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Growth of meromorphic solutions of complex differential and difference equations

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ABSTRACT: In this article, we investigate some properties of meromorphic solution of the following differential-difference equation

$$\sum_{j=0}^{n} c_j f(z+j) + \sum_{j=0}^{n} l_j f^{(j)}(z+j) + \sum_{j=0}^{m} d_j f^{(j)}(z) - c = (f(z+\eta) - f(z) - b)\mathrm{e}^{Q},$$

where $n, m \geqslant 1$ are two integers, η is a constant with $|\eta| \neq n$, Q is a polynomial. We study the growth of solutions of a more general differential-difference equation given by Lü et al [Rest Math **74** (2019):1–18]. Meantime we obtain the relation between $\deg Q$ and $\rho(f)$.

KEYWORDS: meromorphic solution, complex differential-difference, value distribution, growth

MSC2020: 30D35 39A10

INTRODUCTION AND MAIN RESULTS

We assume that the reader is familiar with the basic notation and fundamental results of Nevanlinna theory [1]. Moreover, we use the notation $\rho(f)$ to denote the order of growth of f, and $\lambda(f)$ to denote the exponents of the zeros of f.

Recently, Li and Saleeby [2] considered existence and uniqueness of solutions of the following functional-difference equations

$$f'(z) = af(g(z)) + bf(z) + c, \tag{1}$$

with $a \neq 0, b, c$ are constants. Such equations can be thought of as generalizations of differential-difference equations, and so they appear as models in a large amount of different settings – for example, in the study of wave motion, cell growth, wavelets, etc. Studies of equations of a more general type than (1) have appeared.

The purpose of this article is to study the growth of solutions of a more general differential-difference equation

$$\sum_{j=0}^{n} c_j f(z+j) + \sum_{j=0}^{n} l_j f^{(j)}(z+j) + \sum_{k=0}^{m} d_k f^{(k)}(z) - c = (f(z+\eta) - f(z) - b)\mathrm{e}^{Q}, \tag{2}$$

where $n, m \geqslant 1$ are two integers, $\eta\,(|\eta| \neq n)$ is a constant. And we obtian the following results.

Theorem 1 *Let f be a transcendental entire function with $\lambda(f-a) < \rho(f) = \rho < \infty$, where a is an entire function satisfying $\rho(a) < \rho$, and let b, c, c_j, l_j ($j = 0, 1, 2, \dots, n$), d_k ($k = 0, 1, 2, \dots, m$) be entire function such that $T(r, b) = S(r, f)$, $T(r, c) = S(r, f)$, $\rho(c_j) < \rho - 1$, $\rho(l_j) < \rho - 1$, $\rho(d_k) < \rho - 1$. If f is a solution of (2), then* $\deg Q = \rho(f) - 1$.

Remark 1 The condition $\rho(c_j) < \rho - 1$ in Theorem 1 cannot not be deleted. For example, the equation

$$\mathrm{e}^{-z-\frac{3}{4}} f(z+1) - \mathrm{e}^{-4z-4} f(z+2) = f(z+\tfrac{1}{2}) - f(z),$$

has a solution $f(z) = \mathrm{e}^{z^2}$, where $c_1 = \mathrm{e}^{-z-\frac{3}{4}}$, $c_2 = \mathrm{e}^{-4z-4}$ and $\mathrm{e}^{Q} = 1$. Here $\rho(c_1) = 1 = \rho(f) - 1$, $\rho(c_2) = 1 = \rho(f) - 1$. But, $\deg Q = 0 \neq 1 = \rho(f) - 1$.

Some idea of the proof of Theorem 1 is based on [3].

Theorem 2 *Let $c_j(z), l_j(z)$ ($j = 0, 1, 2, \dots, n$), d_k ($k = 0, 1, 2, \dots, m$) be meromorphic functions, and set*

$$\sigma = \max\{\sigma(c_j), \sigma(l_j), \sigma(d_k)\}.$$

If $f(z) (\not\equiv 0)$ is a finite order transcendental meromorphic solution of equation

$$\sum_{j=0}^{n} c_j f(z+j) + \sum_{j=0}^{n} l_j f^{(j)}(z+j) + \sum_{k=0}^{m} d_k f^{(k)}(z) + c = 0, \tag{3}$$

where c is a meromorphic function such that $c \not\equiv 0$ and $T(r, c) = S(r, f)$, then we have
(i) *if $\sigma \geqslant \lambda_f$, then $\sigma \geqslant \sigma(f) - 1$;*
(ii) *if $\sigma < \lambda_f$, then $\lambda_f \geqslant \sigma(f) - 1$,*
where $\lambda_f = \max\{\lambda(f), \lambda(1/f)\}$.

