RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2023.022

Neighbor sum distinguishing total choosability of triangle-free IC-planar graphs

Fugang Chao, Chao Li*, **Donghan Zhang**

School of Mathematics and Computer Application, Shangluo University, Shangluo 726000 China

*Corresponding author, e-mail: lichao_math@163.com

Received 21 Feb 2022, Accepted 13 Jan 2023
Available online

ABSTRACT: A proper total coloring ϕ of G with $\sum_{z\in E_G(u)\cup\{u\}}\phi(z)\neq\sum_{z\in E_G(v)\cup\{v\}}\phi(z)$ for each $uv\in E(G)$ is called a neighbor sum distinguishing (NSD) total coloring. Pilśniak and Woźniak conjectured that every graph with maximum degree Δ exists an NSD total $(\Delta+3)$-coloring. In this paper, we improve the results of Song et al [Acta Math Sin (Engl Ser) **36**(2020):292–304] to the list version by applying the combinatorial nullstellensatz.

KEYWORDS: IC-planar graphs, neighbor sum distinguishing total choosability, combinatorial nullstellensatz

MSC2020: 05C15

INTRODUCTION

We only consider the finite, simple and undirected graphs here. For undefined terminology and notations, we follow [3].

Let $G=(V(G),E(G))$ represent a graph. For any $u\in V(G)$, set $E_G(u)=\{uv|\ uv\in E(G)\}$, $N_G(u)=\{v|\ uv\in E(G)\}$ and $d_G(u)=|E_G(u)|$. Denote by $\Delta=\Delta(G)$ the maximum degree of G. A cycle with length t (at least t, at most t) is called a t-cycle (t^+-cycle, t^--cycle). In particular, a 3-cycle is called a triangle.

Let $k\geqslant 0$ be an integer number. A mapping $\phi: V(G)\cup E(G)\to\{1,2,\dots,k\}$ is called a neighbor sum distinguishing (NSD) total coloring of G if it satisfies the below conditions:

(i) $\phi(u)\neq\phi(v)$, $\phi(u)\neq\phi(uv)$ and $\phi(v)\neq\phi(uv)$ for every $uv\in E(G)$;

(ii) $\sum\limits_{z\in E_G(u)\cup\{u\}}\phi(z)\neq\sum\limits_{z\in E_G(v)\cup\{v\}}\phi(z)$ for each $uv\in E(G)$.

Denoted by $\chi_\Sigma^t(G)=\min\{k|\ G$ has an NSD k-total coloring$\}$, the NSD total chromatic number of G. In 2015, Pilśniak and Woźniak [5] first introduced the coloring and given an important conjecture below.

Conjecture 1 ([5]) *For graph G, $\chi_\Sigma^t(G)\leqslant\Delta(G)+3$.*

Conjecture 1 was confirmed for some special classes of graphs, such as complete graphs, bipartite graphs, subcubic graphs [5], planar graphs with $\Delta\geqslant 10$ [11] and planar graphs with $\Delta\geqslant 7$ without triangles [10].

An IC-planar graph is a graph that can be embedded in the plane such that each vertex is incident with at most one crossing. In 2008, Alberson [1] introduced the concept of the IC-plane graph and studied its coloring. In 2020, Song et al [8] given below theorem about IC-planar graphs.

Theorem 1 ([8]) *For each IC-planar graph G with $\Delta\geqslant 7$, if it contains no triangle, then $\chi_\Sigma^t(G)\leqslant\Delta(G)+3$.*

A mapping L is called a k-list total assignment of G if it assigns a set $L(z)$ consisting of k real numbers to each member $z\in V(G)\cup E(G)$. For a k-list total assignment L of G, a mapping ϕ is called an NSD total L-coloring of G if it satisfies the below conditions:

(i) ϕ is an NSD total coloring of G;

(ii) $\phi(z)\in L(z)$ for each $z\in V(G)\cup E(G)$.

Denoted by $\mathrm{ch}_\Sigma^t(G)=\min\{k|\ G$ has an NSD k-total coloring for any k-list total assignment $L\}$ the NSD total choice number of G. Obviously, $\chi_\Sigma^t(G)\leqslant\mathrm{ch}_\Sigma^t(G)$.

There is the below conjecture on the NSD total choice number.

