

# A two-step smoothing Levenberg-Marquardt method for nonlinear complementarity problems

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Received 15 Mar 2022, Accepted 27 Nov 2022

Available online

**ABSTRACT:** By reformulating the nonlinear complementarity problem into a system of nonsmooth equations, we proposed a two-step smoothing Levenberg-Marquardt method with global convergence in this paper, where not only one step but also an approximate step is computed at each iteration. Under local error bound condition, the local convergence result of the present algorithm is also obtained. Finally, numerical experiments illustrate the effectiveness of the present method when compare it with one-step smoothing Levenberg-Marquardt methods.

**KEYWORDS:** nonsmooth analysis, generalized Jacobi, smoothing Levenberg-Marquardt method, two-step iterations

**MSC2020:** 90C33

## INTRODUCTION

Consider the nonlinear complementarity problem (NCP) such that

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) \geq 0,$$

where  $x \in \mathbb{R}^n$ ,  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuously differentiable. It has a large number of important applications and has attracted many interests in last decades [1–4]. For NCP, a popular method is to reformulate it into a system of nonsmooth equations such that

$$H(x) = (\varphi(x_1, F_1(x)), \dots, \varphi(x_n, F_n(x)))^T = 0, \quad (1)$$

where  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  is an NCP function [5, 6]. And a Levenberg-Marquardt (LM) algorithm always solves it by introducing a perturbation item  $\lambda_k$ , then generating the iterative direction  $d_k$  by solving the equation such that

$$(J_k^T J_k + \lambda_k I) d_k = -J_k^T H(x_k), \quad (2)$$

where  $\lambda_k > 0$  is called LM parameter,  $J_k$  denotes the generalized Jacobi of the function  $H$  in nonsmooth LM algorithms [7–9], the Jacobi of the function  $H_\varepsilon$ , which is a smoothing approximating function of  $H$  in smoothing LM algorithms. Since it is always difficult or time consuming to obtain the generalized Jacobi of the function  $H$  for nonsmooth algorithms, the smoothing ones for NCP have been paid more attention for a long time [10–12]. By using smoothing technique to convert NCP to smoothing nonlinear systems, Yu and Pu present a smoothing Levenberg–Marquardt method. And Under the local error bound condition, which is much weaker than nonsingularity assumption or the strictly complementarity condition, they obtain the local superlinear convergence [11]. By investigating an element of related functions’ B-differential, a

smoothing Levenberg-Marquardt was proposed based on a Chen-Harker-Kanzow-Smale (CHKS) smoothing function, which satisfies a property called strongly Jacobi consistency [12]. These algorithms are effective in numerical experiments. However, they are both based on one-step LM iterative step. Considering that the two-step technique has been adopted for complementarity problems in recent years, such as Zheng and Liu proposed a two-step modulus-based matrix splitting iteration method for solving a class of nonlinear complementarity problems [13], Cao and Wang investigated the implicit complementarity problems by a two-step modulus-based matrix splitting iteration method [14], a two-step modulus-based matrix splitting iteration method for the horizontal nonlinear complementarity problem was proposed in [15] and so on, maybe it is also effective when applied into the LM algorithms. And maybe it is valuable to investigate the function  $F$  in a general form, since these methods with two-step technology are based on the property of matrix, and the correlation functions are expressed in the form of matrices.

Actually, for a smooth equation such that

$$G(x) = 0,$$

the LM algorithm always has quadratic convergence if the Jacobi matrix of the smooth function  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is nonsingular and Lipschitzian continuous. As the nonsingularity is a strong condition, Yamashita and Fukushima investigated the convergence rate under the local error bound condition [16] when the LM parameter  $\lambda_k$  is chosen as  $\|G(x_k)\|^2$ . Fan and Pan obtained the convergence order when  $\lambda_k = \|G(x_k)\|^\delta$  for  $\delta \in (0, 2]$  [17]. In [18], Fan introduced a two-step Levenberg-Marquardt method (MLM) with cubic convergence when  $\delta \in [1, 2]$ , where each iteration

$d_k = d_{k_1} + d_{k_2}$  is obtained by solving two linear systems, the one denoted as  $d_{k_1}$  is the solution of the equation in the form of Eq. (2), where  $H(x_k)$  is replaced by  $G(x_k)$ , the other one  $d_{k_2}$  is obtained by solving the linear system such that

$$(J_k^T J_k + \lambda_k I)d = -J_k^T G(y_k),$$

where  $y_k = x_k + d_{k_1}$ . Amini and Rostami proposed a three-step modified Levenberg-Marquardt method (MMLM) with a new line search, where the addition of the LM step and two approximate LM steps are adopted as the trial step at every iteration [19], the convergence order of the algorithm is biquadratic. It is noticeable that both the MLM and the MMLM algorithms have better convergence and they do not need compute more, but only the Jacobi of  $G(x_k)$  once in the  $k$ -th iteration. Inspired by these, a two-step smoothing LM method and its local convergence under the error bound condition for the nonlinear complementarity problem are investigated in this paper.

**PRELIMINARIES**

In this section, we recall some definitions and propositions necessary in this paper.

In nonsmooth analysis, there exists some generalized differentials with different forms. Let  $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be locally Lipschitzian,  $\Omega_G$  denotes a set where  $G$  is not differential, B-differential of  $G$  at  $x \in \mathbb{R}^n$  is defined by

$$\partial_B G(x) = \{\lim JG(x_i) : x_i \rightarrow x, x_i \notin \Omega_G\}.$$

Clarke generalized Jacobi of  $G$  at  $x \in \mathbb{R}^n$  is defined as the convex hull of  $\partial_B G(x)$  such that

$$\partial_{Cl} G(x) = \text{conv } \partial_B G(x).$$

Denote

$$H(x) = H_{\min}(x) = (\varphi_{\min}(x_1, F_1(x)), \dots, \varphi_{\min}(x_n, F_n(x)))^T, \quad (3)$$

in Eq. (1), we use the CHKS smoothing function

$$\varphi_\varepsilon(a, b) = \frac{a + b - \sqrt{\varepsilon^2 + (a - b)^2}}{2},$$

to approximate to the NCP function “ $\varphi_{\min}$ ” throughout this paper. Then, the smoothing approximating function  $H_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$  of the function  $H$  has some good properties, such as the strongly Jacobi consistency [12]

$$\lim_{\varepsilon \rightarrow 0} \text{dist}(\nabla H_\varepsilon(x), \partial_{Cl} H_{\min}(x)) = 0.$$

We next list some others, which are necessary in the convergence analysis.