Example 1 The equation

$$\frac{1}{e}e^{-2z}f(z+1)+\frac{1}{e}e^{-2z}f'(z+1)+\frac{1}{e^4}e^{-4z}f''(z+2)-f''(z)-9f'(z)-19f(z)-19+\frac{1}{e^{2z+1}}=0$$

has a solution $f(z)=e^{z^2}-1$. Here, $2=\lambda_f>\sigma=1$ and $\lambda_f>\sigma(f)-1=1$, the equation and its solution satisfy Theorem 2(ii).

Theorem 3 *Let $c_j(z), l_j(z) (j=0,1,2,\dots,n)$, $d_k (k=0,1,2,\dots,m)$, c be meromorphic functions, such that $T(r,c_j)=S(r,f)$, $T(r,l_j)=S(r,f)$, $T(r,d_k)=S(r,f)$ and $T(r,c)=S(r,f)$. If $f(z)$ is a finite order transcendental meromorphic solution of (3), and $d\sum_{j=0}^{n}c_j+d_0d-c\neq 0$, then $\lambda(f-d)=\sigma(f)$, where d is a constant.*

Example 2 The equation

$$\frac{1}{e}f(z+1)+\frac{1}{e}f'(z+1)+\frac{1}{e^2}f''(z+2)-f(z)-2f'(z)-\frac{1}{e(z+1)^2}+\frac{1}{e(z+1)}+\frac{2}{e^2(z+2)^3}-\frac{1}{z}+\frac{2}{z^2}=0$$

has a solution $f(z)=e^z-\frac{1}{z}$. Here, $\frac{d}{e}-d+\frac{1}{e(z+1)^2}-\frac{1}{e(z+1)}-\frac{2}{e^2(z+2)^3}+\frac{1}{z}-\frac{2}{z^2}\not\equiv 0$ and $\lambda(f)=\lambda(f-d)=\sigma(f)=1$. The equation and its solution satisfy Theorem 3.

PRELIMINARY LEMMAS

Lemma 1 ([1]) *Suppose that $f_1,f_2,\dots,f_n$ $(n\geqslant 2)$ are meromorphic functions and $g_1,g_2,\dots,g_n$ are entire functions satisfying the following conditions*

(i) $\sum_{j=1}^{n}f_je^{g_j}\equiv 0$;

(ii) *g_j-g_k are not constants for $1\leqslant j<k\leqslant n$;*

(iii) *for $1\leqslant j\leqslant n$, $1\leqslant h<k\leqslant n$, $T(r,f_j)=o\{T(r,e^{g_h-g_k})\}$ $(r\to\infty, r\notin E)$, where E is a set of $r\in(0,\infty)$ with finite linear measure.*

Then $f_j\equiv 0$ $(j=1,2,\dots,m)$.

Lemma 2 ([4]) *Let f be a non-constant meromorphic function with $\rho_2(f)<1$ and let c be a non-zero complex number and k be a positive integer. Then*

$$m\left(r,\frac{f^{(k)}(z+c)}{f(z)}\right)=S(r,f),$$

outside of a possible exceptional set with finite logarithmic measure.

Lemma 3 ([5]) *Let f be a non-constant meromorphic function with $\rho_2(f)<1$ and η be a nonzero constant. Then*

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)=S(r,f),\quad m\left(r,\frac{f(z)}{f(z+\eta)}\right)=S(r,f).$$

