

Formal deformations of associative and Lie H -pseudoalgebras

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ABSTRACT: The aim of this paper is to study the 1-parameter formal deformations of associative (resp. Lie) H -pseudoalgebras and describe them with the cohomology groups. As an application, we derive Poisson H -pseudoalgebra from the first-order term of a deformation of the commutative associative H -pseudoalgebra.

KEYWORDS: associative H -pseudoalgebra, Lie H -pseudoalgebra, deformation

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INTRODUCTION

The theory of pseudotensor category was introduced by Beilinson and Drinfeld [1] as a way of expressing notions such as Lie algebras and representations in purely categorical terms. A pseudotensor category is a category equipped with an extra pseudotensor structure formed by a collection of functors $P_n\{L_i\}_{1 \leq i \leq n} \rightarrow M$ and a way to compose them. Let H be a cocommutative Hopf algebra, then $\mathcal{M}^*(H)$ is a pseudotensor category with the same objects as ${}_H\mathcal{M}$ (the category of left H -modules), and the pseudotensor structure is defined by

$$\text{Lin}(\{L_i\}_{i \in I}, M) = \text{Hom}_{H^{\otimes I}}(\boxtimes_{i \in I} L_i, H^{\otimes I} \otimes_H M),$$

where I is a finite non-empty set and $\boxtimes_{i \in I}$ is the tensor product functor ${}_H\mathcal{M}^I \rightarrow_{H^{\otimes I}} \mathcal{M}$. An algebra in this pseudotensor category is called an H -pseudoalgebra or simply a pseudoalgebra. When H is the polynomial algebra $\mathbb{C}[\partial]$, it is actually the conformal algebra [2–4], which is closely related to the vertex algebra. Moreover, conformal algebra provide an axiomatic description of the operator product expansion (OPE) of chiral fields in conformal field theory.

In [5], Bakalov, D’Andrea and Kac studied Lie (resp. associative) algebras in pseudotensor category $\mathcal{M}^*(H)$, called Lie (resp. associative) H -pseudoalgebras, which are closely related to the differential Lie algebras of Ritt and Hamiltonian formalism in the theory of nonlinear evolution equation [6–8]. They established representation theory, cohomology theory, and simultaneously solved classification problems in a series of papers [5, 9–11]. Other algebraic structures in pseudotensor category $\mathcal{M}^*(H)$ have been introduced afterwards, such as Jordan pseudoalgebra [12], Leibniz H -pseudoalgebra [13], left symmetric pseudoalgebra [14] and so on. Moreover, Boyallian and Liberati [15] introduced the notions of Lie H -coalgebra and Lie H -pseudobialgebra from

the point of pseudo-dual. Later, Liu and Wang et al. studied the infinitesimal H -pseudobialgebra and associative pseudo-Yang-Baxter equation [16]. Further research on pseudoalgebras can be found in [17–20].

The deformation is a tool to study a mathematical object by deforming it into a family of the same kind of objects depending on a certain parameter. The deformation theory of algebras was introduced by Gerstenhaber for rings and algebras in [21], and by Nijenhuis and Richardson for Lie algebras in [22]. They studied 1-parameter formal deformations and connected them with Hochschild cohomology and Chevalley-Eilenberg cohomology, respectively. The deformation problem of other algebraic structures, such as Lie triple systems, Yamaguti algebras, Novikov algebras, Hom-type algebras, etc., has been intensively studied in the literature recently, see [23–26].

Inspired by the above results, we consider the deformation theory of associative H -pseudoalgebras and Lie H -pseudoalgebras.

Throughout this paper, we denote by K a fixed algebraically closed field of characteristic zero, and H is a cocommutative Hopf algebra over K . For a comultiplication $\Delta : C \rightarrow C \otimes C$ on a linear space C , we use Sweedler’s notation [27] $\Delta(c) = c_1 \otimes c_2, \forall c \in C$. For the composition of two maps f and g , we write either $g \circ f$ or simply gf . For any vector space V , we will define $\sigma(f \otimes g) = g \otimes f, (123)(f \otimes g \otimes h) = h \otimes f \otimes g$ and $(23)(f \otimes g \otimes h) = f \otimes h \otimes g$, for all $f, g, h \in V$. Similarly, we have symbols (132), (12), (13) and so on.

PRELIMINARIES

In this section, we recall some basic definitions about associative and Lie H -pseudoalgebras, see, for example [5, 16, 20].

Definition 1 An H -pseudoalgebra $(A, \mu = *)$ is a left H -module A together with an operation

$$\mu : A \otimes A \rightarrow (H \otimes H) \otimes_H A, \quad a \otimes b \mapsto \mu(a \otimes b) = a * b,$$

called the pseudoproduct, and satisfying H -bilinearity: for all $a, b \in A$ and $f, g \in H$,

$$f a * g b = (f \otimes g \otimes_H 1)(a * b).$$

If $a * b = \sum_i f_i \otimes g_i \otimes_H e_i$, then $f a * g b = \sum_i f f_i \otimes g g_i \otimes_H e_i$.

In order to define the associativity of an H -pseudoalgebra, we extend the pseudoproduct from $A \otimes A \rightarrow H^{\otimes 2} \otimes_H A$ to maps $(H^{\otimes 2} \otimes_H A) \otimes A \rightarrow H^{\otimes 3} \otimes_H A$ and $A \otimes (H^{\otimes 2} \otimes_H A) \rightarrow H^{\otimes 3} \otimes_H A$ by letting

$$(f \otimes_H a) * b = \sum_i (f \otimes 1)(\Delta \otimes \text{id})(f_i \otimes g_i) \otimes_H e_i,$$

and

$$a * (f \otimes_H b) = \sum_i (1 \otimes f)(\text{id} \otimes \Delta)(f_i \otimes g_i) \otimes_H e_i,$$

where $f \in H^{\otimes 2}$ and $a * b = \sum_i f_i \otimes g_i \otimes_H e_i$.

The H -pseudoalgebra $(A, *)$ is associative if it satisfies $(a * b) * c = a * (b * c)$ for all $a, b, c \in A$, and $(A, *)$ is commutative if $a * b = (\sigma \otimes_H \text{id})(b * a)$ holds.

Definition 2 Let $(A, *)$ and $(B, *)$ be two associative H -pseudoalgebras.

- (1) A left A -module is a pair (M, ρ_M) , where M is a left H -module, $\rho_M \in \text{Hom}_{H \otimes H}(A \otimes M, (H \otimes H) \otimes_H M)$ (we denote $\rho_M(a \otimes m) = a * m$), satisfying $(a * b) * m = a * (b * m)$, for all $a, b \in A$ and $m \in M$.
- (2) A right A -module is a pair (N, ρ_N) , where N is a left H -module, $\rho_N \in \text{Hom}_{H \otimes H}(N \otimes A, (H \otimes H) \otimes_H N)$ (we denote $\rho_N(n \otimes a) = n * a$), satisfying $n * (a * b) = (n * a) * b$, for all $a, b \in A$ and $n \in N$.
- (3) (M, ρ_M) is called an A - B -bimodule, if M is a left A -module and right B -module such that $(a * m) * b = a * (m * b)$ for all $a \in A, b \in B$ and $m \in M$. An A - A -bimodule is simply called an A -bimodule.

Remark 1 An associative H -pseudoalgebra $(A, *)$ is an A -bimodule with the pseudoproduct.

