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On the divisibility $F_k \mid F_x^2 + F_x + 1$

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ABSTRACT: Let F_n and L_n be the nth Fibonacci and Lucas numbers, respectively. We show that if $F_k \mid F_x^2 + F_x + 1$, then $k \in \{4, 7\}$; if $L_k \mid F_x^2 + F_x + 1$, then $k \in \{2, 4\}$.

KEYWORDS: Fibonacci number, Lucas number, divisibility, Diophantine equation, factorization

MSC2010: 11B39 11D99

INTRODUCTION AND MAIN RESULTS

As usual, the sequence of Fibonacci numbers is defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$ with the initial values $F_0 = 0$, $F_1 = 1$. This sequence can be extended for all integers n if one applies the recursive rule backward. This approach provides $F_{-n} = (-1)^{n+1} F_n$ for all $n \geqslant 0$. The associated sequence of Lucas numbers is given by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$. Its extension to $\mathbb{Z}$ satisfies $L_{-n} = (-1)^n L_n$ for all $n \geqslant 0$. From a large number of identities involving Fibonacci numbers, we first recall

$$F_u \mp 1 = \begin{cases} F_{\frac{u\pm 2}{2}} L_{\frac{u\mp 2}{2}}, & \text{if } u \equiv 0 \pmod 4; \\ F_{\frac{u\mp 1}{2}} L_{\frac{u\pm 1}{2}}, & \text{if } u \equiv 1 \pmod 4; \\ F_{\frac{u\mp 2}{2}} L_{\frac{u\pm 2}{2}}, & \text{if } u \equiv 2 \pmod 4; \\ F_{\frac{u\pm 1}{2}} L_{\frac{u\mp 1}{2}}, & \text{if } u \equiv 3 \pmod 4, \end{cases} \tag{1}$$

which provides a natural factorization of a Fibonacci number plus/minus 1. The verification of (1) can be done by using the well-known Binet formula and a straightforward calculation. Bravo, Komatsu, and Luca [1], and Luca and Szalay [3] used this formula in their calculations for the distance between the integers of the forms $F_n F_{n+1} \cdots F_{n+k}$ and F_n^m, and for the Fibonacci Diophantine tuples, respectively. Pongsriiam [7] also applied it in his solution to certain Diophantine equations, and a generalization of these equations is solved by Szalay [9]. Clearly, the equality $F_u^2 - 1 = (F_u - 1)(F_u + 1)$ gives a kind of factorization into 4 factors, but after regrouping them, we obtain

$$F_u^2 - 1 = \begin{cases} F_{u-2} F_{u+2}, & \text{if } u \equiv 0 \pmod 2; \\ F_{u-1} F_{u+1}, & \text{if } u \equiv 1 \pmod 2. \end{cases} \tag{2}$$

It is easy to see that there exists a similar formula for $F_u^2 + 1$ as follows:

$$F_u^2 + 1 = \begin{cases} F_{u-1} F_{u+1}, & \text{if } u \equiv 0 \pmod 2; \\ F_{u-2} F_{u+2}, & \text{if } u \equiv 1 \pmod 2. \end{cases} \tag{3}$$

The formulas (2) and (3), and their generalizations are a tool in solving a variation of Brocard-Ramanujan equation as shown in the work of Pink and Szikszai [5], Pongsriiam [6], Sahukar and Panda [8], and Szalay [9]. Since $F_u^4 - 1 = (F_u^2 - 1)(F_u^2 + 1)$, we obtain a factorization of $F_u^4 - 1$ from (2) and (3), but the question arises naturally: is there any Fibonacci or Lucas factorization of $F_u^3 - 1$? The identity $F_u^3 - 1 = (F_u - 1)(F_u^2 + F_u + 1)$ shows that we need only to analyze the second factor.

It turned out that the situation is completely different from the other cases. The precise result is the following.

