beta)\alpha \in Y\alpha = \operatorname{ran} \alpha_{|Y}$, which is a contradiction. □

Theorem 3 *Suppose that $|[x]_\rho \cap Y| > 1$ for some $[x]_\rho \in X/\rho$. A function $\alpha \in E(X,Y,\rho)$ is a right magnifying element if and only if $\alpha_{|Y}$ is onto Y.*

Proof: The necessity is obtained by Lemma 5. Conversely, assume that $\alpha \in E(X,Y,\rho)$ and $\alpha_{|Y}$ is onto Y. Let γ be a function in $E(X,Y,\rho)$. For all $x \in X$, $x\gamma \in Y$. By assumption, for each $x\gamma \in Y$, we can choose $a_{x\gamma} \in Y$ such that $[a_{x\gamma}]_\rho\alpha = \{x\gamma\}$. Let $M = \{\beta \in E(X,Y,\rho) \mid \beta$ is not onto $Y\}$. Since $|[x]_\rho \cap Y| > 1$ for some $[x]_\rho \in X/\rho$ and $\alpha_{|Y}$ is onto Y, Y and X/ρ are infinite. Then, M is a proper subset of $E(X,Y,\rho)$. Define $\beta : X \to Y$ by $x\beta = a_{x\gamma}$ for all $x \in X$(if there are two distinct elements $a_{x_1}, a_{x_2} \in Y$ such that $[a_{x_1}]_\rho\alpha = [a_{x_2}]_\rho\alpha$, then choose only one of them). Clearly, ran $\beta \subseteq Y$. Let $x, y \in X$ be such that $(x,y) \in \rho$. Then $x\gamma = y\gamma$, and hence $x\beta = a_{x\gamma} = a_{y\gamma} = y\beta$. Since $|[x]_\rho \cap Y| > 1$ for some $[x]_\rho \in X/\rho$, there are distinct elements $y_1, y_2 \in [x]_\rho \cap Y$. Then $y_1 \notin \operatorname{ran}\beta$ or $y_2 \notin \operatorname{ran}\beta$. So β is not onto Y. Therefore, $\beta \in M$. For all $x \in X$, $x\beta\alpha = a_{x\gamma}\alpha = x\gamma$. Thus, α is a right magnifying element. □

The idea of Theorem 3 is demonstrated by the next example.

Example 1 Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}\backslash\{2,4\} = \{6,8,10,12,\ldots\}$. Let $X/\rho = \{\{1,2,3,4\}, \{5,6,7,8\}, \{9,10,11,12\}, \ldots\} = \{[1]_\rho,[5]_\rho,[9]_\rho,[13]_\rho,\ldots\} = \{[x]_\rho \mid x = 4n-3 \text{ where } n \in \mathbb{N}\}$. Since $[5]_\rho \cap Y = \{6,8\}$, $|[5]_\rho \cap Y| > 1$. Define a function $\alpha : X \to Y$ by, for all $x \in X$,

$$x\alpha = \begin{cases} 6 & \text{if } x \in [4n-3]_\rho \text{ for all } n \in \{1,2\}, \\ 2n+2 & \text{if } x \in [4n-3]_\rho \text{ for all } n \in \{3,4,\ldots\}. \end{cases}$$

For convenience, we write α as

$$\alpha = \begin{pmatrix} [1]_\rho & [5]_\rho & [9]_\rho & [13]_\rho & [17]_\rho & [21]_\rho & \cdots \\ 6 & 6 & 8 & 10 & 12 & 14 & \cdots \end{pmatrix}.$$

Since $\alpha_{|Y}$ is onto Y, α is a right magnifying element, by Theorem 3. Let γ be a function in $E(X,Y,\rho)$ defined by

$$x\gamma = \begin{cases} 8, & \text{if } x \in [1]_\rho, \\ 6, & \text{if } x \in [5]_\rho, \\ 4n-2 & \text{if } x \in [4n-3]_\rho \text{ for all } n \in \{3,4,\ldots\}. \end{cases}$$

For convenience, we write γ as

$$\gamma = \begin{pmatrix} [1]_\rho & [5]_\rho & [9]_\rho & [13]_\rho & [17]_\rho & [21]_\rho & \cdots \\ 8 & 6 & 10 & 14 & 18 & 22 & \cdots \end{pmatrix}.$$