Definition 3 Let $(A, *)$ be an H -pseudoalgebra. A is called a Lie H -pseudoalgebra (in this case, $a * b$ is denoted by $[a * b]$ and called the pseudobracket) if $[\cdot * \cdot]$ satisfies the following conditions:

- (1) Skew-commutativity, $[a * b] = -(\sigma \otimes_H \text{id})[b * a]$.
- (2) Jacobi identity, $[[a * b] * c] = [a * [b * c]] - ((12) \otimes_H \text{id})[b * [a * c]]$ in $H^{\otimes 3} \otimes_H A$, for all $a, b, c \in A$.

Definition 4 Let $(L, [\cdot * \cdot])$ be a Lie H -pseudoalgebra. A left L -module is a pair (N, ρ_N) , where N is a left H -module, $\rho_N \in \text{Hom}_{H \otimes H}(L \otimes N, (H \otimes H) \otimes_H N)$. For all $a, b \in L$ and $n \in N$, we denote $\rho_N(a \otimes n) = a * n$, satisfying $[a * b] * n = a * (b * n) - ((12) \otimes_H \text{id})(b * (a * n))$.

Remark 2 A Lie H -pseudoalgebra $(L, [\cdot * \cdot])$ is a left L -module with the pseudobracket.

The cohomology of associative H -pseudoalgebras

Let $(A, *)$ be an associative H -pseudoalgebra and M an A -bimodule. The space of n -cochains $C^n(A, M)$ consists of all maps

$$\varphi \in \text{Hom}_{H^{\otimes n}}(A^{\otimes n}, H^{\otimes n} \otimes_H M).$$

Explicitly, for all $h_i \in H$ and $a_i \in A$ ($i = 1, 2, \dots, n$), φ has the following defining property: H -polylinearity,

$$\varphi(h_1 a_1 \otimes \dots \otimes h_n a_n) = ((h_1 \otimes \dots \otimes h_n) \otimes_H 1) \varphi(a_1 \otimes \dots \otimes a_n).$$

For $n = 0$, we put $C^0(A, M) = K \otimes_H M \simeq M / H_+ M$, where $H_+ = \{h \in H | \varepsilon(h) = 0\}$ is the augmentation ideal of H . The differential $d_0 : C^0(A, M) \rightarrow C^1(A, M) = \text{Hom}_H(A, M)$ is given by

$$(d_0(1 \otimes_H m))(a) = \sum_i (\text{id} \otimes \varepsilon)(h_i) m_i - \sum_j (\varepsilon \otimes \text{id})(f_j) n_j,$$

for all $a \in A, m \in M$, where $a * m = \sum_i h_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M$ and $m * a = \sum_j f_j \otimes_H n_j \in H^{\otimes 2} \otimes_H M$.

For $n \geq 1$, the differential $d_n : C^n(A, M) \rightarrow C^{n+1}(A, M)$ is defined by

$$\begin{aligned} d_n \varphi(a_1 \otimes \dots \otimes a_{n+1}) &= a_1 * \varphi(a_2 \otimes \dots \otimes a_{n+1}) + (-1)^{n+1} \varphi(a_1 \otimes \dots \otimes a_n) * a_{n+1} \\ &+ \sum_{i=1}^n (-1)^i \varphi(a_1 \otimes \dots \otimes a_{i-1} \otimes a_i * a_{i+1} \otimes a_{i+2} \otimes \dots \otimes a_{n+1}). \end{aligned} \quad (1)$$

We use the following convention in the above equation. If $a * m = \sum_i f_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M, m * a = \sum_j g_j \otimes_H n_j \in H^{\otimes 2} \otimes_H M$ for all $a \in A$ and $m \in M$, then for all $f \in H^{\otimes n}$, we set

$$a * (f \otimes m) = \sum_i (1 \otimes f)(\text{id} \otimes \Delta^{(n-1)})(f_i) \otimes_H m_i \in H^{\otimes(n+1)} \otimes_H M,$$

and

$$(f \otimes m) * a = \sum_j (f \otimes 1)(\Delta^{(n-1)} \otimes \text{id})(g_j) \otimes_H n_j \in H^{\otimes(n+1)} \otimes_H M,$$

where $\Delta^{(n-1)} = (\text{id} \otimes \dots \otimes \text{id} \otimes \Delta) \dots (\text{id} \otimes \Delta) \Delta : H \rightarrow H^{\otimes n}$ is the iterated comultiplication for $n > 1$, and $\Delta^{(0)} := \text{id}$. Note that (1) holds also for $n = 0$ if we define $\Delta^{(-1)} = \varepsilon$.

For $g \in H^{\otimes 2}$ and $\varphi \in C^n(A, M)$, we set

$$\begin{aligned} \varphi(a_1 \otimes \dots \otimes a_{i-1} \otimes (g \otimes_H a_i) \otimes a_{i+1} \otimes \dots \otimes a_n) &= ((\text{id}^{\otimes(i-1)} \otimes g \otimes \text{id}^{\otimes(n-i)}) \otimes_H \text{id}) \varphi(a_1 \otimes \dots \otimes a_n) \\ &\in H^{\otimes(n+1)} \otimes_H M. \end{aligned}$$

In particular, by setting $n = 1$ and $n = 2$ in (1), respectively, we have

$$d_1 \varphi(a \otimes b) = a * \varphi(b) - \varphi(a * b) + \varphi(a) * b,$$

and

$$d_2\varphi(a\otimes b\otimes c) = a*\varphi(b,c) - \varphi(a*b,c) + \varphi(a,b*c) - \varphi(a,b)*c.$$

It can be shown that $d_{n+1}d_n = 0$ by a direct computation. A $\varphi \in C^n(A, M)$ is called an n -cocycle if $d_n\varphi = 0$, and φ is called an n -coboundary if there exists an $(n-1)$ -cochain ψ such that $d_{n-1}\psi = \varphi$. Denote by $Z^n(A, M)$ and $B^n(A, M)$ the subspaces of n -cocycles and n -coboundaries, respectively. The quotient space $H^n(A, M) = Z^n(A, M)/B^n(A, M)$ is called the n -th cohomology group of A with coefficients in M .

The cohomology of Lie H -pseudoalgebras

Let $(L, [\cdot * \cdot])$ be a Lie H -pseudoalgebra and M a left L -module. For $n \geq 1$, $C^n(L, M)$ consists of all $\gamma \in \text{Hom}_{H^{\otimes n}}(L^{\otimes n}, H^{\otimes n} \otimes_H M)$. Explicitly, γ has the following defining properties:

(1) H -polylinearity,

$$\gamma(h_1 a_1 \otimes \dots \otimes h_n a_n) = ((h_1 \otimes \dots \otimes h_n) \otimes_H 1) \gamma(a_1 \otimes \dots \otimes a_n).$$

(2) Skew-symmetry,

$$\begin{aligned} \gamma(a_1 \otimes \dots \otimes a_{i+1} \otimes a_i \otimes \dots \otimes a_n) \\ = -(\sigma_{i,i+1} \otimes_H \text{id}) \gamma(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_n), \end{aligned}$$

where $\sigma_{i,i+1} : H^{\otimes n} \rightarrow H^{\otimes n}$ is the transposition of the i -th and $(i+1)$ -st factors.