Theorem 1 *If $k \geqslant 3$ and $F_k \mid F_n^2 + F_n + 1$, then $k = 4$ ($F_4 = 3$) or $k = 7$ ($F_7 = 13$). Moreover*

(i) $3 \mid F_x^2 + F_x + 1$ *if and only if* $x = 8t + 1$ *or* $x = 8t + 2$ *or* $x = 8t + 7$ *for some* $t \in \mathbb{Z}$;

(ii) $13 \mid F_x^2 + F_x + 1$ *if and only if* $x = 28t + 4$ *or* $x = 28t + 10$ *for some* $t \in \mathbb{Z}$.

We also investigated a related problem, where the divisor F_k is replaced by the Lucas number L_k.

Theorem 2 *If* $k \geqslant 2$ *and* $L_k \mid F_x^2 + F_x + 1$, *then* $k = 2$ *(*$L_2 = 3$*) or* $k = 4$ *(*$L_4 = 7$*). Moreover*

(i) $3 \mid F_x^2 + F_x + 1$ *if and only if* $x = 8t+1$ *or* $x = 8t+2$ *or* $x = 8t + 7$ *for some* $t \in \mathbb{Z}$*;*

(ii) $7 \mid F_x^2 + F_x + 1$ *if and only if* $x = 16t + 3$ *or* $x = 16t + 12$ *or* $x = 16t + 13$ *for some* $t \in \mathbb{Z}$.

An easy consequence of the theorems above is

Corollary 1 $F_x^2 + F_x + 1$ *is never divisible by two distinct Lucas numbers larger than* 1. *In addition,* $F_x^2 + F_x + 1$ *is divisible by two distinct Fibonacci numbers larger than* 1 *if and only if* $x = 56t + 10$ *(*$t \in \mathbb{Z}$*), and in this case the two Fibonacci factors are* 3 *and* 13.

To prove the main results, we recall the following two lemmas appearing in the proof of Theorem 1 of Németh, Soydan, and Szalay [4], and in Lemma 1 of Komatsu, Luca, and Tachiya [2].

Lemma 1 (Németh, Soydan, and Szalay [4]) *The sequence* $(F_n)_{n\in\mathbb{Z}}$ *is periodic modulo* F_k *with period* $4k$.

Lemma 2 (Komatsu, Luca, and Tachiya [2]) *Let* $X \geqslant 3$ *be a real number. Let* $a, b \in \mathbb{N}$ *with* $\max\{a, b\} \leqslant X$. *Then there exist integers* u, v *not both zero such that* $\max\{|u|, |v|\} \leqslant \sqrt{X}$ *and* $|au + bv| \leqslant 3\sqrt{X}$.

PROOF OF Theorem 1

Assume that $F_k \mid F_x^2 + F_x + 1$. By Lemma 1, reducing x modulo $4k$ in the relation $F_k \mid F_x^2 + F_x + 1$, we may assume that $|x| \leqslant 2k$. Now we write $\omega = e^{2\pi i/3}$ and

$$F_x^2 + F_x + 1 = (F_x - \omega)(F_x - \overline{\omega}).$$

Writing $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$, we have the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{valid for all } n \in \mathbb{Z}.$$

Note that $\beta = -\alpha^{-1}$. Clearly

$$\begin{aligned} F_x - \omega &= \frac{\alpha^x - \beta^x}{\sqrt{5}} - \omega = \frac{\alpha^x - (-1)^x\alpha^{-x} - \sqrt{5}\omega}{\sqrt{5}} \\ &= \frac{\alpha^{-x}}{\sqrt{5}}(\alpha^{2x} - \sqrt{5}\omega\alpha^x - (-1)^x), \end{aligned}$$

and similarly

$$F_x - \overline{\omega} = \frac{\alpha^{-x}}{\sqrt{5}}(\alpha^{2x} - \sqrt{5}\overline{\omega}\alpha^x - (-1)^x).$$

Thus,

$$F_k \,\Big|\, \frac{(\alpha^{2x} - \sqrt{5}\omega\alpha^x - (-1)^x)}{\sqrt{5}} \cdot \frac{(\alpha^{2x} - \sqrt{5}\overline{\omega}\alpha^x - (-1)^x)}{\sqrt{5}}.$$

The factors in the right–hand side are in the field $\mathbb{K} := \mathbb{Q}(\omega, \sqrt{5}) \subset \mathbb{Q}(\zeta^{2\pi i/15})$, which is a class number 1 field of degree 4. Thus, all ideals in $\mathcal{O}_{\mathbb{K}}$ are principal.