Conjecture 2 ([5]) *For graph G, $\mathrm{ch}_\Sigma^t(G)\leqslant\Delta(G)+3$.*

For Conjecture 2, there are many classes of graphs satisfying it; refer to [4, 6, 7, 12] for details.

Here, we extend the result of Theorem 1 to the list version and obtain the below result.

Theorem 2 *For each IC-planar graph G with $\Delta\geqslant 7$, if it contains no triangle, then $\mathrm{ch}_\Sigma^t(G)\leqslant\Delta(G)+3$.*

PRELIMINARIES

Set $uv\in E(G)$. We call u an ℓ-vertex (resp., ℓ^+-vertex, ℓ^--vertex) and an ℓ-neighbor (resp., ℓ^+-neighbor, ℓ^--neighbor) of v if $d_G(u)=\ell$ ($d_G(u)\geqslant\ell$, $d_G(u)\leqslant\ell$). Let $n_G^\ell(u)$ ($n_G^{\ell^+}(u)$, $n_G^{\ell^-}(u)$) denote the number of ℓ-vertices (resp., ℓ^+-vertices, ℓ^--vertices) adjacent to u in G.

Below, we introduce two important lemmas, which will be applied to the proof of the main result.

Lemma 1 ([2]) *For an arbitrary field $\mathbb{F}$, let $P\in\mathbb{F}[x_1,\dots,x_n]$ with $\deg(P)=\sum_{k=1}^n i_k$, where $i_k\geqslant 0$ is an integer. If the coefficient $c_P(x_1^{i_1},\dots,x_n^{i_n})$ of the monomial $x_1^{i_1}\cdots x_n^{i_n}$ in P is nonzero, and if $S_1,\dots,S_n$ are subsets of $\mathbb{F}$ with $|S_k|>i_k$, then there are $s_1\in S_1,\dots,s_n\in S_n$ such that $P(s_1,\dots,s_n)\neq 0$.*

Lemma 2 ([7]) *Assume that S_i is a set of real numbers with $|S_i| = p_i$, $1 \leqslant i \leqslant t$. Let $S = \{\sum_{i=1}^{t} x_i \mid x_i \in S_i$ and $x_i \neq x_j$ for $1 \leqslant i < j \leqslant t\}$. Then $|S| \geqslant \sum_{i=1}^{t} p_i - t^2 + 1$.*

PROOF OF Theorem 2

Unavoidable configurations

Let G be an example not satisfying Theorem 2 with minimal $|E(G)|$. Set $k = \max\{\Delta(G)+3, 10\}$. By the minimality, any subgraph G' of G has an NSD total L-coloring ϕ' for any k-list total assignment L of G. We will extend the NSD total L-coloring ϕ' of G' to an NSD total L-coloring ϕ of G to obtain a contradiction. Let $m(u) = \sum_{z \in E_G(u) \cup \{u\}} \phi(z)$. In the coloring ϕ', the definition of $m'(u)$ is the same as $m(u)$. Not stated otherwise, $\phi(z) = \phi'(z)$ for any $z \in (V(G) \cup E(G)) \cap (V(G') \cup E(G'))$. For any $z \in V(G) \cup E(G)$, Let $S(z)$ denote the set of the available colors for z.

For a 3^--vertex v, since there are at most 9 forbidden colors for it and $|L(v)| \geqslant 10$, it can always be colored properly. For simplicity, we will omit the colors of all 3^--vertices in the following proof.

Claim 1 *If v is a 3^--vertex of G, then $n_G^{4^-}(v) = 0$.*

Proof: Suppose to be contrary that G has an edge vv_1 with $d_G(v) \leqslant 3$ and $d_G(v_1) \leqslant 4$. Let $G' = G - vv_1$. Then G' has an NSD total L-coloring ϕ'. In order to get an NSD total L-coloring ϕ of G. We first delete the colors on v and v_1. Then $|S(v_1)| \geqslant 10 - 2(4-1) = 4$ and $|S(vv_1)| \geqslant 10 - (3-1) - (4-1) = 5$. By Lemma 2, we have at least six different color combinations to color v_1 and vv_1 properly. Thus there is at least one color combination such that $m(v_1) \neq m(z)$ for each $z \in N_G(v_1) \backslash \{v\}$. Note that v is a 3^--vertex. So the ϕ' can be extended to an NSD total L-coloring ϕ of G, a contradiction. □