For $n \geq 1$, the differential $d_n : C^n(L, M) \rightarrow C^{n+1}(L, M)$ is defined as follows:

$$\begin{aligned} (d_n \gamma)(a_1 \otimes a_2 \otimes \dots \otimes a_n \otimes a_{n+1}) \\ = \sum_{1 \leq i \leq n+1} (-1)^{i+1} (\sigma_{1 \rightarrow i} \otimes_H \text{id}) a_i * \gamma(a_1 \otimes \dots \otimes \hat{a}_i \otimes \dots \otimes a_{n+1}) \\ + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} (\sigma_{1 \rightarrow i, 2 \rightarrow j} \otimes_H \text{id}) \\ \times \gamma([a_i * a_j] \otimes a_1 \otimes \dots \otimes \hat{a}_i \otimes \dots \otimes \hat{a}_j \otimes \dots \otimes a_{n+1}), \end{aligned}$$

where $\sigma_{1 \rightarrow i}$ is the permutation $h_1 \otimes h_1 \otimes \dots \otimes h_{i-1} \otimes h_{i+1} \otimes \dots \otimes h_{n+1} \mapsto h_1 \otimes \dots \otimes h_{n+1}$, $\sigma_{1 \rightarrow i, 2 \rightarrow j}$ is the permutation $h_i \otimes h_j \otimes h_1 \otimes \dots \otimes h_{i-1} \otimes h_{i+1} \otimes \dots \otimes h_{j-1} \otimes h_{j+1} \otimes \dots \otimes h_{n+1} \mapsto h_1 \otimes \dots \otimes h_{n+1}$, and \hat{a}_i means omitted. In particular, for $n = 1$, we have

$$\begin{aligned} (d_1 \gamma)(a_1 \otimes a_2) \\ = a_1 * \gamma(a_2) - ((12) \otimes_H \text{id}) a_2 * \gamma(a_1) - \gamma([a_1 * a_2]), \end{aligned}$$

and for $n = 2$, we get

$$\begin{aligned} (d_2 \gamma)(a_1 \otimes a_2 \otimes a_3) &= a_1 * \gamma(a_2 \otimes a_3) - ((12) \otimes_H \text{id}) a_2 * \gamma(a_1 \otimes a_3) \\ &+ ((132) \otimes_H \text{id}) a_3 * \gamma(a_1 \otimes a_2) - \gamma([a_1 * a_2] \otimes a_3) \\ &+ ((23) \otimes_H \text{id}) \gamma([a_1 * a_3] * a_2) - ((123) \otimes_H \text{id}) \gamma([a_2 * a_3] \otimes a_1). \end{aligned}$$

One can routinely verify that $d_{n+1}d_n = 0$. A $\gamma \in C^n(L, M)$ is called an n -cocycle if $d_n\gamma = 0$, and γ is called an n -coboundary if there exists an

$(n-1)$ -cochain ψ such that $d_{n-1}\psi = \gamma$. Denote by $Z^n(L, M)$ and $B^n(L, M)$ the subspaces of n -cocycles and n -coboundaries, respectively. The quotient space $H^n(L, M) = Z^n(L, M)/B^n(L, M)$ is called the n -th cohomology group of L with coefficients in M .

DEFORMATIONS OF ASSOCIATIVE AND LIE H -PSEUDOALGEBRAS

In this section, we define the deformations of associative and Lie H -pseudoalgebras and characterize them in terms of cohomology groups.

The deformations of associative H -pseudoalgebras

Definition 5 Let $(A, *)$ be an associative H -pseudoalgebra and

$$\begin{aligned} G_t(x, y) &= \sum_{i \geq 0} g_i(x, y) t^i \\ &= g_0(x, y) + g_1(x, y)t + g_2(x, y)t^2 + \dots, \end{aligned}$$

where t is a formal variable, each $g_i : A \otimes A \rightarrow H^{\otimes 2} \otimes_H A$ is an H -bilinear map and $g_0(x, y) = x * y$. We call G_t the 1-parameter formal deformation (we simply call it deformation) of $(A, *)$ if (A, G_t) is a family of associative H -pseudoalgebras. More precisely, we have

$$G_t(x, G_t(y, z)) = G_t(G_t(x, y), z), \quad \forall x, y, z \in A, \quad (2)$$

which is called the deformation equation of associative H -pseudoalgebra.

Example 1 $A = H\{e_1, e_2\}$ is an associative H -pseudoalgebra with the pseudoproduct given by $e_1 * e_1 = e_1 * e_2 = 0, e_2 * e_1 = 1 \otimes 1 \otimes_H e_1, e_2 * e_2 = 1 \otimes 1 \otimes_H e_2$. Define

$$\begin{aligned} G_1(e_1, e_1) &= G_1(e_1, e_2) = 0, \\ G_1(e_2, e_1) &= 1 \otimes 1 \otimes_H (1+t)e_1, \\ G_1(e_2, e_2) &= 1 \otimes 1 \otimes_H (1+t)e_2, \end{aligned}$$

where $t \in K$. It is easy to check that (2) holds. Hence G_1 is a deformation of A .

Now we focus on the deformation equation of associative H -pseudoalgebra. Note that (2) is equivalent to

$$\sum_{i, j \geq 0} (g_i(x, g_j(y, z)) - g_i(g_j(x, y), z)) t^{i+j} = 0.$$

Here we denote $g_i \hat{\circ} g_j(x, y, z) = g_i(x, g_j(y, z)) - g_i(g_j(x, y), z)$, then the above equation may be written as

$$\sum_{i, j \geq 0} (g_i \hat{\circ} g_j) t^{i+j} = 0,$$

which is equivalent to

$$\begin{aligned} \sum_{k \geq 0} t^k \sum_{i=0}^k g_i \hat{\circ} g_{k-i} &= 0 \quad \text{or} \\ \sum_{i=0}^k g_i \hat{\circ} g_{k-i} &= 0, \quad k = 0, 1, 2, \dots \end{aligned}$$

For $k = 0$, $g_0 \hat{\circ} g_0 = 0$, this corresponds to the associativity of A .

For $k = 1$, $g_0 \hat{\circ} g_1 + g_1 \hat{\circ} g_0 = 0$, i.e.,

$$a * g_1(b, c) - g_1(a, b) * c + g_1(a, b * c) - g_1(a * b, c) = 0, \quad (3)$$

For $k = 2$, $g_0 \hat{\circ} g_2 + g_1 \hat{\circ} g_1 + g_2 \hat{\circ} g_0 = 0$, i.e., $d_2 g_2 = g_0 \hat{\circ} g_2 + g_2 \hat{\circ} g_0 = -g_1 \hat{\circ} g_1$.

⋮

For $k = n$ ($n = 3, 4, \dots$), $d_2 g_n = g_0 \hat{\circ} g_n + g_n \hat{\circ} g_0 = -(g_0 \hat{\circ} g_{n-1} + g_2 \hat{\circ} g_{n-2} + \dots + g_{n-1} \hat{\circ} g_0)$.

Condition (3) implies that $d_2 g_1 = 0$. By the definition of n -cocycle, we have the following result.

Proposition 1 Let $(A, *)$ be an associative H -pseudoalgebra and $G_t(x, y) = \sum_{i \geq 0} g_i(x, y)t^i$ the deformation of A . Then g_1 is a 2-cocycle of the cohomology of A .