Let $M = \{\beta \in E(X,Y,\rho) \mid \beta \text{ is not onto}\}$. Clearly, M is a proper subset of $E(X,Y,\rho)$. Define a function $\beta : X \to Y$ by, for all $x \in X$,

$$x\beta = \begin{cases} 12, & \text{if } x \in [1]_\rho, \\ 8, & \text{if } x \in [5]_\rho, \\ 8n-8 & \text{if } x \in [4n-3]_\rho \text{ for all } n \in \{3,4,\ldots\}. \end{cases}$$

For convenience, we write β as

$$\beta = \begin{pmatrix} [1]_\rho & [5]_\rho & [9]_\rho & [13]_\rho & [17]_\rho & [21]_\rho & \cdots \\ 12 & 8 & 16 & 24 & 32 & 40 & \cdots \end{pmatrix}.$$

Then β is not onto Y since $6 \in Y\backslash \operatorname{ran}\beta$. It is obvious that $\beta \in M$, and $x\beta\alpha = x\gamma$ for all $x \in X$.

Theorem 4 *Suppose that $|[x]_\rho \cap Y| = 1$ for each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap Y \neq \varnothing$. A function $\alpha \in E(X,Y,\rho)$ is a right magnifying element if and only if $\alpha_{|Y}$ is onto Y but not one-to-one.*

Proof: Let $\alpha \in E(X,Y,\rho)$. Assume that α is a right magnifying element. By Lemma 5, $\alpha_{|Y}$ is onto Y. Suppose that $\alpha_{|Y}$ is one-to-one. Since $|[x]_\rho \cap Y| = 1$ for each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap Y \neq \varnothing$, there is a unique element $a_x \in [x]_\rho \cap Y$. So if $[x]_\rho \cap Y \neq \varnothing$, we then set $[x]_\rho = [a_x]_\rho$. Since $\alpha_{|Y}$ is onto Y, there is a unique $y_{a_x} \in Y$ such that $[y_{a_x}]_\rho\alpha = \{a_x\}$. Let $y' \in Y$. We then define a function $\alpha' : X \to Y$ by, for each $x \in X$,

$$x\alpha' = \begin{cases} y_{a_x}, & \text{if } [x]_\rho \cap Y \neq \varnothing, \\ y' & \text{if } [x]_\rho \cap Y = \varnothing. \end{cases}$$

Clearly, $\operatorname{ran}\alpha' \subseteq Y$. Let $a,b \in X$ be such that $(a,b) \in \rho$. Then $a,b \in [x]_\rho$ for some $[x]_\rho \in X/\rho$. Thus $a\alpha' = y_{a_x} = b\alpha'$ if $[x]_\rho \cap Y \neq \varnothing$, and $a\alpha' = y' = b\alpha'$ if $[x]_\rho \cap Y = \varnothing$. Therefore, $\alpha' \in E(X,Y,\rho)$. Since α is a right magnifying element, there is a proper subset M of $E(X,Y,\rho)$ such that $M\alpha = E(X,Y,\rho)$. Clearly, $M\alpha \subseteq E(X,Y,\rho)\alpha$ and $E(X,Y,\rho)\alpha \subseteq E(X,Y,\rho) = M\alpha$. So $M\alpha = E(X,Y,\rho)\alpha$. Therefore, $M\alpha\alpha' = E(X,Y,\rho)\alpha\alpha'$. Let $\gamma \in E(X,Y,\rho)$. It is obvious that $\operatorname{dom}\gamma = \operatorname{dom}\gamma\alpha\alpha' = X$. Let $x \in X$. Since $|[x]_\rho \cap Y| = 1$ for each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap Y \neq \varnothing$, $a_{x\gamma\alpha} = x\gamma\alpha$ and hence $(y_{a_{x\gamma\alpha}})\alpha = a_{x\gamma\alpha} = x\gamma\alpha$. This implies that $y_{a_{x\gamma\alpha}} = x\gamma$ because $\alpha_{|Y}$ is one-to-one. So we have $x\gamma\alpha\alpha' = (x\gamma\alpha)\alpha' = y_{a_{x\gamma\alpha}} = x\gamma$. Thus, $E(X,Y,\rho)\alpha\alpha' = E(X,Y,\rho)$. Similarly, $M\alpha\alpha' = M$ and hence $M = M\alpha\alpha' = E(X,Y,\rho)\alpha\alpha' = E(X,Y,\rho)$, a contradiction.