In fact, the two factors $F_x - \omega$ and $F_x - \overline{\omega}$ are almost coprime since their greatest common divisor divides $\omega - \overline{\omega} = \mathrm{i}\sqrt{3}$ and $\mathrm{i}\sqrt{3}$ is prime in $\mathcal{O}_{\mathbb{K}}$ since the prime 3 is inert in $\mathbb{Q}(\sqrt{5})$. Thus, these two factors are not coprime only if $F_x \equiv 1 \pmod{\mathrm{i}\sqrt{3}}$ since otherwise $F_x^2 + F_x + 1$ is coprime to 3. Then

$$\begin{aligned} F_x - \omega \equiv 1 - \omega &= \frac{3}{2} - \frac{\sqrt{3}}{2}\mathrm{i} = -\mathrm{i}\sqrt{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) \\ &\equiv 0 \pmod{\mathrm{i}\sqrt{3}}. \end{aligned}$$

In this case $F_x^2 + F_x + 1 \equiv 3 \pmod 9$, and the factor 3 gets split into two pieces associated to $\mathrm{i}\sqrt{3}$ each, one which goes into $F_x - \omega$ and the other goes into $F_x - \overline{\omega}$, and the quotients remain coprime.

Hence, if $F_x \not\equiv 1 \pmod 3$, or $F_x \equiv 1 \pmod 3$ but F_k is not a multiple of 3, then

$$\begin{aligned} F_k = \gcd\left(F_k, \frac{\alpha^{2x} - \sqrt{5}\omega\alpha^x - (-1)^x}{\sqrt{5}}\right) \\ \times \gcd\left(F_k, \frac{\alpha^{2x} - \sqrt{5}\overline{\omega}\alpha^x - (-1)^x}{\sqrt{5}}\right). \end{aligned}$$

Otherwise, if $F_x \equiv 1 \pmod 3$ and $3 \mid F_k$, then we divide across by 3 and get

$$\begin{aligned} \frac{F_k}{3} = \gcd\left(\frac{F_k}{3}, \frac{\alpha^{2x} - \sqrt{5}\omega\alpha^x - (-1)^x}{i\sqrt{15}}\right) \\ \times \gcd\left(\frac{F_k}{3}, \frac{\alpha^{2x} - \sqrt{5}\overline{\omega}\alpha^x - (-1)^x}{\mathrm{i}\sqrt{15}}\right) \end{aligned}$$

and now $(\alpha^{2x} - \sqrt{5}\omega\alpha^x - (-1)^x)/(\mathrm{i}\sqrt{15})$ and $(\alpha^{2x} - \sqrt{5}\overline{\omega}\alpha^x - (-1)^x)/(\mathrm{i}\sqrt{15})$ are coprime.

To unify the two branches we introduce $\varepsilon = 1$ or $\mathrm{i}\sqrt{3}$ according to the previous two cases. It follows that

$$\frac{F_k}{\varepsilon^2}\mathcal{O}_{\mathbb{K}} = \mathcal{U} \cdot \mathcal{V} = (U\mathcal{O}_{\mathbb{K}}) \cdot (V\mathcal{O}_{\mathbb{K}}), \tag{4}$$

where $\mathcal{U}$ and $\mathcal{V}$ are the greatest common divisor ideals of

$$\frac{F_k}{\varepsilon^2} \quad \text{and} \quad \frac{(\alpha^{2x}-\sqrt{5}\omega\alpha^x-(-1)^x)}{\sqrt{5}\varepsilon},$$

