RESEARCH ARTICLE

doi: 10.2306/scienceasia1513-1874.2026.079

ScienceAsia 52 (4): 2026: ID 2026079: **1–6**

Entire solutions of a class of delay-differential equations

Changwen Peng[a], **Zhijin Zhou**[a], **Huawei Huang**[b,*], **Lei Tao**[c]

[a] School of Science, Guiyang University, Guiyang 550005 China
[b] School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025 China
[c] School of Mathematics and Big Data, Guizhou Education University, Guiyang 550018 China

*Corresponding author, e-mail: hwhuang7809@163.com

Received 7 Aug 2025, Accepted 11 Aug 2026
Available online 2 Sep 2026

ABSTRACT: This study focuses on the delay-differential equation $\bar{g}\underline{g}+B(z)g'/g=R(z,g)=\frac{P(z,g)}{Q(z,g)}$, where $P(z,g)$ and $Q(z,g)$ are polynomials in g with rational coefficients in z and no common roots, $B(z)$ is rational and $g\equiv g(z)$, $\bar{g}\equiv g(z+1)$, $\underline{g}\equiv g(z-1)$. Reduced forms of this equation arise when it admits a transcendental entire solution of hyper-order less than one. Furthermore, we examine the properties of entire solutions for these reduced forms.

KEYWORDS: delay-differential equation, hyper-order, entire solution

MSC2020: 30D35 34M55

INTRODUCTION AND KEY FINDINGS

Basic results from the Nevanlinna theory (refer to [1–3]) are used in the subsequent analysis. Consider a meromorphic function g. Additionally, standard notations such as $\sigma(g)$ for the order, $\lambda(g)$ for the convergence exponent regarding zeros of g, and similar conventions are utilized. In what follows, for convenience indeed, we set $g\equiv g(z)$, $\bar{g}\equiv g(z+1)$, $\underline{g}\equiv g(z-1)$. Here, we reintroduce the definition of hyper-order for g as follows

$$\sigma_2(g)=\limsup_{r\to\infty}\frac{\log\log T(r,g)}{\log r}.$$

Furthermore, a meromorphic function α is called a small function with respect to g if $T(r,\alpha)=S(r,g)$, where $S(r,g)$ denotes any quantity satisfying $S(r,g)=o(T(r,g))$ as $r\to\infty$, possibly outside a set of finite logarithmic measure.

Halburd and Korhonen [4], along with Chiang and Feng [5], and Laine and Yang [6], pioneered the progression of the difference analogue of Nevanlinna theory. This theoretical framework has been employed to investigate the characteristics of solutions to difference equations. Numerous significant findings have been achieved through this approach, for example, see [5, 7, 8] and so on.

The Nevanlinna theory is extensively applied in the examination of delay-differential equations (see [9–19]). In particular, Quispel, Capel and Sahadevan [16] analyzed the equation

$$g(\bar{g}-\underline{g})+ag'=bg,$$

where a and b are constants. This equation results from applying symmetry reduction to the Kac-Van Moerbeke equation and admits a formal continuum limit connecting it to the first Painlevé equation $g''=6g^2+t$.

In 2017, Halburd and Korhonen [12] considered an extended version of the equation $g(\bar{g}-\underline{g})+ag'=bg$, which is expressed in the form $\bar{g}-\underline{g}+B(z)g'/g=P(z,g)/Q(z,g)$. From their analysis, the following result follows.

Theorem 1 ([12]) *Let g be a transcendental meromorphic solution of*

$$\bar{g}-\underline{g}+B(z)g'/g=R(z,g)=\frac{P(z,g)}{Q(z,g)},\qquad(1)$$

where $P(z,g)$ and $Q(z,g)$ are polynomials in g with rational coefficients in z, the roots of $Q(z,g)$ are nonzero rational functions of z and are distinct from those of $P(z,g)$, and $B(z)$ is rational. If $\sigma_2(g)<1$, then

$$\deg_g(P)=\deg_g(Q)+1\leqslant 3 \textit{ or } \deg_g(R)\leqslant 1.$$

Subsequently, Liu and Song [14] investigated a particular delay-differential equation. This equation combines a difference equation $\bar{g}\underline{g}=R(z,g)$ with a certain differential equation $g'=R(z,g)$. They obtained the following Theorem 2 analogous to Theorem 1.

Theorem 2 ([14]) *Let g be a transcendental meromorphic solution of*

$$\bar{g}\underline{g}+B(z)g'/g=R(z,g)=\frac{P(z,g)}{Q(z,g)},\qquad(2)$$

where $P(z,g)$ and $Q(z,g)$ are polynomials in g with rational coefficients in z, the roots of $Q(z,g)$ are nonzero rational functions of z and are distinct from those of $P(z,g)$, and $B(z)$ is rational. If $\sigma_2(g)<1$, then

$$\deg_g(Q)+1=\deg_g(P)\leqslant 3,$$

or the degree of $R(z,g)$ as a rational function in g equals 1.