Conversely, assume that $\alpha_{|Y}$ is onto Y but not one-to-one. Let γ be a function in $E(X,Y,\rho)$. For all $x \in X$, $x\gamma \in Y$. By assumption, for each $x\gamma \in Y$, we can choose an element $a_{x\gamma} \in Y$ such that $[a_{x\gamma}]_\rho\alpha = \{x\gamma\}$. (If there are two distinct elements $a_{x_1}, a_{x_2} \in Y$ such that $[a_{x_1}]_\rho\alpha = [a_{x_2}]_\rho\alpha$, then choose only one of them.) Let $M = \{\beta \in E(X,Y,\rho) \mid \beta \text{ is not onto } Y\}$. Since $|[x]_\rho \cap Y| = 1$ for each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap Y \neq \varnothing$, and $\alpha_{|Y}$ is onto Y but not one-to-one, we then have that Y and X/ρ are infinite. Thus M is a proper subset of $E(X,Y,\rho)$. We then define a function $\beta : X \to Y$ by $x\beta = a_{x\gamma}$ for all $x \in X$. Clearly, $\operatorname{ran}\beta \subseteq Y$. Let $x,y \in X$ be such that $(x,y) \in \rho$. Then $x\gamma = y\gamma$. Thus $x\beta = a_{x\gamma} = a_{y\gamma} = y\beta$. So $\beta \in E(X,Y,\rho)$. Since $\alpha_{|Y}$ is not one-to-one, there are two distinct elements $a_{x_1}, a_{x_2} \in Y$ such that $[a_{x_1}]_\rho\alpha = [a_{x_2}]_\rho\alpha$ and hence either $a_{x_1} \notin \operatorname{ran}\beta$ or $a_{x_2} \notin \operatorname{ran}\beta$. So $\beta \in M$. For each $x \in X$, $x\beta\alpha = a_{x\gamma}\alpha = x\gamma$. Therefore, $M\alpha = E(X,Y,\rho)$. □

In other words, one may consider the subset Y of X in Theorem 4 as a transversal of the equivalence relation $\rho \cap Y \times Y$. In particular, if Y is a transversal of the equivalence relation ρ, then $E(X,Y,\rho) = T_\rho(X,Y)$ and hence the result is obtained by [17].

The idea of Theorem 4 is demonstrated by the next example.

Example 2 Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Let $X/\rho = \{\{1,2\}, \{3,4\}, \{5,6\}, \ldots\} = \{[1]_\rho, [3]_\rho, [5]_\rho, [7]_\rho, \ldots\} = \{[x]_\rho \mid x = 2n-1 \text{ where } n \in \mathbb{N}\}$. Then $|[x]_\rho \cap Y| = 1$ for each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap Y \neq \varnothing$. Define a function $\alpha : X \to Y$ by, for all $x \in X$,

$$x\alpha = \begin{cases} 2, & \text{if } x \in \{1,2,3,4\}, \\ x-1 & \text{if } x > 4 \text{ and } x \text{ is odd}, \\ x-2 & \text{if } x > 4 \text{ and } x \text{ is even}. \end{cases}$$

For convenience, we write α as

$$\alpha = \begin{pmatrix} [1]_\rho & [3]_\rho & [5]_\rho & [7]_\rho & [9]_\rho & [11]_\rho & \cdots \\ 2 & 2 & 4 & 6 & 8 & 10 & \cdots \end{pmatrix}.$$

Since $2\alpha = 2 = 4\alpha$, $\alpha_{|Y}$ is not one-to-one. It is not hard to see that for all $x \in Y$, there exists $x+2 \in Y$ such that $(x+2)\alpha = x$. Hence, $\alpha_{|Y}$ is onto Y. By Theorem 4, α is a right magnifying element. Let γ be a function in $E(X,Y,\rho)$ defined by, for all $x \in X$,

$$x\gamma = \begin{cases} x, & \text{if } x \text{ is even,} \\ x+1, & \text{if } x \text{ is odd.} \end{cases}$$