(i) For arbitrary  $\varepsilon_1 > 0, \varepsilon_2 > 0$ , there is

$$\|H_{\varepsilon_1}(x) - H_{\varepsilon_2}(x)\| \leq \kappa |\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}|,$$

where  $\kappa = \sqrt{2n}$ .

(ii) For arbitrary  $\varepsilon > 0$ , we have

$$\|H(x) - H_\varepsilon(x)\| \leq \kappa \sqrt{\varepsilon}.$$

(iii) Denote  $\alpha(x) = \{i : x_i = F_i(x)\}$ . Suppose  $x \in \mathbb{R}^n$  is not a solution of NCP. Define a function  $\bar{\varepsilon} : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\bar{\varepsilon}(x, \delta) = \begin{cases} 1, & \text{if } \frac{n\tau^2(x)}{\delta^2} - \rho(x) \leq 0, \\ \frac{\rho(x)\delta}{\sqrt{n\tau^2(x) - \delta^2\rho(x)}}, & \text{otherwise} \end{cases} \quad (4)$$

where  $\rho(x) = \min\{(x_i - F_i(x))^2 : i \notin \alpha(x)\}$ ,  $\tau(x) = \frac{1}{2} \max\{\|(x_i - F_i(x))(e_i - \nabla F_i(x))\| : i \notin \alpha(x)\}$  and  $\bar{\delta} > 0$  is a given constant. Then, there is

$$\text{dist}(\nabla H_\varepsilon(x), \partial_{Cl} H_{\min}(x)) \leq \delta,$$

for all  $\varepsilon$  such that  $0 \leq \varepsilon \leq \bar{\varepsilon}(x, \delta)$ .

Semismooth functions include many types of functions in nonsmooth analysis, such as smooth functions, convex functions, maximum functions and so on. The function  $H_{\min}$  is semismooth, it has some properties as follows

$$H(y) - H(x) - V^T(y - x) = o(\|y - x\|), \quad \forall V \in \partial_{Cl} H(x),$$

which means that

$$\|H(y) - H(x) - V^T(y - x)\| \leq C\|y - x\|^{1+p}, \quad (5)$$

where  $C > 0$  and  $p \geq 0$  are constants. Suppose  $V \in \partial_{Cl} H(x)$  is Lipschitz continuous, there is

$$\|H(y) - H(x)\| \leq O(\|y - x\|^{(1+p)}). \quad (6)$$

Especially, we have  $p \geq 1$  in Eq. (5) when  $H$  is strongly semismooth.

**ALGORITHM AND ITS GLOBAL CONVERGENCE**

In this section, a two-step smoothing Levenberg-Marquardt method for nonlinear complementarity problems is presented, and its global convergence is also shown.

Define  $\Phi(x)$  as the merit function of (1) such that

$$\Phi(x) = \frac{1}{2} \|H(x)\|^2,$$

and also its approximating merit function  $\Phi_\varepsilon(x)$  such that

$$\Phi_\varepsilon(x) = \frac{1}{2} \|H_\varepsilon(x)\|^2.$$

In what follows, we present the two-step smoothing Levenberg-Marquardt method.

**Algorithm 1**

Step 0: Choose an initial point  $x_0 \in \mathbb{R}^n$  and  $\eta \in (0, 1), \alpha \in (0, 1), s \in (0, 1), m \in (0, 1), \sigma \in (0, 1 - \alpha/4)$ ,

$\gamma > 0, \varepsilon > 0$ . Set  $\kappa = \sqrt{2n}$ ,  $\beta_0 = \|H(x_0)\|$ ,  $\varepsilon_0 = (\alpha\beta_0/2\kappa)^2, k = 1$ .

Step 1: If  $\|V(x_k)^T H(x_k)\| \leq \varepsilon$ , where  $V(x_k) \in \partial_{\text{Cl}} H(x_k)$ , then terminates. Otherwise, go to Step 2.

Step 2: Set  $\lambda_k = \|H(x_k)\|^{\delta_k}$ , where

$$\delta_k = \begin{cases} \frac{1}{\|H(x_k)\|}, & \text{if } \Phi(x_k) \geq 1, \\ 1 + \frac{1}{k}, & \text{otherwise.} \end{cases}$$

Obviously,  $\delta_k \in (0, 2]$ .

Step 3: Solve the following linear system

$$(J(x_k)^T J(x_k) + \lambda_k I) d_k = -J(x_k)^T H_{\varepsilon_k}(x_k), \quad (7)$$

to obtain its solution  $d_{k_1}$ , where  $J(x_k)$  is the Jacobi of the function  $H_{\varepsilon_k}(x_k)$ . Set  $y_k = x_k + d_{k_1}$ .

Step 4: Solve the linear system

$$(J(x_k)^T J(x_k) + \lambda_k I) d_k = -J(x_k)^T H_{\varepsilon_k}(y_k), \quad (8)$$

to obtain  $d_{k_2}$ . Set  $d_k = d_{k_1} + d_{k_2}$ .