Zhang and Huang [19] studied Eq. (1) and reached the following conclusion.

Theorem 3 ([19]) *Let $R(z,g) \not\equiv 0$ be an irreducible rational function in g with rational coefficients in z, and let $B(z)$ be rational. If the Eq. (1) admits a transcendental entire solution g satisfying $\sigma_2(g) < 1$, then (1) transforms into*

$$\bar{g} - \underline{g} + B(z)g'/g = B_1(z)g + B_0(z), \tag{3}$$

or $$\bar{g} - \underline{g} + B(z)g'/g = \frac{B_2(z)g^2 + B_1(z)g + B_0(z)}{g},$$

where $B_j(z)(j=0,1,2)$ are rational functions.

Combining Theorem 2 and Theorem 3, the subsequent theorem arises.

Theorem 4 *Assume that $P(z,g)$ and $Q(z,g)$ are relatively prime polynomials in g, with rational coefficients in z, and let $B(z)(\not\equiv 0)$ be rational. Suppose Eq. (2) admits a transcendental entire solution g satisfying $\sigma_2(g) < 1$.*

(i) *If g possesses finitely many zeros, then (2) transforms into*

$$\bar{g}\,\underline{g} + B(z)g'/g = B_2(z)g^2 + B_0(z), \tag{4}$$

where $B_2(z)$, $B_0(z)$ are rational functions.

(ii) *If g possesses infinitely many zeros, then (2) transforms into*

$$\bar{g}\,\underline{g} + B(z)\frac{g'}{g} = \frac{B_3(z)g^3 + B_2(z)g^2 + B_1(z)g + B_0(z)}{g}, \tag{5}$$

where $B_i(z)$ (for $i=0,1,2,3$) are rational functions.

Remark 1 (i) Considering $R(z,g)(\not\equiv 0)$ as an irreducible rational function in g, we obtain $B_0(z) \not\equiv 0$ and $B_2(z) \not\equiv 0$ in (4), while in (5) neither $B_3(z)$ nor $B_0(z)$ vanishes.

(ii) In Theorem 2, $\deg_g(Q) + 1 = \deg_g(P) \leqslant 3$. In Theorem 4, $\deg_g(P) = \deg_g(Q) + 2$ in (4) and (5).

Thus, Theorem 4 stands independently from Theorem 2.

By combining Theorem 2 and Theorem 4, Corollary 1 can be readily derived.

Corollary 1 *Assume $B(z)$ is rational, and let $P(z,g)$ and $Q(z,g)$ be polynomials in g with rational coefficients in z, such that the roots of $Q(z,g)$ are nonzero rational functions of z and are distinct from those of $P(z,g)$. Then Eq. (2) admits no transcendental entire solution g satisfying $\sigma_2(g) < 1$.*

In their work [19], Zhang and Huang investigated entire solutions to Eq. (3) with $\sigma_2(g) < 1$, and their analysis yielded Theorem 5, stated as follows.

Theorem 5 ([19]) *Consider rational functions $B(z)$, $B_0(z)$ and $B_1(z)$, with at least one of $B_1(z)$ or $B_0(z)$ nonzero. Suppose g is a transcendental entire solution of (3) satisfying $\sigma_2(g) < 1$.*

(i) *When $B(z) \equiv 0$, one has $\sigma(g) \geqslant 1$.*

(ii) *When $B(z) \not\equiv 0$, one obtains $g(z) = L(z)\,\mathrm{e}^{lz}$, where $L(z)(\not\equiv 0)$ is a polynomial and $l \in \mathbb{C}/\{0\}$. In particular, if $B_1(z)$ is a polynomial with $B_1(z) \not\equiv \pm 2\mathrm{i}$, then $g(z) = C_0\,\mathrm{e}^{lz}$, where $C_0 \in \mathbb{C}/\{0\}$; if $B_1 \equiv \pm 2\mathrm{i}$, then $g(z) = (A_1 z + A_0)\,\mathrm{e}^{(2n\pm\frac{1}{2})\pi \mathrm{i} z}$, where $n \in \mathbb{Z}$ and $A_1, A_0 \in \mathbb{C}$ satisfying $|A_1| + |A_0| \neq 0$.*

Inspired by the results established in Theorem 5, we now investigate the entire solutions to the reduced form (4), and aim to determine the specific forms that such solutions must take.