Claim 2 *If v is a 4-vertex of G, then $n_G^{4}(v) \leqslant 1$.*

Proof: Suppose to be contrary that G has a 4-vertex v with two 4-neighbors v_1 and v_2. Let $N_G(v) = \{v, v_2, v_3, v_4\}$, $N_G(v_1) = \{v, v_{11}, v_{12}, v_{13}\}$ and $N_G(v_2) = \{v, v_{21}, v_{22}, v_{23}\}$. Then G' has an NSD total L-coloring ϕ', where $G' = G - vv_1 - vv_2$. To extend the coloring ϕ' to an NSD total L-coloring ϕ of G, we first erase the colors on v, v_1 and v_2. Then $|S(v)| \geqslant 10-2(4-2) = 6$, $|S(vv_i)| \geqslant 10-(4-2+4-1) = 5$ and $|S(v_i)| \geqslant 10 - 2(4-1) = 4$ $(i = 1,2)$. We associate v, vv_1, vv_2, v_1 and v_2 with the variables x_1, x_2, x_3, x_4, x_5, respectively.

Let $m(v) = \sum_{i=1}^{3} x_i + m'(v) - \phi'(v)$, $m(v_i) = \sum_{j \in \{i+1, i+3\}} x_j + m'(v_i) - \phi'(v_i)$ $(i = 1,2)$ and

$$P(x_1, \cdots, x_5) = \prod_{i=1}^{2} \big(m(v) - m(v_i)\big)\big(m(v) - m'(v_{i+2})\big)$$
$$\times \prod_{i=1}^{2}\prod_{j=1}^{3} \big(m(v_i) - m'(v_{ij})\big) \prod_{1 \leqslant i < j \leqslant 3} (x_i - x_j) \prod_{4 \leqslant i \leqslant 5} (x_1 - x_i) \prod_{2 \leqslant i \leqslant 3} (x_i - x_{i+2}).$$

With the help from MATHEMATICA, we have $c_P(x_1 x_2 \prod_{i=1}^{5} x_i^3) = 16$. By Lemma 1, there exists a color $s_1 \in S(v)$, a color $s_2 \in S(vv_1)$, a color $s_3 \in S(vv_2)$, a color $s_4 \in S(v_1)$ and a color $s_5 \in S(v_2)$ such that $P(s_1, \ldots, s_5) \neq 0$. Hence, the ϕ' can be extended to an NSD total L-coloring ϕ of G by coloring vv_1, vv_2 with colors s_2, s_3 and recoloring v, v_1, v_2 with colors s_1, s_4, s_5, respectively. It is a contradiction. The claim holds. □

The following Claims 3–5 follow from the similar proof of Claim 2.

Claim 3 *If v is an ℓ-vertex of G with $5 \leqslant \ell \leqslant 6$, then*
(i) $n_G^{2^-}(v) \leqslant \ell - 5$.
(ii) $n_G^{3}(v) \leqslant \ell - 4$.
(iii) $n_G^{3}(v) = 0$ *when* $n_G^{2^-}(v) = 1$.
(iv) $n_G^{4}(v) = 0$ *when* $n_G^{3^-}(v) = \ell - 4$.

Claim 4 *If v is a 7-vertex of G, then*
(i) $n_G^{2^-}(v) \leqslant 2$.
(ii) $n_G^{3}(v) \leqslant 5$.
(iii) $n_G^{3^-}(v) \leqslant 2$ *when* $n_G^{2^-}(v) \geqslant 1$.

Claim 5 *If v is an ℓ-vertex of G with $8 \leqslant \ell \leqslant 15$, then*
(i) $n_G^{2^-}(v) \leqslant \lfloor \frac{3\ell}{7} \rfloor$.
(ii) $n_G^{3^-}(v) \leqslant \lfloor \frac{3\ell}{7} \rfloor$ *when* $n_G^{2^-}(v) \geqslant 1$.

The following Claim 6 holds by applying the similar proof of Claim 1.

Claim 6 *If v is an ℓ-vertex of G with $\ell \geqslant 16$, then $n_G^{2^-}(v) \leqslant \lceil \frac{\ell}{2} \rceil - 1$.*

Set $V_{2^-}(G) = \{v \mid d_G(v) \leqslant 2,\ v \in V(G)\}$ and $H = G - V_{2^-}(G)$.