Definition 6 Let $(A, *)$ be an associative H -pseudoalgebra. Two deformations $G_t(x, y) = \sum_{i \geq 0} g_i(x, y)t^i$ and $G'_t(x, y) = \sum_{i \geq 0} g'_i(x, y)t^i$ of $(A, *)$ are equivalent if there exists a family $\Phi_t : A \rightarrow A[[t]]$ of H -linear maps of the form $\Phi_t = \sum_{i \geq 0} \phi_i t^i = \phi_0 + \phi_1 t^1 + \phi_2 t^2 + \dots$, where $\phi_i : A \rightarrow A$ is an H -linear map for $i = 1, 2, \dots$ and $\phi_0 = \text{id}$ such that

$$\Phi_t(G_t(x, y)) = G'_t(\Phi_t(x), \Phi_t(y)).$$

More precisely,

$$\sum_{i \geq 0} \phi_i \left(\sum_{j \geq 0} g_j(x, y)t^j \right) t^i = \sum_{i \geq 0} g'_i \left(\sum_{j \geq 0} \phi_j(x)t^j, \sum_{k \geq 0} \phi_k(y)t^k \right) t^i.$$

The deformation G_t is called trivial if G_t is equivalent to g_0 . An associative H -pseudoalgebra $(A, *)$ is called rigid, if every 1-parameter formal deformation G_t is trivial.

Theorem 1 ([28]) Let $(A, *)$ be an associative H -pseudoalgebra. There is a one-to-one correspondence between the elements of $H^2(A, A)$ and the infinitesimal deformation of A defined by $G_t(x, y) = g_0(x, y) + g_1(x, y)t$.

Theorem 2 Let $(A, *)$ be an associative H -pseudoalgebra and $H^2(A, A) = 0$. Then $(A, *)$ is rigid.

Proof: Let G_t be the deformation of $(A, *)$. Suppose that $G_t = g_0 + \sum_{i \geq 1} g_i t^i$, then

$$d_2 g_n = -(g_1 \hat{\circ} g_{n-1} + g_2 \hat{\circ} g_{n-2} + \dots + g_{n-1} \hat{\circ} g_1) = 0.$$

Hence $g_n \in Z^2(A, A) = B^2(A, A)$. It follows that there exists $\phi \in C^1(A, A)$ such that $g_n = d_1 \phi$.

Let $\Phi_t = \text{id} - \phi t^n : (A, G'_t) \rightarrow (A, G_t)$. Note that Φ_t is an isomorphism of H -modules with

$$\Phi_t \circ \sum_{i \geq 0} \phi^i t^{in} = \sum_{i \geq 0} \phi^i t^{in} \circ \Phi_t = \text{id}_A.$$

Define $G'_t(x, y) = \Phi_t^{-1} G_t(\Phi_t(x), \Phi_t(y))$. It is immediate that G'_t is a formal deformation of $(A, *)$ and G_t is equivalent to G'_t . Suppose that $G'_t(x, y) = g'_0(x, y) + \sum_{i > 0} g'_i(x, y)t^i$. Then

$$\begin{aligned} & (\text{id} - \phi t^n) \left(g'_0(x, y) + \sum_{i > 0} g'_i(x, y)t^i \right) \\ &= \left(g_0(x, y) + \sum_{i \geq n} g_i(x, y)t^i \right) (x - \phi(x)t^n, y - \phi(y)t^n), \end{aligned}$$

that is,

$$\begin{aligned} x * y + \sum_{i > 0} g'_i(x, y)t^i - \phi \left(x * y + \sum_{i > 0} g'_i(x, y)t^i \right) t^n \\ &= x * y - (x * \phi(y) + \phi(x) * y)t^n + (\phi(x) * \phi(y))t^{2n} \\ &+ \sum_{i \geq n} g_i(x, y)t^i - \sum_{i \geq n} (g_i(\phi(x), y) + g_i(x, \phi(y)))t^{i+n} \\ &+ \sum_{i \geq n} g_i(\phi(x), \phi(y))t^{i+2n}. \end{aligned}$$

Then we have $g'_1 = g'_2 = \dots = g'_{n-1} = 0$ and the coefficients of t^n are identical, i.e.,

$$g'_n - \phi(x * y) = -(\phi(x) * y + x * \phi(y)) + g_n(x, y).$$

It follows that $g'_n = g_n - d_1 \phi = 0$ and $G'_t(x, y) = g'_0(x, y) + \sum_{i \geq n} g'_i(x, y)t^i$. By induction, this procedure ends with $G'_t = g'_0$. So G_t is equivalent to g_0 , as desired. \square

The deformations of Lie H -pseudoalgebras

Definition 7 Let $(L, [\cdot * \cdot])$ be a Lie H -pseudoalgebra. A 1-parameter formal deformation (we simply call it deformation) of L is given by

$$[x * y]_t = \sum_{i \geq 0} [x, y]_i t^i = [x, y]_0 + [x, y]_1 t + [x, y]_2 t^2 + \dots,$$

where t is a formal variable, $[\cdot, \cdot]_i : L \otimes L \rightarrow H^{\otimes 2} \otimes_H L$ is an H -bilinear map and $[x, y]_0 = [x * y]$ such that $(L, [\cdot * \cdot]_t)$ is a family of Lie H -pseudoalgebras, that is, for all $x, y, z \in L$, the following conditions hold:

(skew-commutativity):

$$[x * y]_t = -[y * x]_t, \quad (4)$$

(Jacobi identity):

$$[x * [y * z]_t]_t - ((12)_H \text{id})[y * [x * z]_t]_t = [[x * y]_t, * z]_t. \quad (5)$$

Conditions (4) and (5) are called the deformation equations of a Lie H -pseudoalgebra. Note that the skew-commutativity of $[\cdot * \cdot]_t$ is equivalent to the skew-symmetry of all $[\cdot, \cdot]_i$ for $i = 0, 1, 2, \dots$

Example 2 We consider the 3-dimensional Lie algebra L with the bracket

$$\begin{aligned} [x, y]_t &= 2y - kt y, & [x, z]_t &= -2z + (ly + kz)t, \\ [y, z]_t &= x - \frac{k}{2}xt, & \forall k, l &\in K. \end{aligned}$$

Suppose that $H = sp\{1, g\}$ is the group Hopf algebra with $g^2 = 1, \Delta(g) = g \otimes g, S(g) = g = g^{-1}$. Then $[\cdot * \cdot]_1$ defines a deformation of Lie H -pseudoalgebra $H \otimes L$ with the action

$$\begin{aligned} [(f \otimes x) * (h \otimes y)]_1 &= f \otimes g \otimes_H (1 \otimes [x, y]_t), \\ [(f \otimes x) * (h \otimes z)]_1 &= f \otimes g \otimes_H (1 \otimes [x, z]_t), \\ [(f \otimes y) * (h \otimes z)]_1 &= f \otimes g \otimes_H (1 \otimes [y, z]_t), \end{aligned}$$

where $f, h \in \{1, g\}$.

Note that the deformation equation (5) can be rewritten as

$$\begin{aligned} \sum_{i=0}^k \left([x, [y, z]_i]_{k-i} - ((12) \otimes_H \text{id})[y, [x, z]_i]_{k-i} \right. \\ \left. - [[x, y]_i, z]_{k-i} \right) = 0, \quad k = 0, 1, 2, \dots \end{aligned}$$

For $k = 0$, it is nothing else but the Jacobi identity of $(L, [\cdot * \cdot])$.

For $k = 1$, we have

$$\begin{aligned} [x, [y * z]_1] - ((12) \otimes_H \text{id})[y, [x * z]_1] - [[x * y]_1, z]_1 \\ + [x * [y, z]_1] - ((12) \otimes_H \text{id})[y * [x, z]_1] - [[x, y]_1 * z] = 0, \end{aligned}$$

which implies that $d_2[\cdot, \cdot]_1 = 0$. Hence we have the following result.