Step 5: Set  $\sigma_k = \min\{\sigma, \lambda_k/4\}$ . Select the smallest  $m_k$  from  $\mathbb{N}$  which satisfies the following inequality

$$\Phi_{\varepsilon_k}(x_k + s^{m_k} d_k) - \Phi_{\varepsilon_k}(x_k) \leq -\sigma_k s^{m_k} \|d_k\|^2. \quad (9)$$

Set  $t_k = s^{m_k}, x_{k+1} = x_k + t_k d_k$ .

Step 6: If

$$\|H(x_{k+1})\| \leq \max\{\eta\beta_k, \frac{\|H(x_{k+1}) - H_{\varepsilon_k}(x_{k+1})\|}{\alpha}\},$$

set  $\beta_{k+1} = \|H(x_{k+1})\|$  and choose  $\varepsilon_{k+1}$  such that

$$0 < \varepsilon_{k+1} \leq \min\{(\frac{\alpha}{2\kappa}\beta_{k+1})^2, m\varepsilon_k, \bar{\varepsilon}(x_{k+1}, \gamma\beta_{k+1})\},$$

where  $\bar{\varepsilon}(\cdot, \cdot)$  is defined by (4). Otherwise, set  $\beta_{k+1} = \beta_k, \varepsilon_{k+1} = m\varepsilon_k$ .

Step 7: Set  $k := k + 1$ , return to Step 1.

**Remark 1** Algorithm 1 is well defined since the line search condition such that

$$\Phi_{\varepsilon_k}(x_k + t_k d_k) - \Phi_{\varepsilon_k}(x_k) \leq -\sigma_k t_k \|d_k\|^2,$$

where  $\sigma_k = \min\{\sigma, \lambda_k/4\} > 0$  and  $t_k > 0$  small enough.

**Remark 2** It is important to force the smoothing parameter  $\varepsilon_k$  to go to zero in Algorithm 1. Since  $\varepsilon_{k+1} = m\varepsilon_k$ , the sequence  $\varepsilon_k$  generated is not increasing.

We next investigate the global convergence of Algorithm 1.

**Assumption 1** The solution set  $X$  of NCP is not empty.

**Assumption 2** There exists a neighborhood  $\Omega \subset X$  such that both  $H_\varepsilon(x)$  and its Jacobi  $J(x)$  are Lipschitz continuous on it, i.e., there exists constant  $L_1 > 0, L_2 > 0$  such that  $\|J(y) - J(x)\| \leq L_1 \|y - x\|$ , for all  $x, y \in \Omega$ , and

$$\|H_\varepsilon(y) - H_\varepsilon(x)\| \leq L_2 \|y - x\|, \quad \forall x, y \in \Omega.$$

Then, by the Lipschitzness of the Jacobi, there is

$$\|H_\varepsilon(y) - H_\varepsilon(x) - J(x)(y - x)\| \leq L_1 \|y - x\|^2, \quad (10)$$

and

$$\|J(x)\| \leq L_2. \quad (11)$$

In Algorithm 1, if  $\|V(x_k)^T H(x_k)\| \leq \varepsilon$  occurs at some iteration  $k$ , then Algorithm 1 terminates finitely a solution of the problem (1). Otherwise, Algorithm 1 generates a sequence  $\{(x_k, \varepsilon_k)\}$ , where  $\{\varepsilon_k\}$  is not increasing, based on which, we give the following assumption.

**Assumption 3** The sequence generated by Algorithm 1 has at least one accumulation point, which is a solution of NCP.

By the virtue of [11], we have Lemma 1 to Lemma 3 as follows.

**Lemma 1** For all  $k \geq 0$ , there is

$$\|H(x_k) - H_{\varepsilon_k}(x_k)\| \leq \alpha \|H(x_k)\|^2.$$

**Lemma 2** For all  $x \in \mathbb{R}^n, k \in \mathbb{N}$ , the following statement holds

$$\|H_{\varepsilon_{k+1}}(x_{k+1})\| + \kappa \sqrt{\varepsilon_{k+1}} \leq \|H_{\varepsilon_k}(x_k)\| + \kappa \sqrt{\varepsilon_k},$$

where  $\kappa = \sqrt{2n}$ .

Denote an index set  $K$  such that

$$K = \{0\} \cup \left\{ k \in \mathbb{N} : \|H(x_k)\| \leq \max\left\{ \eta\beta_{k-1}, \frac{\|H(x_k) - H_{\varepsilon_{k-1}}(x_k)\|}{\alpha} \right\} \right\}.$$

**Lemma 3** If the index set  $K$  is infinite, each accumulation point of  $\{x_k\}$  is a solution of NCP.

**Lemma 4** ([7]) Let  $\{x_k\} \subseteq \mathbb{R}^n$  and  $\{\varepsilon_k\}$  be two sequences with  $x_k \rightarrow x^*$  for some  $x^* \in \mathbb{R}^n$  and  $\varepsilon_k \downarrow 0$ . Then, there exists a subsequence  $\{\nabla\Phi_{\varepsilon_k}(x_k)\}_L$  such that  $\lim_{k \rightarrow \infty, k \in L} \nabla\Phi_{\varepsilon_k}(x_k) \in \partial_{\text{Cl}}\Phi(x^*)$ .

**Theorem 1** If  $\{x_k\}$  is a sequence generated by Algorithm 1 and  $x^*$  is an accumulation point of  $\{x_k\}$ , then  $0 \in \partial_{\text{Cl}}\Phi(x^*)$ .

*Proof:* If  $K$  is infinite, the conclusion follows by Lemma 3 immediately. Hence, we just need considering the case when  $K$  contains finitely many indices. Suppose  $K$  is finite,  $\tilde{k}$  is the largest element in  $K$ , and  $x^*$  is an accumulation point of the sequence such that  $x_k \rightarrow x^*$ ,  $k \in L$ . Without loss of generality, we suppose  $K \cap L = \emptyset$ . Then, from Step 6 of Algorithm 1, there is

$$\beta_k = \beta_{\tilde{k}} = \|H(x_{\tilde{k}})\|,$$

and

$$\|H(x_k)\| > \eta\beta_k > 0,$$

which implies that

$$\|\Phi(x_k)\| > \eta^2\|\Phi(x_{\tilde{k}})\| > 0, \quad k > \tilde{k}.$$

Next, we prove the statement by contradiction. Assume that  $x^*$  is not a stationary point of  $\Phi(x)$ , which implies that  $\Phi(x^*) \neq 0$ .