PROOF OF Theorem 1

Let $g=f-a$. Then $\lambda(g)=\lambda(f-a)<\rho(f)=\rho(g)=\rho$. Hence

$$g=f-a=I(z)e^{S(z)},$$

where $I(z)$ is an entire function and $S(z)$ is a nonzero polynomial such that $\rho(I)<\rho(f)=\rho=\deg S$. Substituting $f=a+I(z)e^{S(z)}$ into (2), we have

$$\sum_{j=0}^{n}c_ja(z+j)+\sum_{j=0}^{n}c_jI(z+j)e^{S(z+j)}+\sum_{j=0}^{n}l_ja^{(j)}(z+j)+\sum_{j=0}^{n}l_j\tilde{S}_j(I(z+j))e^{S(z+j)}+\sum_{k=0}^{m}d_kS_j(I)e^{S(z)}+\sum_{k=0}^{m}d_ka^{(k)}-c$$
$$=(I(z+\eta)e^{S(z+\eta)}+a(z+\eta)-I(z)e^{S(z)}-a(z)-b)e^{Q(z)}.\quad(4)$$

where

$$S_k(I)=I^{(k)}+\lambda_{k-1}I^{(k-1)}+\dots+\lambda_0I,$$

λ_j $(0\leqslant j\leqslant k-1)$ are polynomials. And

$$\widetilde{S_k}(I(z+k))=\tilde{I}^{(k)}(z+k)+\tilde{\lambda}_{k-1}\tilde{I}^{(k-1)}(z+k)+\dots+\tilde{\lambda}_0\tilde{I}(z+k),$$

$\tilde{\lambda}_j$ $(0\leqslant j\leqslant k-1)$ are polynomials. Eq. (4) implies that

$$\phi_1e^{S(z)}+\phi_2=(\phi_3e^{S(z)}+\phi_4)e^{Q(z)},\quad(5)$$

where

$$\phi_1=\sum_{j=0}^{n}c_jI(z+j)e^{S(z+j)-S(z)}+\sum_{j=0}^{n}l_j\widetilde{S}_j(I(z+j))e^{S(z+j)-S(z)}+\sum_{k=0}^{m}d_kS_k(I),$$
$$\phi_2=\sum_{j=0}^{n}c_ja(z+j)+\sum_{k=0}^{m}d_ka^{(k)}+\sum_{j=0}^{n}l_ja^{(j)}(z+j)-c,$$
$$\phi_3=I(z+\eta)e^{S(z+\eta)-S(z)}-I(z),$$
$$\phi_4=a(z+\eta)-a(z)-b.$$

Since $\rho(I)<\rho=\rho(e^S)$ and $\rho(e^{S(z+j)-S(z)})<\rho=\rho(e^S)$, we have $\rho(S_k(I))<\rho$, $\rho(\widetilde{S}_k(I(z+k)))<\rho$. Hence $\rho(\phi_i)<\rho$ $(i=1,2,3,4)$. We assume that $\phi_3\neq 0$. Otherwise, if $\phi_3=0$, then $e^{S(z+\eta)-S(z)}=\frac{I(z)}{I(z+\eta)}$. Hence we have

$$m(r,e^{\rho(z+\eta)-\rho(z)})=O(r^{\rho(f)-1}),$$
$$m\left(r,\frac{I(z)}{I(z+\eta)}\right)=O(r^{\rho(I)-1+\varepsilon}).$$

It is impossible. Next, we divide the proof into the following two cases.

Case 1: $\phi_2=0$.

We assume that $\phi_1\neq 0$. Otherwise, Eq. (5) becomes $e^{S(z)}=-\frac{\phi_4}{\phi_3}$, so $\rho=\rho(e^{S(z)})=\rho(-\frac{\phi_4}{\phi_3})<\rho$, which is impossible. By $\phi_1\neq 0$ and (5), we have

$$\phi_1e^{S(z)}=(\phi_3e^{S(z)}+\phi_4)e^{Q(z)}.$$

If $\phi_4 \neq 0$, then $\phi_3 e^{S(z)} + \phi_4$ has the same zeros with ϕ_1. By the Nevanlinna's second fundamental theorem, we have

$$T(r, e^S) \leqslant \bar{N}(r, e^S) + \bar{N}(r, \tfrac{1}{e^S}) + \bar{N}\left(r, \frac{1}{e^S + \frac{\phi_4}{\phi_3}}\right) + S(r, e^S)$$
$$\leqslant N(r, \tfrac{1}{\phi_1}) + S(r, e^S) \leqslant T(r, \phi_1) + S(r, e^S) = S(r, e^S),$$