By Claims 1–6, we can directly obtain the following Eq. (1).

$$\begin{cases} n_G^{2^-}(u) = 0 & \text{if } d_G(u) \leqslant 5; \\ n_G^{2^-}(u) \leqslant d_G(u) - 5 & \text{if } d_G(u) \in \{6,7,8\}; \\ n_G^{2^-}(u) \leqslant d_G(u) - 6 & \text{if } d_G(u) \in \{9,10\}; \\ n_G^{2^-}(u) \leqslant d_G(u) - 7 & \text{if } d_G(u) \in \{11,12\}; \\ n_G^{2^-}(u) \leqslant d_G(u) - 8 & \text{if } d_G(u) \geqslant 13. \end{cases} \quad (1)$$

Claim 7 *If v is a vertex of H, then*
(i) $d_H(v) \geqslant 3$.
(ii) $d_H(v) = d_G(v)$ *when* $3 \leqslant d_G(v) \leqslant 5$.
(iii) $n_H^{4}(v) \leqslant 1$ *when* $d_H(v) = 4$.
(iv) $n_H^{3}(v) \leqslant \ell - 4$ *when* $d_H(v) = \ell$ *with* $4 \leqslant \ell \leqslant 6$.
(v) $n_H^{4}(v) = 0$ *when* $d_H(v) = \ell$ *and* $n_H^{3}(v) = \ell - 4$ *with* $5 \leqslant \ell \leqslant 6$.
(vi) $n_H^{3}(v) \leqslant 5$ *when* $d_H(v) = 7$.

Proof: Note that $d_G(v) \geqslant 3$ by the definition of H. (i) and (ii) follow from Eq. (1). We prove (iii) as follows.

Suppose that $n_H^{4}(v) \geqslant 2$ when $d_H(v) = 4$. Then $d_G(v) \geqslant 4$ by the definition of H. In the following, we show $d_G(v) = 4$. Assume that $d_G(v) \geqslant 5$. Then

$d_H(v) = d_G(v) - n_G^{2^-}(v) \geqslant 5$ by Eq. (1) and the definition of H, a contradiction. Thus, $d_G(v) = d_H(v) = 4$. And so $n_G^4(v) \leqslant 1$ by Claim 2. Therefore, v has at least one 4-neighbor v_1 in H with $d_G(v_1) \geqslant 6$ by (ii). And so $d_H(v_1) = d_G(v_1) - n_G^{2^-}(v_1) \geqslant 5$ by Eq. (1) and the definition of H, a contradiction. Hence, the statement (iii) holds.

With a similar proof to that of (iii), we can prove that (iv)–(vi) holds. □

Discharging process

In a plane graph, a t-face (resp., a t^+-face, a t^--face) is a face with degree t (resp., at least t, at most t).

Below, let G embed on a plane so that every vertex is incident with at most one crossing and the number of crossings is minimal. By turning all crossings of G into new 4-vertices on the plane, we obtain a planar graph $G^\times$ and call it the associated planar graph of G. To avoid confusion, a vertex in $G^\times$ is called *false* if it is not a vertex of G and *real* otherwise. We call a face f in $G^\times$ a *false* face if it is incident with one false vertex and a *real* face otherwise.

Let $H^\times$ be the associated planar graph of H. Set $\omega(v) = d_{H^\times}(v) - 4$ for any $z \in V(H^\times)$ and $\omega(f) = d_{H^\times}(f) - 4$ for any $f \in F(H^\times)$. By

$$|V(H^\times)| - |E(H^\times)| + |F(H^\times)| = 2$$

and

$$\sum_{v \in V(H^\times)} d_{H^\times}(v) = \sum_{f \in F(H^\times)} d_{H^\times}(f) = 2|E(H^\times)|.$$

One can obtain

$$\sum_{v \in V(H^\times)} (d_{H^\times}(v) - 4) + \sum_{f \in F(H^\times)} (d_{H^\times}(f) - 4) = -8.$$

Next, we make some discharging rules to redistribute charges among vertices and faces under the total charges unchanged. For simplicity, a real ℓ-vertex is still called an ℓ-vertex in the following. The rules are given as follows.

(R1) Assume that z is a false 4-vertex and x is a neighbor of z in $H^\times$.

(R1.1) If $d_{H^\times}(x) = 5$ and $n^3_{H^\times}(x) = 0$, then z receives 1 from x.