(i): We first show that there exists an index set  $L$  such that  $\{d_k\}_{k \in L} \subset \{d_k\}$ , and it is bounded. From the LM Eq. (7), we have

$$\|d_{k_2}\| = \|(J(x_k)^T J(x_k) + \lambda_k I)^{-1} J(x_k)^T H_{\varepsilon_k}(y_k)\|,$$

together with (10), there is

$$\begin{aligned} \|d_{k_2}\| &\leq \|(J(x_k)^T J(x_k) + \lambda_k I)^{-1} J(x_k)^T H_{\varepsilon_k}(x_k)\| \\ &\quad + \|(J(x_k)^T J(x_k) + \lambda_k I)^{-1} J(x_k)^T d_{k_1}\| \\ &\quad + L_1 \|d_{k_1}\|^2 \|(J(x_k)^T J(x_k) + \lambda_k I)^{-1} J(x_k)^T\|. \end{aligned} \quad (12)$$

Since  $d_{k_1}$  is the solution of the LM Eq. (7), there is

$$d_{k_1} = \|(J(x_k)^T J(x_k) + \lambda_k I)^{-1} J(x_k)^T H_{\varepsilon_k}(x_k)\|. \quad (13)$$

Suppose the singular value decomposition (SVD) of  $J(x_k)$  is as follows

$$J(x_k) = U_k \Sigma_k S_k^T = U_k \begin{pmatrix} \sigma_{k,1} & & & \\ & \sigma_{k,2} & & \\ & & \ddots & \\ & & & \sigma_{k,n} \end{pmatrix} S_k^T, \quad (14)$$

where  $U_k, S_k \in \mathbb{R}^{n \times n}$  are orthogonal matrices and  $\sigma_{k,1} \geq \dots \geq \sigma_{k,n} \geq 0$ , there is

$$\begin{aligned} \|(J(x_k)^T J(x_k) + \lambda_k I)^{-1}\| &= \|S_k(\Sigma_k^2 + \lambda_k I)^{-1} S_k^T\| \\ &= \|(\Sigma_k^2 + \lambda_k I)^{-1}\| \\ &= \max_{i=1,2,\dots,n} (\sigma_{k,i}^2 + \lambda_k)^{-1} \leq \lambda_k^{-1}. \end{aligned} \quad (15)$$

Hence, from (12), (13) and (15), we have

$$\begin{aligned} \|d_{k_2}\| &\leq \|d_{k_1}\| + \|(J(x_k)^T J(x_k) + \lambda_k I)^{-1}\| \|J(x_k)^T\| \|d_{k_1}\| \\ &\quad + L_1 \|d_{k_1}\|^2 \|(J(x_k)^T J(x_k) + \lambda_k I)^{-1}\| \|J(x_k)^T\| \\ &\leq \|d_{k_1}\| + \frac{L_2}{\lambda_k} \|d_{k_1}\| + \frac{L_1 L_2}{\lambda_k} \|d_{k_1}\|^2. \end{aligned} \quad (16)$$

Furthermore, since

$$\|d_{k_1}\| \leq \frac{\|J(x_k)H_{\varepsilon_k}(x_k)\|}{\|\lambda_k\|}, \quad (17)$$

from the LM Eq. (7) and  $\{\lambda_k\}$  is bounded for any  $k \in L$ , there exists an index set denoted as  $L$  that  $\{d_{k_1}\}_{k \in L}$  is bounded. Then, combining (16) with (17), we have  $d_k = d_{k_1} + d_{k_2}$  are bounded. Conveniently, suppose that there exists a convergent subsequence  $\{d_k\}_{k \in L}$  such that  $d_k \rightarrow d^*$ .

(ii): We next show that  $\liminf t_k = 0$ . If  $\liminf_{k \in L} t_k = t^* > 0$ , the line search rules in Algorithm 1 shows that

$$\Phi_{\varepsilon_k}(x_{k+1}) - \Phi_{\varepsilon_k}(x_k) \leq -t_k \sigma_k \|d_k\|^2,$$

for all  $k \in L$ . Since

$$d_k = d_{k_1} + d_{k_2},$$

It follows from LM Eqs. (7) and (8) that

$$\begin{aligned} d_{k_1} &= -[J(x_k)^T J(x_k) + \lambda_k I]^{-1} J(x_k) H_{\varepsilon_k}(x_k), \\ d_{k_2} &= -[J(x_k)^T J(x_k) + \lambda_k I]^{-1} J(x_k) H_{\varepsilon_k}(y_k), \end{aligned}$$

then

$$d_k = -[J(x_k)^T J(x_k) + \lambda_k I]^{-1} J(x_k) [H_{\varepsilon_k}(x_k) + H_{\varepsilon_k}(y_k)]. \quad (18)$$

According to (14) such that

$$J(x_k) = U_k \Sigma_k S_k^T = U_k \begin{pmatrix} \sigma_{k,1} & & & \\ & \sigma_{k,2} & & \\ & & \ddots & \\ & & & \sigma_{k,n} \end{pmatrix} S_k^T,$$

where  $U_k, S_k \in \mathbb{R}^{n \times n}$  are orthogonal matrices and  $\sigma_{k,1} \geq \dots \geq \sigma_{k,n} \geq 0$ , we obtain

$$(J(x_k)^T J(x_k) + \lambda_k I)^{-1} = S_k (\Sigma_k^2 + \lambda_k I)^{-1} S_k^T,$$

then

$$d_k = -S_k \Lambda_k S_k^T J(x_k) [H_{\varepsilon_k}(x_k) + H_{\varepsilon_k}(y_k)]. \quad (19)$$

where

$$\Lambda_k = \begin{pmatrix} \frac{1}{\sigma_{k,1}^2 + \lambda_k} & & & \\ & \frac{1}{\sigma_{k,2}^2 + \lambda_k} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_{k,n}^2 + \lambda_k} \end{pmatrix}.$$