Proposition 2 The first-order term $[\cdot, \cdot]_1$ of the deformation $[\cdot * \cdot]_t = \sum_{i \geq 0} [\cdot, \cdot]_i t^i$ of Lie H -pseudoalgebra $(L, [\cdot * \cdot])$ is a 2-cocycle of the cohomology of L .

Definition 8 Let $(L, [\cdot * \cdot])$ be a Lie H -pseudoalgebra. Two deformations $[x * y]_t = \sum_{i \geq 0} [x, y]_i t^i$ and $[x * y]'_t = \sum_{i \geq 0} [x, y]'_i t^i$ of L are equivalent if there exists a family $\Phi_t : L \rightarrow L[[t]]$ of H -linear maps of the form $\Phi_t = \sum_{i \geq 0} \phi_i t^i = \phi_0 + \phi_1 t^1 + \phi_2 t^2 + \dots$, where $\phi_i : L \rightarrow L$ is an H -linear map for $i \in \mathbb{N}_+$ and $\phi_0 = \text{id}$ such that

$$\Phi_t([x * y]_t) = [\Phi_t(x), \Phi_t(y)]'_t.$$

More precisely,

$$\sum_{i \geq 0} \phi_i \left(\sum_{j \geq 0} [x, y]_j t^j \right) t^i = \sum_{i \geq 0} \left[\sum_{j \geq 0} \phi_j(x) t^j, \sum_{k \geq 0} \phi_k(y) t^k \right]'_i t^i.$$

A formal deformation $[\cdot * \cdot]_t$ is called trivial if $[\cdot * \cdot]_t$ is equivalent to $[\cdot * \cdot]$. A Lie H -pseudoalgebra $(L, [\cdot * \cdot])$ is called rigid, if every 1-parameter formal deformation $[\cdot * \cdot]_t$ is trivial.

Similar to the results obtained in Theorems 1 and 2, we have the following results for Lie H -pseudoalgebras.

Theorem 3 ([5]) Let $(L, [\cdot * \cdot])$ be a Lie H -pseudoalgebra. There is a one-to-one correspondence between the elements of $H^2(L, L)$ and the infinitesimal deformation of L defined by $[x * y]_t = [x * y] + [x, y]_1 t$.

Theorem 4 Let $(L, [\cdot * \cdot])$ be a Lie H -pseudoalgebra and $H^2(L, L) = 0$. Then $(L, [\cdot * \cdot])$ is rigid.

POISSON H -PSEUDOALGEBRAS

In this section we mainly construct Poisson H -pseudoalgebra from the first-order term of deformation for a commutative associative H -pseudoalgebra. First we give the definition of Poisson H -pseudoalgebras.

Definition 9 A Poisson H -pseudoalgebra is a triple $(P, [\cdot * \cdot], *)$, such that $(P, [\cdot * \cdot])$ is a Lie H -pseudoalgebra, $(P, *)$ is a commutative associative H -pseudoalgebra and they satisfy the compatible condition

$$[x * (y * z)] = [x * y] * z + ((12) \otimes_H \text{id})y * [x * z], \quad (6)$$

for all $x, y, z \in P$.

Remark 3

- (1) When $H = k$, a Poisson H -pseudoalgebra is actually a Poisson algebra.
- (2) The compatible condition (6) is equivalent to

$$[(x * y) * z] = x * [y * z] + ((23) \otimes_H \text{id})[x * z] * y.$$

In fact,

$$\begin{aligned} [(x * y) * z] &= -((132) \otimes_H \text{id})[z * (x * y)] \\ &\stackrel{(6)}{=} -((132) \otimes_H \text{id})([z * x] * y + ((12) \otimes_H \text{id})x * [z * y]) \\ &= ((132)(12) \otimes_H \text{id})[x * z] * y \\ &\quad - ((132)(12) \otimes_H \text{id})x * [z * y] \\ &= ((23) \otimes_H \text{id})[x * z] * y - ((23) \otimes_H \text{id})x * [z * y] \\ &= x * [y * z] + ((23) \otimes_H \text{id})[x * z] * y. \end{aligned}$$

Example 3 Let V be an ordinary Poisson algebra. Then $P = H \otimes V$ equipped with operations

$$\begin{aligned} (f \otimes a) * (g \otimes b) &= f \otimes g \otimes_H (1 \otimes ab) \quad \text{and} \\ [(f \otimes a) * (g \otimes b)] &= f \otimes g \otimes_H (1 \otimes [a, b]), \end{aligned}$$

for all $f, g \in H$ and $a, b \in V$ is a Poisson H -pseudoalgebra.

Proposition 3 Let $(A, *)$ be a commutative associative H -pseudoalgebra. Define $[x * y]_A = 0$ or $[x * y]_A = x * y - (\sigma \otimes_H \text{id})(y * x)$ for all $x, y \in A$, then $(A, *, [\cdot * \cdot]_A)$ is a Poisson H -pseudoalgebra.

Proof: A commutative associative H -pseudoalgebra $(A, *)$ is naturally a Poisson H -pseudoalgebra with the trivial pseudobracket $[x * y]_A = 0$. Suppose $[x * y]_A =$

$x * y - (\sigma \otimes_H \text{id})(y * x)$, then $(A, [*]_A)$ is a Lie H -pseudoalgebra [5]. We only need to check the compatible condition (6). For all $x, y, z \in A$, we have

$$\begin{aligned} & [x * y]_A * z + ((12) \otimes_H \text{id})y * [x * z]_A \\ &= (x * y - (\sigma \otimes_H \text{id})(y * x)) * z \\ & \quad + ((12) \otimes_H \text{id})(y * (x * z - (\sigma \otimes_H \text{id})(z * x))) \\ &= (x * y) * z - ((12) \otimes_H \text{id})(y * x) * z \\ & \quad + ((12) \otimes_H \text{id})(y * (x * z) - ((23) \otimes_H \text{id})y * (z * x)) \\ &= (x * y) * z - ((12) \otimes_H \text{id})(y * x) * z \\ & \quad + ((12) \otimes_H \text{id})(y * (x * z)) - ((123) \otimes_H \text{id})y * (z * x) \\ &= x * (y * z) - ((123) \otimes_H \text{id})(y * z) * x \\ &= [x * (y * z)]_A. \end{aligned}$$

This completes the proof. \square

As in the case of Poisson algebras, the following proposition shows that the tensor product of two Poisson H -pseudoalgebras is still a Poisson H -pseudoalgebra. Since the proof is a straightforward computation, we omit the details.

Proposition 4 *Let H_i be the cocommutative Hopf algebras for $i = 1, 2$. Suppose $(A_i, *_i, [\cdot * \cdot]_i)$ is a Poisson H_i -pseudoalgebra, then the tensor product $(A_1 \otimes A_2, *, [\cdot * \cdot])$ is a Poisson $H_1 \otimes H_2$ -pseudoalgebra with the following structures:*

$$\begin{aligned} (a \otimes b) * (a' \otimes b') &= a *_1 a' \otimes b *_2 b' \\ &= \sum_{i,j} (f_i \otimes f'_j) \otimes (g_i \otimes g'_j) \otimes_H (e_i \otimes e'_j), \end{aligned}$$

and

$$\begin{aligned} [(a \otimes b) * (a' \otimes b')] &= a *_1 a' \otimes [b *_2 b'] + [a *_1 a'] \otimes b *_2 b' \\ &= \sum_{i,j} (f_i \otimes p'_j) \otimes (g_i \otimes q'_j) \otimes_H (e_i \otimes t'_j) \\ & \quad + \sum_{i,j} (p_i \otimes f'_j) \otimes (q_i \otimes g'_j) \otimes_H (t_i \otimes e'_j), \end{aligned}$$

where $a *_1 a' = \sum_i f_i \otimes g_i \otimes_{H_1} e_i$, $[a *_1 a'] = \sum_i p_i \otimes q_i \otimes_{H_1} t_i$, $b *_2 b' = \sum_j f'_j \otimes g'_j \otimes_{H_2} e'_j$ and $[b *_2 b'] = \sum_j p'_j \otimes q'_j \otimes_{H_2} t'_j$ for all $a, a' \in A_1$ and $b, b' \in A_2$.