(R1.2) If $d_{H^\times}(x) = 5$ and $n^3_{H^\times}(x) = 1$, then z receives 2/3 from x.

(R1.3) If $d_{H^\times}(x) \geqslant 6$ and $n^3_{H^\times}(x) = 0$, then z receives 2 from x.

(R1.4) If $d_{H^\times}(x) \geqslant 6$ and $n^3_{H^\times}(x) \geqslant 1$, then z receives 4/3 from x.

(R2) If $d_{H^\times}(z) = 3$, then it receives 1/3 from each $x \in N_{H^\times}(z)$.

(R3) Every false 3-face receives 1 from the false 4-vertex incident with it in $H^\times$.

For each $z \in V(H^\times) \cup F(H^\times)$, we use $\omega'(z)$ to represent the new charge after applying the rules. Then one can obtain

$$\sum_{z \in V(H^\times) \cup F(H^\times)} \omega'(z) = \sum_{v \in V(H^\times)} (d_{H^\times}(v) - 4) + \sum_{f \in F(H^\times)} (d_{H^\times}(f) - 4) = -8 < 0.$$

Thus, there exists a vertex or a face whose charge is negative.

Firstly, we show the new charge of each face is nonnegative. Note that there is no real 3-face in $H^\times$ as G (and thus H) is an IC-planar graph containing no triangle. Pick arbitrarily a face z from $F(H^\times)$. If z is a false 3-face, then $\omega'(z) = 3 - 4 + 1 = 0$ by (R3) as it is incident with a false 4-vertex. If z is a real or false k-face ($k \geqslant 4$), then $\omega'(z) = k - 4 \geqslant 0$ as no rule is applied to it. Thus, the new charge of each face is nonnegative.

Secondly, we prove the new charge of each real vertex is nonnegative. Choose a real vertex z from $V(H)$. Note that each z is adjacent to at most one false 4-vertex as G (and thus H) is an IC-planar graph. Note also that $\delta(H^\times) \geqslant 3$ and $n^3_{H^\times}(z) \leqslant n^3_H(z)$ by the definition of $H^\times$ and Claim 7.

(i) Let $d_{H^\times}(z) = 3$. Then $\omega'(z) = 3 - 4 + 3 \cdot \frac{1}{3} = 0$ by (R2).

(ii) Let $d_{H^\times}(z) = 4$. Then $\omega'(z) = 4 - 4 = 0$ as $n^3_{H^\times}(z) = 0$ by Claim 7.

(iii) Let $d_{H^\times}(z) = 5$. Then, by Claim 7, $n^3_{H^\times}(z) \leqslant n^3_H(z) \leqslant 1$. If $n^3_{H^\times}(z) = 0$, then $\omega'(z) \geqslant 5 - 4 - 1 = 0$ by (R1.1). If $n^3_{H^\times}(z) = 1$, then $\omega'(z) \geqslant 5 - 4 - \frac{1}{3} - \frac{2}{3} = 0$ by (R1.2) and (R2).

(iv) Let $d_{H^\times}(z) = 6$. Then, by Claim 7, $n^3_{H^\times}(z) \leqslant n^3_H(z) \leqslant 2$. If $n^3_{H^\times}(z) = 0$, then $\omega'(z) \geqslant 6 - 4 - 2 = 0$ by (R1.3). If $1 \leqslant n^3_{H^\times}(z) \leqslant 2$, then $\omega'(z) \geqslant 6 - 4 - 2 \times \frac{1}{3} - \frac{4}{3} = 0$ by (R1.4) and (R2).

(v) Let $d_{H^\times}(z) = 7$. Then, by Claim 7, $n^3_{H^\times}(z) \leqslant n^3_H(z) \leqslant 5$. If $n^3_{H^\times}(z) = 0$, then $\omega'(z) \geqslant 7 - 4 - 2 = 1$ by (R1.3). If $1 \leqslant n^3_{H^\times}(z) \leqslant 5$, then $\omega'(z) \geqslant 7 - 4 - 5 \times \frac{1}{3} - \frac{4}{3} = 0$ by (R1.4) and (R2).