Denote  $C(x_k) = J(x_k) [H_{\varepsilon_k}(x_k) + H_{\varepsilon_k}(y_k)]$ , then

$$\|d_k\|^2 = C(x_k)^T S_k \Lambda_k^2 S_k^T C(x_k), \quad (20)$$

Since  $x_k \rightarrow x^*$  ( $k \in L$ ),  $\varepsilon_k \rightarrow 0$  and  $\sigma_k = \min\{\sigma, \lambda_k/4\}$ , there exists some  $m_1 > 0$  such that

$\sigma_{k,i}^2 \leq m_1$  for  $k \in L, i = 1, \dots, n$ , which implies that some convergent subsequences such that

$$\sigma_{k,i}^2 \rightarrow \sigma_i^2,$$

exists. Considering that

$$\lambda_k = \|H(x_k)\|^{\delta_k},$$

where  $\delta \in (0, 2]$  holds for any  $k \in L$ , there exists a convergence subsequence of  $\{\lambda_k\}_{k \in L}$ . Denote it as  $\{\lambda_k\}_{k \in L}$  itself and  $\lambda_k \rightarrow \lambda^*, k \in L \rightarrow \infty$ . Moreover, suppose  $\lambda_k \leq m_2$  for convenience, there is

$$\|d_k\|^2 \geq M \|C(x_k)\|^2, \quad (21)$$

where  $M = 1/(m_1 + m_2)$ . So, we obtain from the line search rule in Step 5 that

$$\begin{aligned} \Phi_{\varepsilon_k}(x_{k+1}) - \Phi_{\varepsilon_k}(x_k) &\leq -\sigma_k t_k \|d_k\|^2 \\ &\leq -\sigma_k t_k M \|C(x_k)\|^2, \end{aligned}$$

for any  $k \in L$  large enough. However, since  $K$  is finite, there is

$$\{\sigma_k\} \rightarrow \sigma^* = \min\{\sigma, \frac{1}{4} \|H(x^*)\|^{\delta_k}\} > 0.$$

Letting  $c = \sigma^* t^* M \|C(x^*)\|^2 > 0$  and combining that  $\{\varepsilon_k\} \rightarrow 0$ , we have

$$\Phi(x_{k+1}) - \Phi(x_k) \leq -\frac{c}{2},$$

then, similar to the proof of Theorem 3.10 in [9], deducing from Lemmas 1 and 2, we obtain

$$\Phi(x_{l_{j+1}}) - \Phi(x_{l_j}) \leq -\frac{c}{4},$$

for all  $l_j$  sufficiently large, which contradicts the non-negativity of  $\Phi(x)$ . Hence, we obtain  $\liminf_{k \in L} t_k = 0$ .

(iii): Assume  $\liminf_{k \in L} t_k = 0$ , the reminder proof is similar to Theorem 3.4 in [12], since

$$\frac{\Phi_{\varepsilon_k}(x_k + s^{m_k-1} d_k) - \Phi_{\varepsilon_k}(x_k)}{s^{m_k-1}} > -\sigma_k \|d_k\|^2.$$

for all  $K \in L$  large enough, we have  $\nabla \Phi(x^*)^T d^* \geq -\sigma^* \|d^*\|^2$ , namely,

$$V^{*T} d^* \geq -\sigma^* \|d^*\|^2,$$

where  $d^*$  is the limit of  $\{d_k\}_L$  and  $\sigma^*$  is the limit of  $\{\sigma_k\}_L$ ,  $V^* \in \partial_{\text{cl}} \Phi(x^*)$ . Suppose  $\{S_k\}_L \rightarrow S^*$ ,  $\Lambda^* = \text{diag}(\frac{1}{\sigma_1^2 + \lambda^*}, \dots, \frac{1}{\sigma_n^2 + \lambda^*})$ , together with (19), one has

$$(V^*)^T S^* (\sigma^* (\Lambda^*)^2 - \Lambda^*) (S^*)^T V^* \geq 0,$$

which implies  $V^* = 0$  and  $\Phi(x^*) = 0$ . This contradicts  $\Phi(x^*) \neq 0$ . The proof is completed.  $\square$

**Remark 3** The Armijo line search plays an important rule in the proof of Theorem 1. Although the two-step and three-step LM methods have been proved with global convergence [18, 19], they are all applied for solving smoothing systems of equations, the global convergence is valuable when the two-step technology is used for solving a nonsmooth system.

### LOCAL CONVERGENCE UNDER LOCAL ERROR BOUND

In this section, we investigate the local convergence of the algorithm under the local error bound.

Give the error bound assumption firstly.

**Assumption 4**  $H$  provides a local error bound on some neighborhood of  $\bar{x} \in X$ , i.e., there exists constants  $c_1 > 0$  such that

$$c_1 \text{dist}(x, X) \leq \|H(x)\|, \quad \forall x \in N(\bar{x}, r), \quad (22)$$

where  $\text{dist}(x, X) = \inf_{y \in X} \|y - \bar{x}\|$  and  $N(\bar{x}, r)$  is a neighborhood of  $\bar{x}$ .