For convenience, we write γ as

$$\gamma = \begin{pmatrix} [1]_\rho & [3]_\rho & [5]_\rho & [7]_\rho & [9]_\rho & [11]_\rho & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & \cdots \end{pmatrix}.$$

Let $M = \{\beta \in E(X,Y,\rho) \mid \beta \text{ is not onto } Y\}$. Clearly, M is a proper subset of $E(X,Y,\rho)$. Define a function $\beta : X \to Y$ by, for all $x \in X$,

$$x\beta = \begin{cases} x+2, & \text{if } x \text{ is even,} \\ x+3, & \text{if } x \text{ is odd.} \end{cases}$$

For convenience, we write β as

$$\beta = \begin{pmatrix} [1]_\rho & [3]_\rho & [5]_\rho & [7]_\rho & [9]_\rho & [11]_\rho & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & \cdots \end{pmatrix}.$$

Then β is not onto Y since $2 \in Y \setminus \text{ran } \beta$. It is obvious that $\beta \in M$ and it is not hard to see that for all $x \in X$, $x\beta\alpha = x\gamma$.

Let $\alpha \in E(X,Y,\rho)$. Assume that $|Y| > 1$, $\rho \neq X \times X$ and $\rho \neq \Delta X$. It is clear that $\alpha_{|Y}$ is onto Y if and only if $y\alpha^{-1} \cap Y \neq \varnothing$ for all $y \in Y$, and $\alpha_{|Y}$ is not one-to-one if and only if $|y\alpha^{-1} \cap Y| > 1$ for some $y \in Y$. In the case $|[x]_\rho \cap Y| > 1$ for some $[x]_\rho \in X/\rho$, we automatically obtain the properties $|y\alpha^{-1} \cap Y| > 1$ for some $y \in Y$ because $\alpha_{|Y}$ is onto Y and α is restricted by an equivalence relation ρ. Therefore, Theorem 3 and Theorem 4 closely resemble the results in [14, Theorem 2.3, p 56] and [17, Theorem 2.6, p 106].

Left magnifying elements

Although $E(X,Y,\rho)$ is a subsemigroup of $T(X,Y)$, the conditions for elements in $E(X,Y,\rho)$ to be left magnifying elements are more complicated than those for elements in $T(X,Y)$[14].

Lemma 6 *Let $\alpha \in E(X,Y,\rho)$. If α is a left magnifying element, then $[x]_\rho\alpha \cap [y]_\rho\alpha = \varnothing$ for all $[x]_\rho$, $[y]_\rho \in X/\rho$.*

Proof: Let $\alpha \in E(X,Y,\rho)$. Assume that there are two distinct equivalence classes $[x]_\rho, [y]_\rho \in X/\rho$ such that $[x]_\rho\alpha \cap [y]_\rho\alpha \neq \varnothing$. Then there exists an element $a \in [x]_\rho\alpha \cap [y]_\rho\alpha$ such that $x\alpha = a = y\alpha$. Suppose that α is a left magnifying element. Then there exists a proper subset M of $E(X,Y,\rho)$ such that $\alpha M = E(X,Y,\rho)$. Since $|Y| > 1$, we choose $a' \in Y$ such that $a' \neq a$. Define a function $\gamma : X \to Y$ by, for all $x' \in X$,

$$x'\gamma = \begin{cases} a, & \text{if } x' \in [x]_\rho, \\ a', & \text{otherwise.} \end{cases}$$

Clearly, $\gamma \in E(X,Y,\rho)$. Then $\alpha\beta = \gamma$ for some $\beta \in M$. Thus $a = x\gamma = x\alpha\beta = a\beta = y\alpha\beta = y\gamma = a'$, a contradiction. □

Lemma 7 *If a function $\alpha \in E(X,Y,\rho)$ is a left magnifying element, then $|[x]_\rho \cap \text{ ran } \alpha| \leqslant 1$ for all $[x]_\rho \in X/\rho$.*