this is impossible. Hence, $\phi_4 = 0$ and $\phi_1 = \phi_3 e^{Q(z)}$, that is

$$\sum_{j=0}^{n} c_j \frac{I(z+j)}{I(z+\eta)} e^{S(z+j)-S(z)} + \sum_{j=0}^{n} l_j \frac{\tilde{S}_j(I(z+j))}{I(z+\eta)} e^{S(z+j)-S(z)}$$
$$+ \sum_{k=0}^{m} d_k \frac{S_k(I)}{I(z+\eta)} = (e^{S(z+\eta)-S(z)} - \frac{I(z)}{I(z+\eta)}) e^{Q(z)}. \quad (6)$$

Let $S(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + m_0$, where $b_m \neq 0, b_{m-1}, \ldots, b_0$ are constants and m is a positive integer. By $\rho(c_j) < \rho - 1$, $\rho(l_j) < \rho - 1$, $\rho(d_j) < \rho - 1$, we have $m = \rho(e^S) = \rho > 1$. Hence $m \geqslant 2$. So

$$e^{S(z+j)-S(z)} = e^{jmb_m z^{m-1}} e^{S_j(z)} = e^{S_j(z)} w^j(z), \quad \leqslant j \leqslant n,$$

where $w = e^{mb_m z^{m-1}}$ and $\deg S_j \leqslant m-2$. By $\rho(I) < \rho(e^S) = \rho(w) + 1$. By Lemma 2 and Lemma 3, we have

$$m\left(r, \frac{I(z+j)}{I(z+\eta)}\right) = S(r, w), \quad m\left(r, \frac{I^{(k)}(z+c)}{I(z)}\right) = S(r, w).$$

Eq. (6) implies that

$$\left(e^{S(z+\eta)-S(z)} - \frac{I(z)}{I(z+\eta)}\right) e^{Q(z)} = \sum_{j=0}^{n} g_{n-j} w^j + M(z)$$
$$= F_n(w) + M(z), \quad (7)$$

where $g_{n-j} = \left(c_j \frac{I(z+j)}{I(z+\eta)} + l_j \frac{\tilde{S}_j(I(z+j))}{I(z+\eta)}\right) e^{S_j}$, $F_n(w) = \sum_{j=0}^{n} g_{n-j} w^j$ and $M(z) = \sum_{k=0}^{m} d_k \frac{S_k(I)}{I(z+\eta)}$.

By $\rho(c_j) < \rho - 1 = \rho(w)$ and $\rho(g_k) < \rho - 1 = \rho(w)$, we have

$$m(r, g_{n-j}) = S(r, w),$$
$$m(r, \tfrac{1}{g_{n-j}}) = S(r, w),$$
$$m(r, M(z)) = S(r, w).$$

Next, we prove $m(r, F_n(w)) = nm(r, w) + S(r, w)$. Since

$$T(r, F_n(w)) = nT(r, w) + S(r, w).$$

It is obviously that

$$N(r, F_n(w)) = nN(r, w) + S(r, w).$$

Hence

$$m(r, F_n(w)) = nm(r, w) + S(r, w)$$
$$= n \frac{|m||b_m|}{\pi} (1 + o(1)) r^{s-1} + S(r, w).$$

Together with $m(r, M(z)) = S(r, w)$, we have

$$m(r, (e^{S(z+\eta)-S(z)} - \frac{I(z)}{I(z+\eta)}) e^Q) = m(r, F_n(w) + M(z))$$
$$= n \frac{|m||b_m|}{\pi} (1 + o(1)) r^{s-1} + S(r, w). \quad (8)$$

Eqs. (6) and (8) imply that $\deg Q \leqslant \rho - 1$. If $\deg Q < \rho - 1$,then

$$m(r, (e^{S(z+\eta)-S(z)} - \frac{I(z)}{I(z+\eta)}) e^Q)$$
$$= |\eta| \frac{|m||b_m|}{\pi} (1 + o(1)) r^{s-1} + S(r, w). \quad (9)$$

Together (8) with (9), we have $|\eta| = n$, this is impossible. Hence $\deg Q = \rho - 1$.

Case 2: $\phi_2 \neq 0$.