We recall the SVD of  $J(x_k)$ , which is the Jacobi of the function  $H_\varepsilon(x_k)$  and is necessary for the discussion of the local convergence. More specially, denote

$$\begin{aligned} U_k &= (U_{k,1}, U_{k,2}, U_{k,3}), \\ \Sigma_k &= \begin{pmatrix} \Sigma_{k,1} & & \\ & \Sigma_{k,2} & \\ & & 0 \end{pmatrix}, \quad S_k = \begin{pmatrix} S_{k,1}^T \\ S_{k,2}^T \\ S_{k,3}^T \end{pmatrix}, \end{aligned}$$

where  $\Sigma_{k,1} = \text{diag}(\sigma_{k,1}, \dots, \sigma_{k,r})$  with  $\sigma_{k,1} \geq \sigma_{k,2} \geq \dots \geq \sigma_{k,r} > 0$ , and  $\Sigma_{k,2} = \text{diag}(\sigma_{k,r+1}, \dots, \sigma_{k,r+q})$  with  $\sigma_{k,r+1} \geq \sigma_{k,r+2} \geq \dots \geq \sigma_{k,r+q} > 0$ , when we neglect the subscription  $k$  in  $\Sigma_{k,i}$  for convenience,  $J(x_k)$  denoted as  $J_k$  is such that

$$J_k = U_1 \Sigma_1 S_1^T + U_2 \Sigma_2 S_2^T.$$

Furhermore, denote  $H_{\varepsilon_k} = H_\varepsilon(x_k)$ . Then, from (7) and (8) in Algorithm 1, we conclude that

$$\begin{aligned} d_{k_1} &= -S_1(\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T H_{\varepsilon_k} \\ &\quad - S_2(\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T H_{\varepsilon_k}, \\ d_{k_2} &= -S_1(\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T H_{\varepsilon_k}(y_k) \\ &\quad - S_2(\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T H_{\varepsilon_k}(y_k), \end{aligned}$$

and

$$\begin{aligned} H_{\varepsilon_k} + J_k d_{k_1} &= H_{\varepsilon_k} - U_1(\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T H_{\varepsilon_k} \\ &\quad - U_2(\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T H_{\varepsilon_k} \\ &= \lambda_k U_1(\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T H_{\varepsilon_k} \\ &\quad + \lambda_k U_2(\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T H_{\varepsilon_k} + U_3 U_3^T H_{\varepsilon_k}, \end{aligned}$$

$$H_{\varepsilon_k}(y_k) + J_k d_{k_2} = \lambda_k U_1 (\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1 H_{\varepsilon_k}(y_k) \\ + \lambda_k U_2 (\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2 H_{\varepsilon_k}(y_k) + U_3 U_3^T H_{\varepsilon_k}(y_k).$$

Then, similar to the proof of Lemma 3.4 in [15], we conclude Lemma 5 and Lemma 6 immediately.

**Lemma 5** Suppose Assumptions 1, 2, and 4 hold. If  $x_k \in N(\bar{x}, r/2)$ , then there is

- (a)  $\|U_1 U_1^T H_{\varepsilon_k}(y_k)\| \leq c_2 \|\bar{x}_k - x_k\|^2$ ,
- (b)  $\|U_2 U_2^T H_{\varepsilon_k}(y_k)\| \leq c_3 \|\bar{x}_k - x_k\|^3$ ,
- (c)  $\|U_3 U_3^T H_{\varepsilon_k}(y_k)\| \leq c_4 \|\bar{x}_k - x_k\|^3$ ,

where  $c_2, c_3, c_4$  are positive constants.

**Lemma 6** Suppose Assumptions 1 and 4 hold. If  $\text{dist}(x_k, X) = \|x_k - \bar{x}_k\|$ ,  $\bar{x}_k \in X$  and  $x_k, y_k \in N(\bar{x}, r/2)$ , there are positive constants  $c_5, c_6, c_7$  such that

$$\|d_{k_1}\| \leq c_5 \|\bar{x}_k - x_k\|, \quad \|d_{k_2}\| \leq c_6 \|\bar{x}_k - x_k\|,$$

and

$$\|H_{\varepsilon_k}(y_k) + J_k d_{k_2}\| \leq c_7 \|\bar{x}_k - x_k\|^{2+\delta_k}.$$

*Proof:* Consider  $d_{k_1}$  is the minimizer of an optimization problem such that

$$\min_{d \in \mathbb{R}^n} q(d) = \|H_{\varepsilon_k} + J_k d\|^2 + \lambda_k \|d\|^2,$$

where  $J_k$  is the Jacobi of  $H_{\varepsilon_k}$ , it is obvious that

$$\|d_{k_1}\|^2 \leq \frac{1}{\lambda_k} q(d_k) \\ \leq \frac{1}{\lambda_k} q(\bar{x}_k - x_k) \\ = \frac{\|H_{\varepsilon_k} + J_k(\bar{x}_k - x_k)\|^2}{\lambda_k} + \|\bar{x}_k - x_k\|^2, \\ \leq \frac{L_1^2 \|\bar{x}_k - x_k\|^4}{\lambda_k} + \|\bar{x}_k - x_k\|^2.$$

Since

$$\lambda_k = \|H(x_k)\|^{\delta_k} \geq c_1^{\delta_k} \|\bar{x}_k - x_k\|^{\delta_k}, \quad (23)$$

we have

$$\|d_{k_1}\|^2 \leq L_1^2 c_1^{-\delta_k} \|\bar{x}_k - x_k\|^{4-\delta_k} + \|\bar{x}_k - x_k\|^2,$$

which gives

$$\|d_{k_1}\| \leq c_5 \|\bar{x}_k - x_k\|,$$

where  $c_5 > 0$ . Then, combine (16) such that

$$\|d_{k_2}\| \leq \|d_{k_1}\| + \frac{L_2}{\lambda_k} \|d_{k_1}\| + \frac{L_1 L_2}{\lambda_k} \|d_{k_1}\|^2,$$

with (23), we have  $\|d_{k_2}\| \leq c_6 \|\bar{x}_k - x_k\|$ , where  $c_6 > 0$ .