Theorem 6 *Let $B_2(z)$, $B_0(z)$ and $B(z)(\not\equiv 0)$ be rational functions, with $B_2(z)$ and $B_0(z)$ both nonzero. Suppose g is a transcendental entire solution of (4) satisfying $\sigma_2(g) < 1$. Then $g(z) = L(z)\,\mathrm{e}^{\lambda z^2 + \mu z}$, where $L(z)(\not\equiv 0)$ is a polynomial and $\lambda, \mu \in \mathbb{C}$ satisfy $|\lambda| + |\mu| \neq 0$. In particular, if $B_2(z)$ is a polynomial, then $g(z) = L\,\mathrm{e}^{\lambda z^2 + \mu z}$ for some $L \in \mathbb{C}\backslash\{0\}$, and $B_2(z) = \mathrm{e}^{2\lambda}$.*

Theorem 6 establishes the explicit form of entire solutions to Eq. (4), however, characterizing its meromorphic solutions remains an open problem.

The Examples 1–3 provided below illustrate the existence of the form (4) as stated in Theorem 4.

Example 1 The equation

$$\bar{g}\,\underline{g} + B(z)\frac{g'}{g} = \frac{z^2-1}{z^2}g^2 + \frac{z+1}{z}B(z)$$

possesses an entire solution $g(z) = ze^z$, where $B(z)$ is a nonzero rational function. It's clear that $\sigma(g) = 1$ and $\sigma_2(g) = 0$.

Example 2 The equation

$$\bar{g}\,\underline{g} + \frac{z-1}{2z^2-z}\,\frac{g'}{g} = \frac{z(z-2)e^2}{(z-1)^2}g^2 + 1$$

possesses an entire solution $g(z) = (z-1)\cdot \mathrm{e}^{z^2+z}$. It's clear that $\sigma(g) = 2$, $\sigma_2(g) = 0$.

Example 3 The equation

$$\bar{g}\,\underline{g} + B(z)\frac{g'}{g} = \mathrm{e}^{-2}g^2 + B(z)(2-2z)$$

is characterized by the transcendental entire solution $g(z) = \gamma\,\mathrm{e}^{-z^2+2z}$, where γ stands as a nonzero constant. It's clear that $\sigma(g) = 2$, $\sigma_2(g) = 0$, $B_2(z) = \mathrm{e}^{-2}$.

Furthermore, an analysis of the characteristics of entire solutions for Eq. (5) leads to Theorem 7, described as:

Theorem 7 *Let $B_i(z)$ for $i = 0, 1, 2, 3$ and $B(z)$ be rational functions, with $B_0(z) \not\equiv 0$ and $B(z) \not\equiv 0$. If g is a transcendental entire solution of (5) satisfying $\sigma_2(g) < 1$, then $\sigma(g) \geqslant 1$ and $\lambda(g) = \sigma(g)$.*

Example 4 provided below illustrates that the situation described in Theorem 4 and Theorem 7 can indeed occur.

Example 4 The equation

$$\bar{g}\underline{g} + z\frac{g'}{g} = \frac{g^3 + 2\pi i z g - 2\pi i z}{g}$$

admits an entire solution given by $g(z) = e^{2\pi i z} + 1$. It's evident that g has an infinite number of zeros, and $\lambda(g) = \sigma(g) = 1$.

When the degree of $Q(z, g)$ with respect to g is zero, Liu and Song derived Theorem 8, which states the following.

Theorem 8 ([14]) *Let g be a nonrational meromorphic solution of*

$$\bar{g}\underline{g} + B(z)g'/g = P(z, g),$$

where $B(z)$ is rational and $P(z, g)$ is a polynomial in g. If $\sigma_2(g) < 1$, then $\deg_g(P) \leqslant 2$.

By combining Theorem 2 with Theorem 8, we establish the ensuing Theorem 9.

Theorem 9 *Let $a_0, \ldots, a_p$, $b_0, \ldots, b_d$, and B be rational functions with $a_p(z)b_d(z) \not\equiv 0$. Suppose g is a nonrational meromorphic solution of*

$$\bar{g}\underline{g} + B(z)\frac{g'}{g} = \frac{P(z, g)}{Q(z, g)} = \frac{a_0(z) + \cdots + a_p(z)g^p}{b_0(z) + \cdots + b_d(z)g^d}, \quad (6)$$

where $P(z, g)$ and $Q(z, g)$ are coprime polynomials in g. If $\sigma_2(g) < 1$, it follows that $\deg_g(P) - \deg_g(Q) = p - d \leqslant 2$.

Example 5 below demonstrates that (6) can indeed have a solution with hyper-order less than one when $\deg_g(P) - \deg_g(Q) = p - d \leqslant 2$.

Example 5 The equation

$$\bar{g}\underline{g} + \frac{g'}{g} = \frac{g^3 + \pi g^2 + \pi}{g}$$

has a meromorphic solution $g(z) = \tan(\pi z)$.

SOME LEMMAS

To substantiate our findings, we rely on the following lemmas.

Lemma 1 ([20]) *Let g be a non-constant meromorphic function and $c \in \mathbb{C}$. For $\sigma_2(g) < 1$ and $\varepsilon > 0$, we have*

$$m\left(r, \frac{g(z+c)}{g(z)}\right) = o\left(\frac{T(r, g)}{r^{1-\sigma_2(g)-\varepsilon}}\right)$$

for all r outside a set of finite logarithmic measure.