Next we show that the first-order element of a deformation induces a Poisson H -pseudoalgebra. We start with some useful lemmas.

Lemma 1 *Let $(A, *)$ be a commutative associative H -pseudoalgebra and ϕ a skew-commutative 2-cochain such that $d_2\phi = 0$. Then*

$$\phi(x, y * z) = \phi(x, y) * z + ((12) \otimes_H \text{id})(y * \phi(x, z)),$$

for all $x, y, z \in A$.

Proof: Since ϕ is skew-commutative (i.e., $\phi(x, y) = -(\sigma \otimes_H \text{id})\phi(y, x)$) and A is commutative, we have

$$\phi(y, x * z) = -((123) \otimes_H \text{id})\phi(x * z, y), \tag{7}$$

$$x * \phi(y, z) = -((23) \otimes_H \text{id})x * \phi(z, y), \tag{8}$$

$$\phi(x, z) * y = ((132) \otimes_H \text{id})(y * \phi(x, z)), \tag{9}$$

$$\phi(x, y) * z = -((12) \otimes_H \text{id})\phi(y, x) * z, \tag{10}$$

for all $x, y, z \in A$. The condition $d_2\phi = 0$ implies that

$$\phi(x, y * z) = \phi(x * y, z) - x * \phi(y, z) + \phi(x, y) * z. \tag{11}$$

Then we obtain

$$\begin{aligned} & ((23) \otimes_H \text{id})\phi(x, z * y) = \\ & ((23) \otimes_H \text{id})(\phi(x * z, y) - x * \phi(z, y) + \phi(x, z) * y), \tag{12} \end{aligned}$$

and

$$\begin{aligned} & ((12) \otimes_H \text{id})(\phi(y * x, z) - \phi(y, x * z)) \\ &= ((12) \otimes_H \text{id})(y * \phi(x, z) - \phi(y, x) * z). \tag{13} \end{aligned}$$

By adding (11)–(13) and considering the fact that ϕ is skew-commutative and A is commutative, we have

$$\begin{aligned} \text{LHS} &= \phi(x, y * z) + ((23) \otimes_H \text{id})\phi(x, z * y) \\ & \quad + ((12) \otimes_H \text{id})(\phi(y * x, z) - \phi(y, x * z)) \\ &= \phi(x, y * z) + \phi(x, y * z) + \phi(x * y, z) \\ & \quad - ((12) \otimes_H \text{id})\phi(y, x * z) \\ &\stackrel{\text{⑦}}{=} 2\phi(x, y * z) + \phi(x * y, z) + ((12)(123) \otimes_H \text{id})\phi(x * z, y) \\ &= 2\phi(x, y * z) + \phi(x * y, z) + ((23) \otimes_H \text{id})\phi(x * z, y), \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= \phi(x * y, z) - x * \phi(y, z) + \phi(x, y) * z \\ & \quad + ((23) \otimes_H \text{id})(\phi(x * z, y) - x * \phi(z, y) + \phi(x, z) * y) \\ & \quad + ((12) \otimes_H \text{id})(y * \phi(x, z) - \phi(y, x) * z) \\ &\stackrel{\text{⑧-⑩}}{=} \phi(x * y, z) + ((23) \otimes_H \text{id})(x * \phi(z, y)) \\ & \quad + \phi(x, y) * z + ((23) \otimes_H \text{id})\phi(x * z, y) \\ & \quad - ((23) \otimes_H \text{id})(x * \phi(z, y)) + ((23)(132) \otimes_H \text{id})(y * \phi(x, z)) \\ & \quad + ((12) \otimes_H \text{id})(y * \phi(x, z)) + \phi(x, y) * z \\ &= \phi(x * y, z) + ((23) \otimes_H \text{id})\phi(x * z, y) + 2\phi(x, y) * z \\ & \quad + 2((12) \otimes_H \text{id})(y * \phi(x, z)). \end{aligned}$$

It follows that

$$\begin{aligned} & 2\phi(x, y * z) + \phi(x * y, z) + ((23) \otimes_H \text{id})\phi(x * z, y) \\ &= \phi(x * y, z) + ((23) \otimes_H \text{id})\phi(x * z, y) \\ & \quad + 2\phi(x, y) * z + 2((12) \otimes_H \text{id})(y * \phi(x, z)), \end{aligned}$$

that is,

$$\phi(x, y * z) = \phi(x, y) * z + ((12) \otimes_H \text{id})(y * \phi(x, z)).$$

This completes the proof. \square

Lemma 2 Let $(A, *)$ be a commutative associative H -pseudoalgebra and G_t the deformation of A . Then

$$x * g_2(y, z) = ((123) \otimes_H \text{id})(g_2(y, z) * x), \quad (14)$$

$$g_2(y, z) * x = ((132) \otimes_H \text{id})(x * g_2(y, z)), \quad (15)$$

$$g_2(y, z * x) = ((23) \otimes_H \text{id})g_2(y, x * z), \quad (16)$$

$$g_2(z * x, y) = ((12) \otimes_H \text{id})g_2(x * z, y), \quad (17)$$

for all $x, y, z \in A$

Proof: It can be verified by a direct computation and we omit the details. \square

Lemma 3 Let $(A, *)$ be a commutative associative H -pseudoalgebra and G_t the deformation of A . Then

$$\begin{aligned} & -g_1 \hat{\circ} g_1(x, y, z) - ((123) \otimes_H \text{id})g_1 \hat{\circ} g_1(y, z, x) \\ & \quad - ((132) \otimes_H \text{id})g_1 \hat{\circ} g_1(z, x, y) \\ & = g_2 \hat{\circ} g_0(x, y, z) + ((123) \otimes_H \text{id})g_2 \hat{\circ} g_0(y, z, x) \\ & \quad + ((132) \otimes_H \text{id})g_2 \hat{\circ} g_0(z, x, y). \end{aligned}$$

for all $x, y, z \in A$

Proof: Since $g_0 \hat{\circ} g_2 + g_1 \hat{\circ} g_1 + g_2 \hat{\circ} g_0 = 0$, we have

$$\begin{aligned} -g_1 \hat{\circ} g_1(x, y, z) & = g_0 \hat{\circ} g_2(x, y, z) + g_2 \hat{\circ} g_0(x, y, z) \\ & = x * g_2(y, z) - g_2(x, y) * z + g_2(x, y * z) - g_2(x * y, z). \end{aligned} \quad (18)$$