Next, we show that  $\|H_{\varepsilon_k}(y_k) + J_k d_{k_2}\| \leq c_7 \|\bar{x}_k - x_k\|^2$ . In fact, similar the proof of the inequality (3.36) in [19], we have

$$\|H_{\varepsilon_k}(y_k) + J_k d_{k_2}\| \\ = \lambda_k U_1 (\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T H_{\varepsilon_k}(y_k) \\ + \lambda_k U_2 (\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T H_{\varepsilon_k}(y_k) + U_3 U_3^T H_{\varepsilon_k}(y_k) \\ \leq \lambda_k \|\Sigma_1^{-1}\| \|U_1 U_1^T H_{\varepsilon_k}(y_k)\| + \lambda_k \|\Sigma_2^{-1}\| \|U_2 U_2^T H_{\varepsilon_k}(y_k)\| \\ + \|U_3 U_3^T H_{\varepsilon_k}(y_k)\| \\ \leq O(\|\bar{x}_k - x_k\|)^{2+\delta_k} + O(\|\bar{x}_k - x_k\|)^{3+\delta_k} + O(\|\bar{x}_k - x_k\|)^3 \\ \leq O(\|\bar{x}_k - x_k\|)^{2+\delta_k}. \quad (24)$$

Since  $\delta_k \in (0, 2]$ , one has

$$\|H_{\varepsilon_k}(y_k) + J_k d_{k_2}\| \leq c_7 \|\bar{x}_k - x_k\|^{2+\delta_k},$$

where  $c_7 > 0$ .  $\square$

**Theorem 2** Suppose  $\{x_k\}$  is generated by Algorithm 1 and Assumptions 1 and 4 hold, then  $\{x_k\}$  converges to some solutions quadratically.

*Proof:* By the virtue of Assumptions 1 and 4 and Lemma 1, there is

$$c_1 \|\bar{x}_{k+1} - x_{k+1}\| \leq \|H(x_{k+1})\| \\ \leq \|H_{\varepsilon_k}(x_{k+1})\| + \alpha \|H(x_k)\|^2 \\ \leq \|H_{\varepsilon_k}(y_k + d_{k_2})\| + \alpha \|H(x_k)\|^2 \\ \leq \|H_{\varepsilon_k}(y_k) + J(y_k) d_{k_2}\| + L_1 \|d_{k_2}\|^2 + \alpha \|H(x_k)\|^2 \\ \leq \|H_{\varepsilon_k}(y_k) + J_k d_{k_2}\| + \|J(y_k) - J_k\| \|d_{k_2}\| + L_1 \|d_{k_2}\|^2 \\ + \alpha \|H(x_k)\|^2,$$

then, together with (24) and (5) such that

$$\|H(x_k)\|^2 \leq O(\|\bar{x}_k - x_k\|^{2(1+p)}),$$

where  $p \geq 0$ , and the Lipschitz continuous of  $J(x)$  that

$$\|J(y_k) - J_k\| \leq L_1 \|\bar{x}_k - x_k\|,$$

we have

$$c_1 \|\bar{x}_{k+1} - x_{k+1}\| \leq O(\|\bar{x}_k - x_k\|^{2+\delta_k}) \\ + O(\|\bar{x}_k - x_k\|^2) + O(\|\bar{x}_k - x_k\|^{2(1+p)}),$$

which implies that  $\|\bar{x}_{k+1} - x_{k+1}\| \leq O(\|\bar{x}_k - x_k\|^2)$ .  $\square$

**Remark 4** In [11], Yu and Pu proposed one-step smoothing LM method for NCP, where the local convergence order is superlinear under the local error bound when  $H$  is strong semismooth. However, we obtained that the local convergence of two-step LM method is quadratic when  $H$  is semismooth.

**NUMERICAL RESULT**

In this section, we give some examples to illustrate the effectiveness of the present algorithm and compare it with the smoothing LM methods with one-step iterative direction proposed in [11] (denoted by SLM) and [12] (denoted by MSLM) at different initial points. Here, we code the algorithm in Matlab R2010a, Windows 10(64), 4G Memory, 2.4 GHZ. In what follows, we set  $\eta = 0.8$ ,  $\alpha = 0.7$ ,  $\sigma = 0.015$ ,  $s = 0.5$ ,  $\gamma = 10$ ,  $m = 0.75$ . The stop criterion is  $\|V(x_k)^T H(x_k)\| \leq 1.0e-6$ .

**Example 1** Let  $F(x) = (f_1(x), f_2(x), f_3(x))^T$ , where

$$\begin{aligned} f_1(x) &= x_1 - 2, \\ f_2(x) &= x_2 - x_3 + x_2^3 + 3, \\ f_3(x) &= x_2 + x_3 + 2x_3^3 - 3. \end{aligned}$$

The problem has the solution  $(2, 0, 1)^T$ .

**Example 2** (Kojima-Shindo Problem) Let  $F(x) = (f_1(x), f_2(x), f_3(x), f_4(x))^T$ , where

$$\begin{aligned} f_1(x) &= 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6, \\ f_2(x) &= 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2, \\ f_3(x) &= 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9, \\ f_4(x) &= x_1^2 + 2x_2^2 + 2x_3 + 3x_4 - 3. \end{aligned}$$

There exist solutions  $(\sqrt{6}/2, 0, 0, 1/2)^T$  and  $(1, 0, 3, 0)^T$  in this problem.