In [19], Zhang and Huang derived the subsequent lemma.

Lemma 2 ([19]) *Let g be a transcendental meromorphic function with $\sigma_2(g) < 1$ that solves*

$$U(z, g)P_1(z, g) = Q_1(z, g),$$

where U is a polynomial involving a difference operator on g, and while P_1 and Q_1 are delay-differential polynomials, all having small meromorphic coefficients. Also, $\deg_g(U) = n$, $\deg_g(Q_1) \leqslant n$, and U is assumed to contain a singular term of maximal total degree in g and its shift. Consequently,

$$m(r, P_1(z, g)) = S(r, g).$$

Lemma 3 ([3]) *Suppose that $s_i(z)$ (for $i = 1, \ldots, n$) are meromorphic functions and $l_i(z)$ (for $i = 1, \ldots, n$) are entire functions satisfying the following conditions.*
(i) $\sum_{i=1}^{n} s_i(z) e^{l_i(z)} \equiv 0$;
(ii) *$l_i(z) - l_k(z)$ are not constants for $1 \leqslant i < k \leqslant n$;*
(iii) *For $1 \leqslant i \leqslant n, 1 \leqslant h < k \leqslant n$,*

$$T(r, s_i) = o\{T(r, e^{l_h - l_k})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ has finite linear measure or finite logarithmic measure.
Then $s_i(z) \equiv 0$ $(i = 1, 2, \ldots, n)$.

Lemma 4 ([2]) *Let g be a meromorphic function and $R(z, g)$ be rational in g with small function coefficients with respect to g. Then*

$$T(r, R(z, g)) = \deg_g(R)T(r, g) + S(r, g).$$

Through a thorough examination of the proof provided for [6, Theorem 2.3], Zhang and Huang [19] arrived at Lemma 5.

Lemma 5 ([19]) *Let g be a meromorphic solution of finite order σ of the equation*

$$g^n P_1(z, g) = Q_1(z, g),$$

where P_1 and Q_1 are polynomials in g with rational coefficients, involving differential and difference operators, and satisfying $\deg_g(Q_1) \leqslant n$. Then, for every $\varepsilon > 0$, we obtain

$$m(r, P_1(z, g)) = O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 6 *Let $R(z,g) \not\equiv 0$ be an irreducible rational function in g with rational coefficients in z, and let $B(z) \not\equiv 0$ be rational. Suppose g is a transcendental entire solution of (2) satisfying $\sigma_2(g) < 1$. If g has only finitely many zeros, then (2) can be rewritten as (4), where $B_2(z)$ and $B_0(z)$ are nonzero rational functions.*

Proof: By the hypotheses and Hadamard factorization theorem, the form of g can be expressed as

$$g(z) = L(z)\,\mathrm{e}^{l(z)}, \tag{7}$$

where $L(z)$ is a nonzero polynomial, and $l(z)$ denotes a non-constant entire function satisfying $\sigma_2(g) = \sigma_2(\mathrm{e}^l) = \sigma(l) < 1$. Plugging (7) into (2), we infer

$$\bar{L}\underline{L}\,\mathrm{e}^{\bar{l}+\underline{l}} + B(z)\left(\frac{L'}{L} + l'\right) = \frac{P(z,g)}{Q(z,g)}, \tag{8}$$

where $L = L(z), \bar{L} = L(z+1), \underline{L} = L(z-1)$ and $l = l(z), \bar{l} = l(z+1), \underline{l} = (z-1)$. Setting $G(z) = \bar{L}\underline{L}\,\mathrm{e}^{\bar{l}+\underline{l}-2l}$. Obviously, $G(z) \not\equiv 0$. We deduce from Lemma 1 and $\sigma_2(\mathrm{e}^l) < 1$ that

$$\begin{aligned} T(r,G) &= m(r,G) \\ &\leqslant m\left(r, \mathrm{e}^{\bar{l}-l}\right) + m\left(r, \mathrm{e}^{\underline{l}-l}\right) + O(\log r) = S(r,\mathrm{e}^l). \end{aligned}$$

Utilizing (8), we derive

$$G(z)\,\mathrm{e}^{2l} + B(z)\left(\frac{L'}{L} + l'\right) = \frac{P(z,g)}{Q(z,g)}. \tag{9}$$

Eq. (9) implies

$$\begin{aligned} T\left(r, \tfrac{P(z,g}{Q(z,g}\right) &\leqslant T(r,\mathrm{e}^{2l}) + S(r,\mathrm{e}^l) \\ &= 2T(r,g) + S(r,g). \end{aligned}$$