and

$$\frac{F_k}{\varepsilon^2} \quad \text{and} \quad \frac{(\alpha^{2x}-\sqrt{5}\overline{\omega}\alpha^x-(-1)^x)}{\sqrt{5}\varepsilon}$$

in $\mathcal{O}_{\mathbb{K}}$, respectively, and U, V are generators of $\mathcal{U}$ and $\mathcal{V}$, respectively. Note that $\mathcal{V}$ is the complex conjugate of $\mathcal{U}$, so we can choose $V=\overline{U}$. Thus,

$$\frac{F_k}{\varepsilon^2}=\lambda U\overline{U},$$

where λ is a unit in $\mathbb{K}$. Since $U\overline{U}$ is real and it is an element of the real subfield of $\mathbb{K}$, which is $\mathbb{Q}(\sqrt{5})$, so is λ. Conjugating the above relation by the only nontrivial automorphism σ of $\mathbb{Q}(\sqrt{5})$, we get that

$$F_k^2=\varepsilon^4|\lambda\lambda^\sigma|UU^\sigma\overline{U}\overline{U}^\sigma, \tag{5}$$

and $|\lambda\lambda^\sigma|=|N_{\mathbb{Q}(\sqrt{5}):\mathbb{Q}}(\lambda)|=1$ as being the norm of the unit λ from $\mathbb{Q}(\sqrt{5})$ to $\mathbb{Q}$. It remains to bound the absolute values of U and its three other conjugates conjugates U^σ, $\overline{U}$, and $\overline{U}^\sigma$. For this, we look at U and write

$$\alpha^{2x}-\sqrt{5}\omega\alpha^x-\mu=(\alpha^x-z_1)(\alpha^x-z_2),$$

$\mu=(-1)^x\in\{\pm1\}$, where

$$z_{1,2}=\frac{\sqrt{5}\omega\pm\sqrt{5\omega^2+4\mu}}{2}.$$

We work in the quadratic extension $\mathbb{L}=\mathbb{K}(z_1)$ of $\mathbb{K}$ and write

$$U\mathcal{O}_{\mathbb{L}}\mid\mathcal{U}_1\mathcal{U}_2,$$

where $\mathcal{U}_i=\gcd(\mathcal{U}\mathcal{O}_{\mathbb{L}},(\alpha^x-z_i)\mathcal{O}_{\mathbb{L}})$ for $i=1,2$. Let us look at $\mathcal{U}_i$. This ideal fulfils the following conditions:

$$\begin{aligned}\alpha^{2k}&\equiv(-1)^k \pmod{\mathcal{U}_i};\\ \alpha^x&\equiv z_i \pmod{\mathcal{U}_i}.\end{aligned} \tag{6}$$

The first relation comes from the fact that $\mathcal{U}_i\mid\mathcal{U}\mid F_k\mid\alpha^k-\beta^k$ and

$$\alpha^k-\beta^k=\alpha^k-(-1)^k\alpha^{-k}=\alpha^{-k}(\alpha^{2k}-(-1)^k),$$

and α^k is a unit. Note that $\max\{2k,|x|\}\leqslant 2k$. By Lemma 2, there are integers a, b not both 0 with $\max\{|a|,|b|\}\leqslant\sqrt{2k}$ such that $|2ka+xb|\leqslant3\sqrt{2k}$.

Raising the first congruence in (6) to a and the second to b and multiplying them, we get

$$\alpha^{2ka+xb}-(-1)^{ka}z_i^b\equiv0 \pmod{\mathcal{U}_i} \quad \text{for } i=1,2.$$

In particular,

$$U\mathcal{O}_{\mathbb{L}}\mid\mathcal{U}_1\mathcal{U}_2\mid(\alpha^{2ka+xb}-(-1)^{ka}z_1^b)(\alpha^{2ka+kb}-(-1)^{ka}z_2^b).$$