(vi) Let $d_{H^\times}(z) = \ell$ with $\ell \geqslant 8$. If $n^3_{H^\times}(z) = 0$, then $\omega'(z) \geqslant \ell - 4 - 2 \geqslant 2$ by (R1.3). If $1 \leqslant n^3_{H^\times}(z) \leqslant \ell$, then $\omega'(z) \geqslant \ell - 4 - \ell \times \frac{1}{3} - \frac{4}{3} \geqslant 0$ by (R1.4) and (R2).

Thus, the new charge of each real vertex is nonnegative. Below, let $f_3(z)$ represent the number of false 3-faces incident with z.

Finally, we show the new charge of each false 4-vertex is nonnegative. Pick a false 4-vertex z from $V(H^\times) \backslash V(H)$. Note that $f_3(z) \leqslant 2$ as G (and thus H) is an IC-planar graph. Note also that $n^3_{H^\times}(z) \leqslant 2$ by the definition of $H^\times$ and Claim 7.

⟨1⟩ Suppose that $n^3_{H^\times}(z) = 0$.

⟨1.1⟩ Assume that $f_3(z)=0$. Then $\omega'(z)\geqslant 4-4=0$ as it gives nothing away.

⟨1.2⟩ Assume that $f_3(z)=1$. Then, by Claim 7, $n^{5^+}_{H^\times}(z)\geqslant 1$. If $n^{5}_{H^\times}(z)=1$ and $n^{4}_{H^\times}(z)=3$, then $\omega'(z)\geqslant 4-4+1-1=0$ by (R1) and (R3) as the 5-neighbor of z is not adjacent to any 3-vertex by Claim 7(5). If $n^{6^+}_{H^\times}(z)=1$, then $\omega'(z)\geqslant 4-4+\frac{4}{3}-1=\frac{1}{3}$ by (R1) and (R3). If $n^{5^+}_{H^\times}(z)\geqslant 2$, then $\omega'(z)\geqslant 4-4+2\cdot\frac{2}{3}-1=\frac{1}{3}$ by (R1) and (R3).

⟨1.3⟩ Assume that $f_3(z)=2$. Then, by Claim 7, $n^{5^+}_{H^\times}(z)\geqslant 2$. If $n^{5}_{H^\times}(z)=2$ and $n^{4}_{H^\times}(z)=2$, then $\omega'(z)\geqslant 4-4+2-2=0$ by (R1) and (R3) as each of the two 5-vertices is not adjacent to any 3-vertex by Claim 7(5). If $n^{5}_{H^\times}(z)\geqslant 3$, then $\omega'(z)\geqslant 4-4+3\times\frac{2}{3}-2=0$ by (R1) and (R3). If z is adjacent to one 5^+-vertex and at least one 6^+-vertex, then $\omega'(z)\geqslant 4-4+\frac{2}{3}+\frac{4}{3}-2=0$ by (R1) and (R3).

⟨2⟩ Suppose that $n^{3}_{H^\times}(z)=1$.

⟨2.1⟩ Assume that $f_3(z)=0$. Then $\omega'(z)\geqslant 4-4+\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$ by (R1) and (R2) as $n^{5^+}_{H^\times}(z)\geqslant 1$ by Claim 7.

⟨2.2⟩ Assume that $f_3(z)=1$. Then $n^{5}_{H^\times}(z)\geqslant 2$ or $n^{6^+}_{H^\times}(z)\geqslant 1$ by Claim 7. Thus, $\omega'(z)\geqslant 4-4+\min\{2\cdot\frac{2}{3},\frac{4}{3}\}-1-\frac{1}{3}=0$ by (R1)–(R3).

⟨2.3⟩ Assume that $f_3(z)=2$. Then $n^{6^+}_{H^\times}(z)\geqslant 2$ or $n^{5}_{H^\times}(z)=2$ and $n^{6^+}_{H^\times}(z)=1$ or $n^{5}_{H^\times}(z)=3$. Note that if $n^{5}_{H^\times}(z)=3$, then at least one of the three 5-vertices is not adjacent to any 3-vertex in $H^\times$ by Claim 7. Thus, $\omega'(z)\geqslant 4-4+\min\{2\cdot\frac{4}{3},2\cdot\frac{2}{3}+\frac{4}{3},1+2\cdot\frac{2}{3}\}-2-\frac{1}{3}=0$ by (R1)–(R3).

⟨3⟩ Suppose that $n^{3}_{H^\times}(z)=2$.