Based on the preceding inequality and Lemma 4, we obtain $\deg_g(P) \leqslant 2$ and $\deg_g(Q) \leqslant 2$. Hence, (9) implies

$$G(z)\,\mathrm{e}^{2l} + B(z)\left(\frac{L'}{L} + l'\right) = \frac{\hat{a}_2(z)L^2\,\mathrm{e}^{2l} + \hat{a}_1(z)L\,\mathrm{e}^{l} + \hat{a}_0(z)}{\hat{b}_2(z)L^2\,\mathrm{e}^{2l} + \hat{b}_1(z)L\,\mathrm{e}^{l} + \hat{b}_0(z)}, \tag{10}$$

where $\hat{a}_i(z), \hat{b}_i(z)$ $(i = 0,1,2)$ are rational functions. (10) shows

$$\hat{b}_2(z)L^2G\,\mathrm{e}^{4l} + \hat{b}_1(z)LG\,\mathrm{e}^{3l} + S_2\,\mathrm{e}^{2l} + S_1\,\mathrm{e}^{l} + S_0 = 0, \tag{11}$$

where

$$\begin{aligned} S_2 &= \hat{b}_2(z)L^2B(z)\left(\frac{L'}{L} + l'\right) + \hat{b}_0(z)G - \hat{a}_2(z)L^2, \\ S_1 &= \hat{b}_1(z)LB(z)\left(\frac{L'}{L} + l'\right) - \hat{a}_1(z)L, \\ S_0 &= \hat{b}_0(z)B(z)\left(\frac{L'}{L} + l'\right) - \hat{a}_0(z). \end{aligned}$$

By (11) and Lemma 3, we obtain $\hat{b}_2(z) \equiv 0$ and $\hat{b}_1(z) \equiv 0$. Since $S_1 \equiv 0$ and $\hat{b}_1(z) \equiv 0$, we have $\hat{a}_1(z) \equiv 0$. Similarly, since $S_2 \equiv 0$ and $\hat{b}_2(z) \equiv 0$, we obtain $\frac{\hat{a}_2(z)}{\hat{b}_0(z)} = \frac{G}{L^2}$. From (10), we deduce that Eq. (2) is of the form

$$\bar{g}\underline{g} + B(z)g'/g = B_2(z)g^2 + B_0(z),$$

where $B_2(z)$ and $B_0(z)$ are rational functions such that $B_2(z) = \frac{\hat{a}_2(z)}{\hat{b}_0(z)} = \frac{G}{L^2} \not\equiv 0$ and $B_0(z) = \frac{\hat{a}_0(z)}{\hat{b}_0(z)} = B(z)\left(\frac{L'}{L} + l'\right) \not\equiv 0$. □

PROOF OF Theorem 4

Suppose g is a transcendental entire solution to (2) such that $\sigma_2(g) < 1$.

Based on Lemma 1 and (2), we can infer that

$$\begin{aligned} T\left(r, \tfrac{P(z,g)}{Q(z,g)}\right) &= m\left(r, \bar{g}\underline{g} + B(z)g'/g\right) + N\left(r, \bar{g}\underline{g} + B(z)g'/g\right) \\ &\leqslant 2m(r,g) + N\left(r, \tfrac{1}{g}\right) + S(r,g) \\ &\leqslant 3T(r,g) + S(r,g). \end{aligned}$$

According to Lemma 4, it follows that $\deg_g(P) \leqslant 3$ and $\deg_g(Q) \leqslant 3$. Therefore,

$$\frac{P(z,g)}{Q(z,g)} = \frac{\hat{a}_3(z)g^3 + \hat{a}_2(z)g^2 + \hat{a}_1(z)g + \hat{a}_0(z)}{\hat{b}_3(z)g^3 + \hat{b}_2(z)g^2 + \hat{b}_1(z)g + \hat{b}_0(z)}, \tag{12}$$

where $\hat{a}_i(z)$ and $\hat{b}_i(z)$ $(i = 0,1,2,3)$ are rational functions.

(i) When g has finitely many zeros, Lemma 6 shows that Eq. (2) reduces to the form (4) with $B_2(z) \not\equiv 0$ and $B_0(z) \not\equiv 0$.

(ii) Now, let's assume that g has an infinite number of zeros. Consider z_0 as one of the zeros of g, and suppose that neither $B(z)$ nor any coefficient in $\frac{P(z,g)}{Q(z,g)}$ exhibits a pole or zero at z_0. If $\hat{b}_0(z) \not\equiv 0$, then z_0 becomes a simple pole of $\bar{g}\underline{g} + B(z)g'/g$, as well as a finite value of $\frac{P(z,g)}{Q(z,g)}$, a contradiction. So, $\hat{b}_0(z) \equiv 0$.