The expression in the right hand side is symmetric in z_1, z_2 so it belongs to $\mathbb{K}$. Let us show that it is not zero. If it were, then

$$\alpha^{4ka+2xb}=z_i^{2b} \tag{7}$$

for some $i=1,2$. We checked that z_i is complex non real for both x even and odd. The same is true for ω replaced by ω^2. If $b=0$, then $\alpha^{4ka}=1$, so $a=0$, which is false since we cannot have both a and b be zero. So, $b\neq0$. We can now complex conjugate the above relation (7) to get that

$$\alpha^{4ka+2xb}=\overline{z_i}^{2b},$$

and taking ratios we get $(z_i/\overline{z_i})^{2b}=1$, so $z_i/\overline{z_i}$ is a root of unity, and this is also false. In fact, it turns out that in all cases $z_i/\overline{z_i}$ is of degree 4 and has two real conjugates, one of absolute value larger than 1 and one of absolute value smaller than 1 so it cannot be a root of unity. Thus, we get that

$$U=\nu^{-1}(\alpha^{2ka+xb}-(-1)^kz_1^b)(\alpha^{2ka+kb}-(-1)^kz_2^b),$$

where ν is some algebraic integer in $\mathbb{K}$. Thus,

$$|U|=|\nu|^{-1}\left|(\alpha^{2ka+xb}-(-1)^kz_1^b)(\alpha^{2ka+kb}-(-1)^kz_2^b)\right|. \tag{8}$$

We computed the absolute values of z_i for $i=1,2$ and also of the analogous numbers with ω replaced by ω^2. We get that they are smaller than $2.5<\alpha^2$. Thus,

$$\begin{aligned}\left|\alpha^{2ka+xb}-(-1)^kz_i^b\right|&\leqslant\alpha^{3\sqrt{2k}}+\alpha^{2\sqrt{2k}}\\&=\alpha^{3\sqrt{2k}}\left(1+\frac{1}{\alpha^{\sqrt{2k}}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\alpha^{2ka+xb}-(-1)^kz_1^b\right|\left|\alpha^{2ka+xb}-(-1)^kz_2^b\right|\\&\qquad\leqslant\alpha^{6\sqrt{2k}}\left(1+\frac{1}{\alpha^{\sqrt{2k}}}\right)^2.\end{aligned} \tag{9}$$

Doing this for $\overline{U}$, U^σ, $\overline{U}^\sigma$ and using (5) as well as (8) and its conjugates and (9) and its conjugates, we get that

$$\begin{aligned}F_k^2&\leqslant9U\overline{U}U^\sigma\overline{U}^\sigma\leqslant9|N_{\mathbb{K}:\mathbb{Q}}(\nu)|^{-1}\left(\alpha^{6\sqrt{2k}}\left(1+\frac{1}{\alpha^{\sqrt{2k}}}\right)^2\right)^4\\&\leqslant9\alpha^{24\sqrt{2k}}\left(1+\frac{1}{\alpha^{\sqrt{2k}}}\right)^8.\end{aligned} \tag{10}$$

Now

$$F_k^2 = \frac{(\alpha^k-\beta^k)^2}{5} \geqslant \frac{\alpha^{2k}}{5}\Big(1-\frac{1}{\alpha^{2k}}\Big)^2 \geqslant \alpha^{2k-4}\Big(1+\frac{2}{\alpha^{2k}}\Big)^{-2}, \qquad (11)$$

which together with (10), and $9 < \alpha^5$ gives

$$\alpha^{2k-4} \leqslant \alpha^{24\sqrt{2k}+5}\left(1+\frac{1}{\alpha^{\sqrt{2k}}}\right)^8\left(1+\frac{2}{\alpha^{2k}}\right)^2.$$

For $k \geqslant 300$, the factor

$$\left(1+\frac{1}{\alpha^{\sqrt{2k}}}\right)^8\left(1+\frac{2}{\alpha^{2k}}\right)^2$$

is smaller than 1.00007. In particular, smaller than α. So,

$$2k < 24\sqrt{2k}+10,$$

which gives $\sqrt{2k} < 25$, so $k \leqslant 312$. For $k \in [3,312]$ and $x \in [0,4k-1]$, we checked $F_k \mid F_x^2+F_x+1$, getting only the following values of (k,x):

$$(4,1),(4,2),(4,7),(4,9),(4,10),(4,15),(7,4),(7,10).$$

So, indeed $k \in \{4,7\}$. The rest of the statements come from the analysis of the sequence $(F_n^2+F_n+1)_{n\in\mathbb{Z}}$ modulo $F_4 = 3$ and modulo $F_7 = 13$, respectively.