Now we compute

$$\begin{aligned} & -g_1 \hat{\circ} g_1(x, y, z) - ((123) \otimes_H \text{id})g_1 \hat{\circ} g_1(y, z, x) \\ & \quad - ((132) \otimes_H \text{id})g_1 \hat{\circ} g_1(z, x, y) \\ & \stackrel{(18)}{=} x * g_2(y, z) - g_2(x, y) * z + g_2(x, y * z) - g_2(x * y, z) \\ & \quad + ((123) \otimes_H \text{id})(y * g_2(z, x) - g_2(y, z) * x + g_2(y, z * x) \\ & \quad - g_2(y * z, x)) + ((132) \otimes_H \text{id})(z * g_2(x, y) - g_2(z, x) * y \\ & \quad + g_2(z, x * y) - g_2(z * x, y)) \\ & \stackrel{(14), (15)}{=} x * g_2(y, z) - g_2(x, y) * z + g_2(x, y * z) \\ & \quad - g_2(x * y, z) + ((123)(123) \otimes_H \text{id})g_2(z, x) * y \\ & - ((123)(132) \otimes_H \text{id})(x * g_2(y, z)) + ((123) \otimes_H \text{id})g_2(y, z * x) \\ & - ((123) \otimes_H \text{id})g_2(y * z, x) + ((132)(123) \otimes_H \text{id})g_2(x, y) * z \\ & \quad - ((132) \otimes_H \text{id})g_2(z, x) * y + ((132) \otimes_H \text{id})g_2(z, x * y) \\ & \quad - ((132) \otimes_H \text{id})g_2(z * x, y) \\ & = g_2(x, y * z) - g_2(x * y, z) \\ & \quad + ((123) \otimes_H \text{id})(g_2(y, z * x) - g_2(y * z, x)) \\ & \quad + ((132) \otimes_H \text{id})(g_2(z, x * y) - g_2(z * x, y)) \\ & = g_2 \hat{\circ} g_0(x, y, z) + ((123) \otimes_H \text{id})g_2 \hat{\circ} g_0(y, z, x) \\ & \quad + ((132) \otimes_H \text{id})g_2 \hat{\circ} g_0(z, x, y). \end{aligned}$$

This completes the proof. \square

Lemma 4 Let $(A, *)$ be a commutative associative H -pseudoalgebra and G_t the deformation of A . Then

$$\begin{aligned} & ((12) \otimes_H \text{id})g_1 \hat{\circ} g_1(y, x, z) + ((13) \otimes_H \text{id})g_1 \hat{\circ} g_1(z, y, x) \\ & \quad + ((23) \otimes_H \text{id})g_1 \hat{\circ} g_1(x, z, y) \\ & = -((12) \otimes_H \text{id})g_2 \hat{\circ} g_0(y, x, z) - ((13) \otimes_H \text{id})g_2 \hat{\circ} g_0(z, y, x) \\ & \quad - ((23) \otimes_H \text{id})g_2 \hat{\circ} g_0(x, z, y). \end{aligned}$$

for all $x, y, z \in A$

Proof: For all $x, y, z \in A$, we have

$$\begin{aligned} & ((12) \otimes_H \text{id})g_1 \hat{\circ} g_1(y, x, z) + ((13) \otimes_H \text{id})g_1 \hat{\circ} g_1(z, y, x) \\ & \quad + ((23) \otimes_H \text{id})g_1 \hat{\circ} g_1(x, z, y) \\ & = -((12) \otimes_H \text{id})(g_0 \hat{\circ} g_2(y, x, z) + g_2 \hat{\circ} g_0(y, x, z)) \\ & \quad - ((13) \otimes_H \text{id})(g_0 \hat{\circ} g_2(z, y, x) + g_2 \hat{\circ} g_0(z, y, x)) \\ & \quad - ((23) \otimes_H \text{id})(g_0 \hat{\circ} g_2(x, z, y) + g_2 \hat{\circ} g_0(x, z, y)) \\ & = ((12) \otimes_H \text{id})(g_2(y, x) * z - y * g_2(x, z) + g_2(y * x, z) - g_2(y, x * z)) \\ & \quad + ((13) \otimes_H \text{id})(g_2(z, y) * x - z * g_2(y, x) + g_2(z * y, x) - g_2(z, y * x)) \\ & \quad + ((23) \otimes_H \text{id})(g_2(x, z) * y - x * g_2(z, y) + g_2(x * z, y) - g_2(x, z * y)) \\ & \stackrel{(14), (15)}{=} ((12)(132) \otimes_H \text{id})z * g_2(y, x) - ((12)(123) \otimes_H \text{id})g_2(x, z) * y \\ & \quad + ((12) \otimes_H \text{id})g_2(y * x, z) - ((12) \otimes_H \text{id})g_2(y, x * z) \\ & \quad + ((13) \otimes_H \text{id})g_2(z, y) * x - ((13) \otimes_H \text{id})z * g_2(y, x) \\ & \quad + ((13) \otimes_H \text{id})g_2(z * y, x) - ((13) \otimes_H \text{id})g_2(z, y * x) \\ & \quad + ((23) \otimes_H \text{id})g_2(x, z) * y - ((23)(123) \otimes_H \text{id})g_2(z, y) * x \\ & \quad + ((23) \otimes_H \text{id})g_2(x * z, y) - ((23) \otimes_H \text{id})g_2(x, z * y) \\ & = ((12) \otimes_H \text{id})g_2(y * x, z) - ((12) \otimes_H \text{id})g_2(y, x * z) \\ & \quad + ((13) \otimes_H \text{id})g_2(z * y, x) - ((13) \otimes_H \text{id})g_2(z, y * x) \\ & \quad + ((23) \otimes_H \text{id})g_2(x * z, y) - ((23) \otimes_H \text{id})g_2(x, z * y) \\ & = -((12) \otimes_H \text{id})g_2 \hat{\circ} g_0(y, x, z) - ((13) \otimes_H \text{id})g_2 \hat{\circ} g_0(z, y, x) \\ & \quad - ((23) \otimes_H \text{id})g_2 \hat{\circ} g_0(x, z, y). \end{aligned}$$

So we complete the proof. \square

Now we give the main result of this section.

Theorem 5 Let $G_t = \sum_{i \geq 0} g_i t^i$ be the deformation of a commutative associative H -pseudoalgebra $(A, *)$. Define $\{x * y\} = g_1(x, y) - (\sigma \otimes_H \text{id})g_1(y, x)$, for all $x, y \in A$. Then $(A, *, \{*\})$ is a Poisson H -pseudoalgebra.

Proof: By Lemma 1, the compatible condition (6) holds. It is easy to prove that the pseudobracket $\{\cdot * \cdot\}$ is skew-commutative. It suffices to prove that $\{\cdot * \cdot\}$ satisfies the Jacobi identity. Using Lemmas 2–4, for all $x, y, z \in A$, we have

$$\begin{aligned} & \{x * y\} * z - \{x * \{y * z\}\} + ((12) \otimes_H \text{id})\{y * \{x * z\}\} \\ & = g_1(g_1(x, y), z) - ((132) \otimes_H \text{id})g_1(z, g_1(x, y)) \\ & \quad - ((12) \otimes_H \text{id})g_1(g_1(y, x), z) + ((13) \otimes_H \text{id})g_1(z, g_1(y, x)) \\ & \quad - g_1(x, g_1(y, z)) + ((123) \otimes_H \text{id})g_1(g_1(y, z), x) \\ & \quad + ((23) \otimes_H \text{id})g_1(x, g_1(z, y)) - ((13) \otimes_H \text{id})g_1(g_1(z, y), x) \\ & \quad + ((12) \otimes_H \text{id})g_1(y, g_1(x, z)) - ((23) \otimes_H \text{id})g_1(g_1(x, z), y) \\ & \quad - ((123) \otimes_H \text{id})g_1(y, g_1(z, x)) + ((132) \otimes_H \text{id})g_1(g_1(z, x), y) \end{aligned}$$