Proof: Let $\alpha \in E(X,Y,\rho)$ be a left magnifying element. Then there exists a proper subset M of $E(X,Y,\rho)$ such that $\alpha M = E(X,Y,\rho)$. Suppose that $|[x]_\rho \cap \text{ran } \alpha| > 1$ for some $[x]_\rho \in X/\rho$. Let $y_1, y_2 \in [x]_\rho \cap \text{ran } \alpha$ be such that $y_1 \neq y_2$. By Lemma 6, there exist distinct equivalence classes $[x_1]_\rho, [x_2]_\rho \in X/\rho$ such that $[x_1]_\rho\alpha = \{y_1\}$ and $[x_2]_\rho\alpha = \{y_2\}$. Define a function $\gamma : X \to Y$ by, for all $x \in X$,

$$x\gamma = \begin{cases} y_1, & \text{if } x \in [x_1]_\rho, \\ y_2, & \text{otherwise.} \end{cases}$$

Clearly, $\gamma \in E(X,Y,\rho)$. Then $\alpha\beta = \gamma$ for some $\beta \in M$. Thus $y_1\beta = x_1\alpha\beta = x_1\gamma = y_1$ and $y_2\beta = x_2\alpha\beta = x_2\gamma = y_2$. Since $(y_1, y_2) \in \rho$ but $y_1\beta \neq y_2\beta$, we have $\beta \notin E(X,Y,\rho)$, which is a contradiction. □

Lemma 8 *Suppose that $[x]_\rho \cap Y \neq \varnothing$ for all $[x]_\rho \in X/\rho$. Let $\alpha \in E(X,Y,\rho)$. If α is a left magnifying element, then $[x]_\rho \cap \text{ran } \alpha = \varnothing$ for some $[x]_\rho \in X/\rho$.*

Proof: Let $\alpha \in E(X,Y,\rho)$ and $[x]_\rho \cap Y \neq \varnothing$ for all $[x]_\rho \in X/\rho$. Assume that α is a left magnifying element. Then there exists a proper subset M of $E(X,Y,\rho)$ such that $\alpha M = E(X,Y,\rho)$. Suppose that $[x]_\rho \cap \text{ ran } \alpha \neq \varnothing$ for all $[x]_\rho \in X/\rho$. By Lemma 6 and Lemma 7, $[x]_\rho\alpha \cap [y]_\rho\alpha = \varnothing$ and $|[x]_\rho \cap \text{ ran } \alpha| = 1$ for all $[x]_\rho, [y]_\rho \in X/\rho$. Let $x \in X$. Since $[x]_\rho \cap \text{ ran } \alpha \neq \varnothing$, there exists $a_x \in [x]_\rho \cap \text{ ran } \alpha$. Since $[x]_\rho\alpha \cap [y]_\rho\alpha = \varnothing$ for all $[x]_\rho, [y]_\rho \in X/\rho$, there exists a unique $[x']_\rho \in X/\rho$ such that $[x']_\rho\alpha = \{a_x\}$. Since $[x']_\rho \cap Y \neq \varnothing$, we can choose $y_x \in [x']_\rho \cap Y$. Then $y_x\alpha = a_x$. We then define a function $\alpha' : X \to Y$ by $x\alpha' = y_x$ for all $x \in X$. Clearly, ran $\alpha' \subseteq Y$. Let $x_1, x_2 \in X$ be such that $(x_1, x_2) \in \rho$. Then $x_1, x_2 \in [x]_\rho$ for some $[x]_\rho \in X/\rho$. So $x_1\alpha' = y_x = x_2\alpha'$. Therefore, $\alpha' \in E(X,Y,\rho)$. It is clear that $\alpha M \subseteq \alpha E(X,Y,\rho)$ and $\alpha E(X,Y,\rho) \subseteq E(X,Y,\rho) = \alpha M$. Thus $\alpha M = \alpha E(X,Y,\rho)$. We next show that $\alpha'\alpha E(X,Y,\rho) = E(X,Y,\rho)$. Clearly, $\alpha'\alpha E(X,Y,\rho) \subseteq E(X,Y,\rho)$. Let $\gamma \in E(X,Y,\rho)$. It is obvious that dom γ = dom $\alpha'\alpha\gamma = X$. Let $x \in X$. Then $x\alpha'\alpha\gamma = y_x\alpha\gamma = a_x\gamma = x\gamma$. Therefore, $\alpha'\alpha\gamma = \gamma$ for all $\gamma \in E(X,Y,\rho)$. Thus $\gamma \in \alpha'\alpha E(X,Y,\rho)$. This shows that $\alpha'\alpha E(X,Y,\rho) = E(X,Y,\rho)$. Similarly, $\alpha'\alpha M = M$. Therefore, $M = \alpha'\alpha M = \alpha'\alpha E(X,Y,\rho) = E(X,Y,\rho)$, which is a contradiction. □