PROOF OF Theorem 2

This proof is very similar to the proof of Theorem 1. Thus, we emphasize only the differences.

Assume that $L_k \mid F_x^2+F_x+1$. First observe that since $F_{2k} = F_kL_k$ we get that the sequence $\{F_n\}_{n\geqslant 0}$ is periodic modulo L_k with period $8k$. Hence we suppose that $|x| \leqslant 4k$. Clearly,

$$L_k \,\Big|\, \frac{(\alpha^{2x}-\sqrt{5}\omega\alpha^x-(-1)^x)}{\sqrt{5}}\cdot\frac{(\alpha^{2x}-\sqrt{5}\overline{\omega}\alpha^x-(-1)^x)}{\sqrt{5}},$$

and we have

$$\frac{L_k}{\varepsilon^2}\mathscr{O}_{\mathbb{K}} = \mathscr{U}\cdot\mathscr{V} = (U\mathscr{O}_{\mathbb{K}})\cdot(V\mathscr{O}_{\mathbb{K}}). \qquad (12)$$

Now

$$\frac{L_k}{\varepsilon^2} = \lambda U\overline{U},$$

and later

$$U\mathscr{O}_{\mathbb{L}} \mid \mathscr{U}_1\mathscr{U}_2,$$

where again $\mathscr{U}_i = \gcd(\mathscr{U}\mathscr{O}_{\mathbb{L}}, (\alpha^x - z_i)\mathscr{O}_{\mathbb{L}})$. In this case, the system of congruences is

$$\begin{aligned} \alpha^{2k} &\equiv -(-1)^k = (-1)^{k+1} \pmod{\mathscr{U}_i};\\ \alpha^x &\equiv z_i \pmod{\mathscr{U}_i}, \end{aligned} \qquad (13)$$

because $\mathscr{U}_i \mid \mathscr{U} \mid L_k \mid \alpha^k+\beta^k$ and

$$\alpha^k+\beta^k = \alpha^{-k}(\alpha^{2k}+(-1)^k).$$

Knowing $\max\{2k,|x|\} \leqslant 4k$, there are integers a, b with $\max\{|a|,|b|\} \leqslant \sqrt{4k} = 2\sqrt{k}$ such that $|2ka+xb| \leqslant 6\sqrt{k}$. Subsequently,

$$|\alpha^{2ka+xb}-(-1)^{(k+1)a}z_i^b| \leqslant \alpha^{6\sqrt{k}}+\alpha^{4\sqrt{k}} = \alpha^{6\sqrt{k}}\left(1+\frac{1}{\alpha^{2\sqrt{k}}}\right).$$

Since $L_k > F_k$, we get

$$\alpha^{2k-4} \leqslant \alpha^{48\sqrt{k}+5}\left(1+\frac{1}{\alpha^{2\sqrt{k}}}\right)^8\left(1+\frac{2}{\alpha^{2k}}\right)^2,$$

and then

$$2k < 48\sqrt{k}+10.$$

Hence $k \leqslant 585$. For $k \in [2,585]$ and $x \in [0,8k-1]$, we checked $L_k \mid F_x^2+F_x+1$, getting only the following values of (k,x):

$$(2,1),(2,2),(2,7),(2,9),(2,10),(2,15),$$
$$(4,3),(4,12),(4,13),(4,19),(4,28),(4,29).$$

The rest of the statements come from the analysis of the sequence $\{F_n^2+F_n+1\}_{n\in\mathbb{Z}}$ modulo $L_2 = 3$ and modulo $L_4 = 7$, respectively.

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