⟨3.1⟩ Assume that $f_3(z)=0$. Then $\omega'(z)\geqslant 4-4+2\cdot\frac{2}{3}-2\cdot\frac{1}{3}=\frac{2}{3}$ by (R1) and (R2) as $n^{5^+}_{H^\times}(z)\geqslant 2$ by Claim 7.

⟨3.2⟩ Assume that $f_3(z)=1$. Then it is adjacent to one 5^+-vertex and one 6^+-vertex or two 5-vertices each of which is not adjacent to any 3-vertex by Claim 7. Thus, $\omega'(z)\geqslant 4-4+\min\{\frac{2}{3}+\frac{4}{3},1+1\}-1-2\cdot\frac{1}{3}=\frac{1}{3}$ by (R1)–(R3).

⟨3.3⟩ Assume that $f_3(z)=2$. Then $n^{6^+}_{H^\times}(z)\geqslant 2$ by Claim 7. Thus, $\omega'(z)\geqslant 4-4+2\cdot\frac{4}{3}-2-2\cdot\frac{1}{3}=0$ by (R1)–(R3).

Thus, the new charge of each false 4-vertex is nonnegative.

In summary, the new charge of each element in $V(H^\times)\cup F(H^\times)$ is nonnegative. It is a contradiction. The proof of Theorem 2 is completed. □

CONCLUSION REMARKS

In this paper, we showed that any triangle-free IC-planar graph with $\Delta\geqslant 7$ satisfies Conjecture 2 by applying the combinatorial nullstellensatz and the discharging method. This is an interesting and challenging problem to expand the class of graphs that satisfy Conjecture 2. According to currently known results, we can easily get the following questions for further research.

Question 1 Is it true that $\mathrm{ch}^t_\Sigma(G)\leqslant\Delta(G)+3$ for any triangle-free IC-planar graph with $\Delta=c\in\{4,5,6\}$?

Question 2 Is it true that $\mathrm{ch}^t_\Sigma(G)\leqslant\Delta(G)+3$ for any IC-planar graph with $\Delta\geqslant 7$ but without adjacent triangles?

Acknowledgements: This work was supported by Shangluo University Doctoral Initiation Fund Project (No. 21SKY108 and No. 22SKY112) and Shangluo University Key Disciplines Project, Discipline name: Mathematics.

REFERENCES

1. Albertson MO (2008) Chromatic number, independent ratio, and crossing number. *ARS Math Contemp* **1**, 1–6.
2. Alon N (1999) Combinatorial nullstellensatz. *Combin Probab Comput* **8**, 7–29.
3. Bondy J, Murty USR (2008) *Graph Theory*, Graduate Texts in Mathematics, Vol 244, Springer, London.
4. Lu Y, Xu C, Miao Z (2018) Neighbor sum distinguishing list total coloring of subcubic graphs. *J Comb Optim* **35**, 778–793.
5. Pilśniak M, Woźniak M (2015) On the total-neighbor-distinguishing index by sums. *Graphs Comb* **31**, 771–782.
6. Qu C, Ding L, Wang G, Yan G (2016) Neighbor distinguishing total choice number of sparse graphs via the Combinatorial Nullstellensatz. *Acta Math Sin (Engl Ser)* **32**, 537–548.
7. Qu C, Wang G, Yan G, Yu X (2016) Neighbor sum distinguishing total choosability of planar graphs. *J Comb Optim* **32**, 906–916.
8. Song W, Duan Y, Miao L (2020) Neighbor sum distinguishing total coloring of triangle free IC-planar graphs. *Acta Math Sin (Engl Ser)* **36**, 292–304.
9. Song W, Miao L, Li J, Zhao Y, Pang J (2018) Neighbor sum distinguishing total coloring of sparse IC-planar graphs. *Discrete Appl Math* **239**, 183–192.
10. Wang J, Ma Q, Han X (2015) Neighbor sum distinguishing total colorings of triangle free planar graphs. *Acta Math Sin (Engl Ser)* **2**, 216–224.
11. Yang D, Sun L, Yu X, Wu J, Zhou S (2017) Neighbor sum distinguishing total chromatic number of planar graphs with maximum degree 10. *Appl Math Comput* **314**, 456–468.
12. Yao J, Yu X, Wang G, Xu C (2016) Neighbor sum (set) distinguishing total choosability of d-degenerate graphs. *Graphs Comb* **32**, 1611–1620.