Assuming $\hat{b}_3(z) \not\equiv 0$, through Eq. (2) and (12), we derive

$$\bar{g}\underline{g} + B(z)\frac{g'}{g} = \frac{\hat{a}_3(z)g^3 + \hat{a}_2(z)g^2 + \hat{a}_1(z)g + \hat{a}_0(z)}{\hat{b}_3(z)g^3 + \hat{b}_2(z)g^2 + \hat{b}_1(z)g}. \tag{13}$$

(13) shows

$$\hat{b}_3(z)g^3\bar{g}\underline{g} = \hat{Q}(z,g), \tag{14}$$

where $\hat{Q}(z,g)$ represents a polynomial involving both differential and difference operators with respect to g, and $\deg_g(\hat{Q}) \leqslant 4$.

Through (14) and Lemma 2, we can establish $T(r,g) = m(r,g) = S(r,g)$. This leads to a contradiction. So, $\hat{b}_3(z) \equiv 0$. Thus, (13) is of the form

$$\bar{g}\underline{g} + B(z)\frac{g'}{g} = \frac{\hat{a}_3g^3 + \hat{a}_2g^2 + \hat{a}_1g + \hat{a}_0}{\hat{b}_2g^2 + \hat{b}_1g}. \tag{15}$$

When $\hat{b}_2(z) \not\equiv 0$, (15) yields

$$\hat{b}_2(z) g^2 \bar{g} \underline{g} = \hat{Q}_1(z, g),$$

where $\hat{Q}_1(z, g)$ is a delay-differential polynomial, and $\deg_g(\hat{Q}_1) \leqslant 3$.

By the same reasoning, we obtain $\hat{b}_2(z) \equiv 0$. Consequently, Eq. (2) can be rewritten in the form (5), where $B_i(z)$ (for $i = 0, 1, 2, 3$) are rational and $B_0(z) \not\equiv 0$.

If $B_3(z) \equiv 0$, then by (5) we get

$$\bar{g} \underline{g} g = B_2(z) g^2 + B_1(z) g + B_0(z) - B(z) g'. \tag{16}$$

According to (16) and Lemma 2, we arrive at the contradiction $T(r, g) = m(r, g) = S(r, g)$. So $B_3(z) \not\equiv 0$.

PROOF OF Theorem 6

Assume that g is a transcendental entire solution of equation (4) and satisfies $\sigma_2(g) < 1$. Similar to the discussion in Theorem 4, we can infer that g possesses only a finite number of zeros. Consequently, it implies $g(z) = L(z) \mathrm{e}^{l(z)}$, where $L(z) \not\equiv 0$ is a polynomial, and $l(z)$ is a non-constant entire function satisfying $\sigma_2(g) = \sigma_2(\mathrm{e}^{l(z)}) = \sigma(l) < 1$. Plugging $g(z) = L(z) \mathrm{e}^{l(z)}$ into (4), we get

$$G(z) \mathrm{e}^{2l} + B(z) \left(\frac{L'}{L} + l' \right) = B_2(z) L^2 \mathrm{e}^{2l} + B_0(z), \tag{17}$$

where $G(z) = \bar{L} \underline{L} \mathrm{e}^{\bar{l} + \underline{l} - 2l} \not\equiv 0$. Lemma 1 implies that $T(r, G) = m(r, G) = S(r, G)$. Applying (17) leads to

$$[G - B_2(z) L^2] \mathrm{e}^{2l} = B_0(z) - B(z) \left(\tfrac{L'}{L} + l' \right). \tag{18}$$

Utilizing Lemma 3 and (18), we derive $G - B_2(z) L^2 \equiv 0$ and $B_0(z) - B(z) \left(\frac{L'}{L} + l' \right) \equiv 0$. Thus, G and l must be polynomials. Furthermore, since G is a polynomial, we conclude that $\bar{l} + \underline{l} - 2l$ is a constant. Thus, we obtain $\deg l \leqslant 2$. Consequently, g can be written as

$$g(z) = L(z) \mathrm{e}^{\lambda z^2 + \mu z}. \tag{19}$$

By (19), $G(z) = \bar{L} \underline{L} \mathrm{e}^{\bar{l} + \underline{l} - 2l}$ and $G - B_2(z) L^2 \equiv 0$, we get

$$\bar{L} \underline{L} \mathrm{e}^{2\lambda} = B_2(z) L^2. \tag{20}$$

Suppose $B_2(z)$ is a polynomial. (20) shows that $B_2(z)$ is a constant. Let's assume

$$L(z) = d_n z^n + d_{n-1} z^{n-1} + d_{n-2} z^{n-2} + \cdots + d_0, \tag{21}$$

where $d_n \neq 0$.

If $n \geqslant 2$, then by (20) and (21), we have

$$d_n d_n \mathrm{e}^{2\lambda} = d_n d_n B_2 \tag{22}$$

and

$$[2d_n (d_n C_n^2 + d_{n-2}) + d_{n-1}^2 - (d_n C_n^1)^2] \mathrm{e}^{2\lambda} = (2 d_n d_{n-2} + d_{n-1}^2) B_2. \tag{23}$$

By (22) and $d_n \neq 0$, we get $\mathrm{e}^{2\lambda} = B_2$. And we get $n(n-1) = n^2$ from (23) and $\mathrm{e}^{2\lambda} = B_2$. This leads to a contradiction. So, $n \leqslant 1$.