$$\begin{aligned}
&= g_1(g_1(x, y), z) - g_1(x, g_1(y, z)) \\
&\quad - ((123) \otimes_H \text{id})(g_1(y, g_1(z, x)) - g_1(g_1(y, z), x)) \\
&\quad - ((132) \otimes_H \text{id})(g_1(z, g_1(x, y)) - g_1(g_1(z, x), y)) \\
&\quad + ((12) \otimes_H \text{id})(g_1(y, g_1(x, z)) - g_1(g_1(y, x), z)) \\
&\quad + ((13) \otimes_H \text{id})(g_1(g_1(z, g_1(y, x)) - g_1(z, y), x)) \\
&\quad + ((23) \otimes_H \text{id})(g_1(x, g_1(z, y)) - g_1(g_1(x, z), y)) \\
&= g_2 \hat{\delta} g_0(x, y, z) + ((123) \otimes_H \text{id}) g_2 \hat{\delta} g_0(y, z, x) \\
&\quad + ((132) \otimes_H \text{id}) g_2 \hat{\delta} g_0(z, x, y) - ((12) \otimes_H \text{id}) g_2 \hat{\delta} g_0(y, x, z) \\
&\quad - ((13) \otimes_H \text{id}) g_2 \hat{\delta} g_0(z, y, x) - ((23) \otimes_H \text{id}) g_2 \hat{\delta} g_0(x, z, y) \\
&= g_2(x, y * z) - g_2(x * y, z) + ((123) \otimes_H \text{id}) g_2(y, z * x) \\
&\quad - ((123) \otimes_H \text{id}) g_2(y * z, x) + ((132) \otimes_H \text{id}) g_2(z, x * y) \\
&\quad - ((132) \otimes_H \text{id}) g_2(z * x, y) - ((12) \otimes_H \text{id}) g_2(y, x * z) \\
&\quad + ((12) \otimes_H \text{id}) g_2(y * x, z) - ((13) \otimes_H \text{id}) g_2(z, y * x) \\
&\quad + ((13) \otimes_H \text{id}) g_2(z * y, x) - ((23) \otimes_H \text{id}) g_2(x, z * y) \\
&\quad + ((23) \otimes_H \text{id}) g_2(x * z, y) \\
&\stackrel{(16), (17)}{=} g_2(x, y * z) - g_2(x * y, z) + ((123)(23) \otimes_H \text{id}) g_2(y, x * z) \\
&\quad - ((123) \otimes_H \text{id}) g_2(y * z, x) + ((132) \otimes_H \text{id}) g_2(z, x * y) \\
&\quad - ((132)(12) \otimes_H \text{id}) g_2(x * z, y) - ((12) \otimes_H \text{id}) g_2(y, x * z) \\
&\quad + ((12) \otimes_H \text{id}) g_2(y * x, z) - ((13)(23) \otimes_H \text{id}) g_2(z, x * y) \\
&\quad + ((13)(12) \otimes_H \text{id}) g_2(y * z, x) - ((23) \otimes_H \text{id}) g_2(x, z * y) \\
&\quad + ((23) \otimes_H \text{id}) g_2(x * z, y) = 0.
\end{aligned}$$

This completes the proof. \square

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REFERENCES

1. Beilinson A, Drinfeld V (2004) *Chiral Algebras*, American Mathematical Society, Providence, RI.
2. D'Andrea A, Kac VG (1998) Structure theory of finite conformal algebras. *Selecta Math* **4**, 377–418.
3. Kac VG, Retakh A (2008) Simple Jordan conformal superalgebras. *J Algebra Appl* **7**, 517–533.
4. Xu MS, Hong YY, Wu ZX (2021) Finite irreducible conformal modules of rank two Lie conformal algebras. *J Algebra Appl* **20**, 2150145.
5. Bakalov B, D'Andrea A, Kac VG (2001) Theory of finite pseudoalgebras. *Adv Math* **162**, 1–140.
6. Dorfman I (1993) *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley and Sons, Chichester.
7. Gel'Fand IM, Dorfman IY (1981) Hamiltonian operators and infinite-dimensional Lie algebras. *Funct Anal Its Appl* **15**, 173–187.
8. Gel'Fand IM, Dorfman IY (1982) Hamiltonian operators and the classical Yang-Baxter equation. *Funct Anal Its Appl* **16**, 241–248.
9. Bakalov B, D'Andrea A, Kac VG (2006) Irreducible modules over finite simple Lie pseudoalgebras I. Primitive pseudoalgebras of type W and S . *Adv Math* **204**, 278–346.
10. Bakalov B, D'Andrea A, Kac VG (2013) Irreducible modules over finite simple Lie pseudoalgebras II. Primitive pseudoalgebras of type K . *Adv Math* **232**, 188–237.
11. Bakalov B, D'Andrea A, Kac VG (2021) Irreducible modules over finite simple Lie pseudoalgebras III. Primitive pseudoalgebras of type H . *Adv Math* **392**, 107963.
12. Kolesnikov P (2009) Simple finite Jordan pseudoalgebras. *Symmetry Integr Geom Methods Appl* **5**, 014.
13. Wu ZX (2015) Leibniz H -pseudoalgebras. *J Algebra* **437**, 1–33.
14. Wu ZX (2015) Graded left symmetric pseudo-algebras. *Comm Algebra* **43**, 3869–3897.
15. Boyallian C, Liberati J (2012) On pseudo-bialgebras. *J Algebra* **372**, 1–34.
16. Liu LL, Wang SH, Shi GD, Guo ZT (2022) Infinitesimal H -pseudobialgebra and associative pseudo-Yang-Baxter equation. *Comm Algebra* **50**, 1071–1091.
17. D'Andrea A, Marchei G (2011) Representations of Lie pseudoalgebras with coefficients. *J Algebra* **329**, 92–106.
18. Retakh A (2004) Unital associative pseudoalgebras and their representations. *J Algebra* **277**, 769–805.
19. Shulman MA (2012) Not every pseudoalgebra is equivalent to a strict one. *Adv Math* **229**, 2024–2041.
20. Sun QX (2012) Generalization of H -pseudoalgebraic structures. *J Math Phys* **53**, 012105.
21. Gerstenhaber M (1964) On the deformation of rings and algebras. *Ann Math* **79**, 59–103.
22. Nijenhuis A, Richardson RW (1966) Cohomology and deformations in graded Lie algebras. *Bull Amer Math Soc* **72**, 1–29.
23. Bai CM, Meng DJ (2001) The realization of non-transitive Novikov algebras. *J Phys A Math Gen* **34**, 6435–6442.
24. Kubo F, Taniguchi Y (2004) A controlling cohomology of the deformation theory of Lie triple systems. *J Algebra* **278**, 242–250.
25. Ma Y, Chen LY, Lin J (2015) One-parameter formal deformations of Hom-Lie-Yamaguti algebras. *J Math Phys* **56**, 011701.
26. Makhoulouf A, Silvestrov S (2010) Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras. *Forum Math* **22**, 715–739.
27. Sweedler ME (1969) *Hopf Algebras*, W. A. Benjamin, New York.
28. Liberati JI (2022) Cohomology of associative H -pseudoalgebras. *arXiv*, 13832v2.