Lemma 9 *Suppose that* $[x]_\rho \cap Y \neq \varnothing$ *for all* $[x]_\rho \in X/\rho$. *Let* $\alpha \in E(X,Y,\rho)$. *Then* α *is a left magnifying element if all of the following conditions hold:*
(i) $[x]_\rho\alpha \cap [y]_\rho\alpha = \varnothing$ *for all* $[x]_\rho, [y]_\rho \in X/\rho$;
(ii) $|[x]_\rho \cap \operatorname{ran}\alpha| \leqslant 1$ *for all* $[x]_\rho \in X/\rho$;
(iii) $[z]_\rho \cap \operatorname{ran}\alpha = \varnothing$ *for some* $[z]_\rho \in X/\rho$.

Proof: Let $\alpha \in E(X,Y,\rho)$. Assume that (i),(ii) and (iii) hold. By (iii), there exists $[\overline{x}]_\rho \in X/\rho$ such that $[\overline{x}]_\rho \cap \operatorname{ran}\alpha = \varnothing$. Let $y_0 \in Y$ and $M = \{\beta \in E(X,Y,\rho) \mid x\beta = y_0$ for all $x \in X$ such that $[x]_\rho \cap \operatorname{ran}\alpha = \varnothing\}$. Clearly, M is a proper subset of $E(X,Y,\rho)$. Let $x \in X$. If $[x]_\rho \cap \operatorname{ran}\alpha \neq \varnothing$, then there exists a unique $y_x \in [x]_\rho \cap \operatorname{ran}\alpha$, by (ii). By (i), there exists a unique $[x']_\rho \in X/\rho$ such that $[x']_\rho\alpha = \{y_x\}$. Let $\gamma \in E(X,Y,\rho)$. We then define a function $\beta : X \to Y$ by, for all $x \in X$,

$$x\beta = \begin{cases} x'\gamma, & \text{if } [x]_\rho \cap \operatorname{ran}\alpha \neq \varnothing, \\ y_0, & \text{if } [x]_\rho \cap \operatorname{ran}\alpha = \varnothing. \end{cases}$$

Claim that $\beta \in E(X,Y,\rho)$. Clearly, $\operatorname{ran}\beta \subseteq Y$. Let $u,v \in X$ be such that $(u,v) \in \rho$. Then $u,v \in [x]_\rho$ for some $[x]_\rho \in X/\rho$. If $[x]_\rho \cap \operatorname{ran}\alpha \neq \varnothing$, then $u\beta = x'\gamma = v\beta$. If $[x]_\rho \cap \operatorname{ran}\alpha = \varnothing$, then $u\beta = y_0 = v\beta$. Therefore, $\beta \in M$. Let $x \in X$. Then $x\alpha = y_t$ for some $y_t \in Y$ and there exists a unique $[t']_\rho \in X/\rho$ such that $[t']_\rho\alpha = \{y_t\}$. Therefore, $x \in [t']_\rho$ and hence $x\alpha\beta = y_t\beta = t'\gamma = x\gamma$. □

By Lemmas 6–9, the next theorem is established.