If $n = 1$, then we have by (20) and (21) that

$$d_1^2 z^2 + 2 d_1 d_0 z + d_0^2 - d_1^2 = d_1^2 z^2 + 2 d_1 d_0 z + d_0^2.$$

From the above equality, we have $d_1 = 0$. This leads to a contradiction. So, we have $L(z) = d_0 \neq 0$ and $B_2 = \mathrm{e}^{2\lambda}$. Hence, $g(z) = L\, \mathrm{e}^{\lambda z^2 + \mu z}$, where $L \in \mathbb{C} \backslash \{0\}$.

PROOF OF Theorem 7

On the contrary, we assume $\sigma(g) < 1$. (5) yields,

$$g P_1(z, g) = B_0(z) - B(z) g', \tag{24}$$

where $P_1(z, g) = \bar{g} \underline{g} - B_3(z) g^2 - B_2(z) g - B_1(z)$. Given that g is transcendental and $B(z) \not\equiv 0$ is rational, we deduce that $B_0(z) - B(z) g' \not\equiv 0$. Thus, $P_1(z, g) \not\equiv 0$. Utilizing Lemma 5 for Eq. (24), we find

$$T(r, P_1(z, g)) = m(r, P_1(z, g)) = O(\log r). \tag{25}$$

Rewrite (24) as

$$-\frac{P_1(z, g)}{B(z)} = \frac{g'}{g} - \frac{B_0(z)}{B(z) g}. \tag{26}$$

Considering (25), (26), and the rationality of $B(z)$, we arrive at $-\frac{P_1(z,g)}{B(z)} \not\equiv 0$ is rational. Thus, we find

$$-\frac{P_1(z, g)}{B(z)} = H z^n (1 + o(1)),$$

where $z \to \infty$, H is a nonzero constant and $n \in \mathbb{Z}$. Based on the Wiman-Valiron theory (see [2, p.51], [21, pp.28–32], or [22, pp.103–105]) and following the same procedure as in [19], we obtain

$$\nu(r) = |H| r^{n+1} (1 + o(1)), \ (|z| = r \notin [0, 1] \cup E,\ r \to \infty), \tag{27}$$

where $E \subset (1, \infty)$ is a set with finite logarithmic measure, and $\nu(r)$ represents the central index of $g(z)$.

By (27), we get $\sigma(g) = \limsup_{r \to \infty} \frac{\log \nu(r)}{\log r} = n + 1 \leqslant 0$, a contradiction. Therefore, we derive $\sigma(g) \geqslant 1$.

From (26) we obtain $m(r, \frac{1}{g}) \leqslant m(r, \frac{g'}{g}) + m(r, P_1) + O(\log r) = S(r, g)$. Consequently, $N(r, \frac{1}{g}) = T(r, g) + S(r, g)$, which yields $\lambda(g) = \sigma(g)$.

PROOF OF Theorem 9

Suppose g is a nonrational meromorphic solution of (6) satisfying $\sigma_2(g) < 1$.

Conversely, let's assume that $s = \deg_g(P) - \deg_g(Q) = p - d \geqslant 3$. Then by (6), we obtain

$$a_p(z) g^p g = H(z, g), \tag{28}$$

where $H(z,g) = b_d(z)g^{d+1}\bar{g}\underline{g} + \cdots + b_0(z)B(z)g' - a_0(z)g$. Obviously, $\deg_g(H) \leqslant \max\{d+3, p\} \leqslant p$ since $p-d \geqslant 3$. By (28) and Lemma 2, we get $m(r,g) = S(r,g)$. So, $N(r,g) = T(r,g) + S(r,g)$, we see that g possesses an infinite number of poles and $\lambda(\frac{1}{g}) = \sigma(g)$.

Following the argument in [15], let z_1 be a pole of g of order $k_0 \geqslant 1$, and suppose that z_1 avoids all zeros and poles of the coefficients in (6). Then (6) implies that g has a pole at either z_1+1 or z_1-1 (or both). If the pole at $z = z_1+1$ has order k_1 satisfying $k_1 \geqslant \frac{sk_0}{2} > k_0$, then shifting (6) gives

$$g(z+2)g(z)+B(z+1)\frac{g'(z+1)}{g(z+1)} = \frac{P(z+1, g(z+1))}{Q(z+1, g(z+1))}. \quad (29)$$

From $s = p-q \geqslant 3$ and (29), we deduce that z_1+2 is a pole of g of order $k_2 = sk_1 - k_0 > sk_1 - k_1 = (s-1)k_1$.