**Example 3** Denote  $G(x) = (g_1(x), g_2(x), \dots, g_n(x))^T$ , where

$$\begin{aligned} g_i(x) &= -(n+1) + x_i + \sum_{j=1}^n x_j, \quad i = 1, \dots, n-1, \\ g_n(x) &= -1 + \prod_{j=1}^n x_j. \end{aligned}$$

**Table 1** Numerical results for Example 1.

Initial point	SLM			MSLM			Algorithm 1		
	Iter	$\ H(x)\ $	CPU (s)	Iter	$\ H(x)\ $	CPU (s)	Iter	$\ H(x)\ $	Cpu (s)
rand(1,3)	6	2.2712e-09	3.86	4	7.6921e-07	3.88	4	2.4252e-15	3.02
5rand(1,3)	8	4.8887e-09	5.13	7	1.1556e-09	3.61	5	1.9281e-12	3.53
10rand(1,3)	14	1.7476e-10	8.53	8	1.0413e-09	4.50	7	1.3958e-07	4.72
100rand(1,3)	39	4.8428e-07	22.19	14	8.0567e-08	8.26	8	1.2880e-07	5.50

**Table 2** Numerical results for Example 2.

Initial point	SLM			MSLM			Algorithm 1		
	Iter	$\ H(x)\ $	CPU (s)	Iter	$\ H(x)\ $	CPU (s)	Iter	$\ H(x)\ $	CPU (s)
$(1, 2, 1, 2)^T$	10	4.4497e-09	8.78	10	1.2116e-09	8.36	6	5.5643e-13	6.17
$(2, 1, 1, 2)^T$	7	1.1143e-09	5.48	6	1.5078e-06	5.19	7	1.6882e-12	4.84
$(10, 10, 10, 10)^T$	14	6.0516e-14	12.44	12	3.4600e-10	10.87	9	2.5746e-17	8.21
$(100, 100, 100, 100)^T$	-	-	-	19	2.0228e-10	15.98	19	8.8180e-11	16.85
$(1000, 1000, 1000, 1000)^T$	23	2.0187e-14	16.95	22	2.9940e-12	18.74	13	2.7798e-08	13.59

Let  $F(x) = (f_1(x), f_2(x), \dots, f_n(x))^T$ , where

$$f_i(x) = \begin{cases} g_i(x) - g_i(x^*) + 1, & \text{if } i \text{ is odd,} \\ g_i(x) - g_i(x^*), & \text{otherwise,} \end{cases}$$

$x^* = (0, 1, 0, 1, \dots)^T \in \mathbb{R}^n$ . Obviously,  $x^*$  is a unregenerate solution of NCP

Numerical results are listed in Table 1–Table 3 respectively, where “-” in the tables means that the number of iterations is more than 50. Meanwhile, we take Kojima-Shindo problem as an example, and describe the trend of  $\|H(x_k)\|$  with the iterative steps. One random comparison results of MSLM and the present algorithm are shown in Fig. 1, when the initial points are chosen from 10rand(1, 4), 100rand(1, 4), and 1000rand(1, 4).

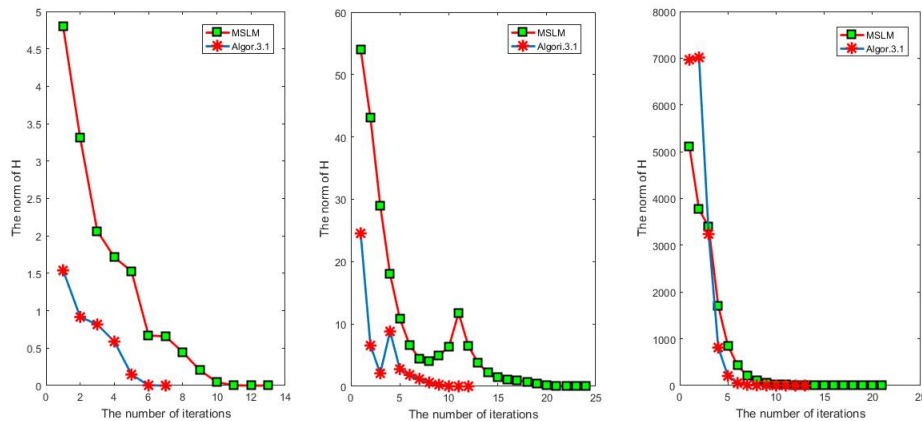
As seen from Table 1–Table 3, the number of iterations and CPU time of the present algorithm are always less, when compared with SLM and MSLM. However, it can be seen that there are two cases, such as the initial points 10rand(1, 3) in Table 1 and  $(100, 100, 100, 100)^T$  in Table 2, where the CPU time of the present algorithm is a little more than MSLM, either the iterative steps is less or the calculation accuracy is improved.

**CONCLUSION**

In this paper, we proposed a two-step smoothing Levenberge-Marquardt method. Compared with the one-step smoothing LM methods proposed in [11] and [12], the present algorithm is effective, and always need less iterations and CPU time. However, a line search rule is adopted in this paper and there also exists cases that the CPU time of the present algorithm is a little more. Maybe a two-step LM algorithm with trust region technology will be better, which is our future topic to research.

**Table 3** Numerical results for Example 3.

Dim	Initial point	SLM			MSLM			Algorithm 1		
		Iter	$\ H(x)\ $	CPU (s)	Iter	$\ H(x)\ $	CPU (s)	Iter	$\ H(x)\ $	CPU (s)
4	$(1, 0, 0, 1)^T$	8	4.8183e-22	6.75	6	8.8448e-07	5.45	3	1.4860e-08	3.12
4	$(10, 10, 10, 10)^T$	14	7.4724e-16	10.22	11	1.4533e-08	9.29	7	3.2901e-08	6.80
5	$(1, 2, \dots, 5)^T$	13	1.8589e-13	14.76	9	2.4660e-08	10.28	7	1.1512e-17	9.05
5	$(10, 10, \dots, 10)^T$	15	3.3329e-13	17.17	11	7.2029e-07	12.31	7	2.5436e-10	8.95
8	$(10, 10, \dots, 10)^T$	14	3.7718e-08	32.14	12	4.8087e-16	25.30	8	4.3555e-16	21.44

**Fig. 1** The comparison of the number of iterations between MSLM and Algorithm 1.

**Acknowledgements:** The authors are grateful to the anonymous referees for their helpful comments and suggestions. This work was supported by the National Science Foundation of China (nos. 12101198 and 72071130).

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