Theorem 5 *Suppose that* $[x]_\rho \cap Y \neq \varnothing$ *for all* $[x]_\rho \in X/\rho$. *Let* $\alpha \in E(X,Y,\rho)$. *Then* α *is a left magnifying element if and only if all of the following conditions hold:*
(i) $[x]_\rho\alpha \cap [y]_\rho\alpha = \varnothing$ *for all* $[x]_\rho, [y]_\rho \in X/\rho$;
(ii) $|[x]_\rho \cap \operatorname{ran}\alpha| \leqslant 1$ *for all* $[x]_\rho \in X/\rho$;
(iii) $[z]_\rho \cap \operatorname{ran}\alpha = \varnothing$ *for some* $[z]_\rho \in X/\rho$.

The idea of Theorem 5 is demonstrated by the next example.

Example 3 Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Let $X/\rho = \{\{1,2,3,4\}, \{5,6,7,8\}, \{9,10,11,12\}, \ldots\} = \{[1]_\rho, [5]_\rho, [9]_\rho, [13]_\rho, \ldots\} = \{[x]_\rho \mid x = 4n-3$ where $n \in \mathbb{N}\}$. Define a function $\alpha : X \to Y$ by $x\alpha = x' + 9$ for all $x \in X$ such that $x \in [x']_\rho$. For convenience, we write α as

$$\alpha = \begin{pmatrix} [1]_\rho & [5]_\rho & [9]_\rho & [13]_\rho & [17]_\rho & [21]_\rho & \cdots \\ 10 & 14 & 18 & 22 & 26 & 30 & \cdots \end{pmatrix}.$$

Clearly, $\operatorname{ran}\alpha = \{10,14,18,22,26,\ldots\}$. It is not hard to check that $[x]_\rho\alpha \cap [y]_\rho\alpha = \varnothing$ and $|[x]_\rho \cap \operatorname{ran}\alpha| \leqslant 1$ for all $[x]_\rho, [y]_\rho \in X/\rho$, and $[z]_\rho \cap \operatorname{ran}\alpha = \varnothing$ for some $[z]_\rho \in X/\rho$. By Theorem 5, α is a left magnifying element. Let γ be a function in $E(X,Y,\rho)$ defined by $x\gamma = x' + 7$ for all $x \in X$ such that $x \in [x']_\rho$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} [1]_\rho & [5]_\rho & [9]_\rho & [13]_\rho & [17]_\rho & [21]_\rho & \cdots \\ 8 & 12 & 16 & 20 & 24 & 28 & \cdots \end{pmatrix}.$$

Let $M = \{\beta \in E(X,Y,\rho) \mid x\beta = 2$ if $[x]_\rho \cap \operatorname{ran}\alpha = \varnothing\}$. Clearly, M is a proper subset of $E(X,Y,\rho)$. Define a function $\beta : X \to Y$ by $x\beta = 2$ if x is a member of $[1]_\rho$ or $[5]_\rho$, and $x\beta = x' - 1$ if $x \in [x']_\rho$ and $x' \in \{9,13,17,21,\ldots\}$. For convenience, we write β as

$$\beta = \begin{pmatrix} [1]_\rho & [5]_\rho & [9]_\rho & [13]_\rho & [17]_\rho & [21]_\rho & \cdots \\ 2 & 2 & 8 & 12 & 16 & 20 & \cdots \end{pmatrix}.$$

It is obvious that $\beta \in M$ and we can see that for all $x \in X$, $x\alpha\beta = x\gamma$.

Theorem 6 *Suppose that* $[x]_\rho \cap Y = \varnothing$ *for some* $[x]_\rho \in X/\rho$. *A function* $\alpha \in E(X,Y,\rho)$ *is a left magnifying element if and only if* $[x]_\rho\alpha \cap [y]_\rho\alpha = \varnothing$ *and* $|[x]_\rho \cap \operatorname{ran}\alpha| \leqslant 1$ *for all* $[x]_\rho, [y]_\rho \in X/\rho$.

Proof: By Lemmas 6 and 7, the necessity is clear. Since there exists $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap Y = \varnothing$, $[x]_\rho \cap \operatorname{ran}\alpha = \varnothing$. The converse is proved by the same argument as in the proof of Lemma 9. □

Theorems 5 and 6 show that although the function α is a left magnifying element in $E(X,Y,\rho)$, α does not need to be one-to-one. This is not the case for elements of $T(X,Y)$ to be left magnifying elements as shown in [14, Theorem 2.3, p 57].

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