Advancing (29) by one yields

$$g(z+3)g(z+1)+B(z+2)\frac{g'(z+2)}{g(z+2)} = \frac{P(z+2, g(z+2))}{Q(z+2, g(z+2))}. \quad (30)$$

(30) shows that z_1+3 is a pole of g of order $k_3 = sk_2 - k_1 > sk_2 - k_2 \geqslant (s-1)^2 k_1$.

By the same argument, we obtain

$$n(|z_1|+m, g) \geqslant (s-1)^{m-1}k_1 + O(1) \quad (31)$$

for all integer $m \geqslant 1$. (31) implies

$$\begin{aligned}\lambda_2\left(\frac{1}{g}\right) &= \limsup_{r\to\infty} \frac{\log\log n(r,g)}{\log r} \\ &\geqslant \limsup_{m\to\infty} \frac{\log\log[(s-1)^{(m-1)}k_1]}{\log(|z_1|+m)} = 1.\end{aligned}$$

This contradicts with our assumption. Thus $\deg_g(P) - \deg_g(Q) \leqslant 2$.

Alternatively, the case with a pole at $z_1 - 1$ is handled similarly to that at z_1+1.

Acknowledgements: This work is supported by the Doctoral Research Start-up Project of Guiyang University (GYU-KY-[2026]), the Guizhou Provincial Basic Research Program(Natural Science)(Nos. QIANKEHEJICHU-MS[2025]281 and [2026]042), the Youth Science and Technology Talents Growth Project of Education Department of Guizhou Province of China (No. QIANJIAO JI [2024]159) and the Doctoral Project under the 2024 University Fund of Guizhou Education University of Guizhou Province of China (No. 2024BS007).

REFERENCES

1. Hayman WK (1964) *Meromorphic Functions*, Clarendon Press, Oxford.
2. Laine I (1993) *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin.
3. Yang CC, Yi HX (2003) *Uniqueness Theory of Meromorphic Functions, Mathematics and Its Applications*, Kluwer Academic Publishers Group, Dordrecht.
4. Halburd RG, Korhonen RJ (2006) Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. *J Math Anal Appl* **314**, 477–487.
5. Chiang YM, Feng SJ (2008) On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. *Ramanujan J* **26**, 105–129.
6. Laine I, Yang CC (2007) Clunie theorems for difference and q-difference polynomials. *J Lond Math Soc* **76**, 556–566.
7. Halburd RG, Korhonen RJ (2007) Finite-order meromorphic solutions and the discrete Painlevé equations. *Proc London Math Soc* **94**, 443–474.
8. Wen ZT (2016) Meromorphic solutions to difference Painlevé equations I and II. *Electron J Differ Eq* **262**, 1–18.
9. Dai HX, Qiao JY, Cao TB (2023) On limiting directions of entire solutions of complex differential-difference equations. *Anal Math* **49**, 381–401.
10. Du YS, Zhang JL (2023) Painlevé III and V types differential difference equations. *Comput Metheds Funct Theory* **23**, 327–345.
11. Du YS, Zhang JL (2023) On uniqueness of meromorphic solutions to delay differential equation. *J Contemp Math Anal* **58**, 142–151.
12. Halburd RG, Korhonen RJ (2017) Growth of meromorphic solutions of delay differential equations. *Proc Amer Math Soc* **145**, 2513–2526.
13. Li ZD, Zhang JL (2023) The growth of meromorphic solutions of a class of delay-differential equations. *Anal Math* **49**, 195–206.
14. Liu K, Song CJ (2019) Non-linear complex differential-difference equations admit meromorphic solutions. *Anal Math* **45**, 569–582.
15. Long JR, Xiang XX (2025) On delay-differential equations with subnormal transcendental meromorphic solutions. *Bull Korean Math Soc* **62**, 513–525.
16. Quispel GRW, Capel HW, Sahadevan R (1992) Continuous symmetries of differential-difference equations: the Kac-van Moerbeke equation and Painlevé reduction. *Phys Lett A* **170**, 379–383.
17. Wang YP, Chen WJ, Huang ZG (2022) On meromorphic solutions of certain type of nonlinear differential-difference equations. *ScienceAsia* **48**, 467–473.
18. Xiang XX, Long JR, Liu J M (2023) On solutions of certain nonlinear delay-differential equations. *J Math Anal Appl* **527**, 1–19.
19. Zhang RR, Huang ZB (2020) Entire solutions of delay differential equations of Malmquist type. *J Appl Anal Comput* **10**, 1720–1740.
20. Halburd RG, Korhonen RJ, Tohge K (2014) Holomorphic curves with shift-invariant hyperplane preimages. *Trans Amer Math Soc* **366**, 4267–4298.
21. He YZ, Xiao XZ (1988) *Algebroid Functions and Ordinary Differential Equations*, Science Press, Beijing.
22. Valiron G (1949) *Lectures on the General Theory of Integral Functions*, Chelsea, New York.