If $\phi_1 = 0$, then

$$0 = \frac{\phi_1}{I(z+\eta)} = F_n(w) + M(z),$$

So

$$0 = m(r, F_n(w) + M(z)) = nm(r, w) + S(r, w)$$
$$= n \frac{|m||b_m|}{\pi} (1 + o(1)) r^{s-1} + S(r, w),$$

which is impossible. Hence $\phi_1 \neq 0$. Eq. (5) implies that $\phi_3 e^{S(z)} + \phi_4$ has the same zeros with $\phi_1 e^{S(z)} + \phi_2$. If $\phi_3(z_0) e^{S(z_0)} + \phi_4(z_0) = 0$, then

$$\phi_1(z_0) e^{S(z_0)} + \phi_2(z_0) = 0$$

and

$$\frac{\phi_4(z_0)}{\phi_3(z_0)} - \frac{\phi_2(z_0)}{\phi_1(z_0)} = 0.$$

If $\frac{\phi_4}{\phi_3} - \frac{\phi_2}{\phi_1} \not\equiv 0$, then by the Nevanlinna's second fundamental theorem, we obtain

$$T(r, e^S) \leqslant \bar{N}(r, e^S) + \bar{N}\left(r, \frac{1}{e^S}\right) + \bar{N}\left(r, \frac{1}{e^S + \frac{\phi_4}{\phi_3}}\right) + S(r, e^S)$$
$$\leqslant N\left(r, \frac{1}{\frac{\phi_4}{\phi_3} - \frac{\phi_2}{\phi_1}}\right) + S(r, e^S) = S(r, e^S),$$

which is impossible. Hence

$$\frac{\phi_4}{\phi_3} - \frac{\phi_2}{\phi_1} = 0,$$

that is

$$\frac{\phi_4}{\phi_3} = \frac{\phi_2}{\phi_1} = t.$$

Substituting $\phi_4 = t\phi_3, \phi_2 = t\phi_1$ into (5), we obtain $\phi_3 e^{Q(z)} = \phi_1$. Using the same method as Case 1, we also obtain $\deg Q = \rho - 1$.

PROOF OF Theorem 2

(i): If $\sigma \geqslant \lambda_f$ and $\sigma < \sigma(f)-1$, then $\lambda_f < \sigma(f)-1$. Hence we have

$$f(z) = \frac{I_1(z)}{I_2(z)} e^{s(z)}, \tag{10}$$

where $I_1(z)$ and $I_2(z)$ are the canonical product formed by zeros and poles of $f(z)$, respectively, and

$$\begin{cases} \lambda(I_1) = \sigma(I_1) = \lambda(f) < \sigma(f)-1, \\ \lambda(I_2) = \sigma(I_2) = \lambda(\frac{1}{f}) < \sigma(f)-1. \end{cases} \tag{11}$$

Set

$$s(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0, \tag{12}$$

where $b_n (\neq 0), \dots, b_0$ are constants and $\deg(s(z)) = n = \sigma(f)$. Since $\lambda_f < \sigma(f)-1$, we obtain $n \geqslant 2$. Eq. (10) implies that

$$\begin{cases} f(z+j) = \frac{I_1(z+j)}{I_2(z+j)} e^{s(z+j)}, \\ f^{(k)}(z+j) = (\frac{I_1(z+j)}{I_2(z+j)} e^{s(z+j)})^{(k)} = \phi_k(z+j) e^{s(z+j)}, \end{cases} \tag{13}$$

where $\phi_k(z+j)$ is a polynomial formed by $\frac{I_1(z+j)}{I_2(z+j)}$, $s(z+j)$ and their derivatives. Eqs. (13) and (3) imply that

$$\sum_{j=0}^{n} (c_j \frac{I_1(z+j)}{I_2(z+j)} + l_j \phi_j(z+j)) e^{s(z+j)} + \sum_{k=0}^{m} d_k \phi_k(z) e^{s(z)} + c = 0. \tag{14}$$

By (11), we have

$$\max\{\sigma(l_j \phi_j(z+j)), \sigma(d_k \phi_k(z))\} \leqslant \max\{\sigma, \sigma(I_1), \sigma(I_2)\} \leqslant \max\{\sigma, \lambda_f\} < n-1,$$

$$\sigma\left(c_j \frac{I_1(z+j)}{I_2(z+j)}\right) \leqslant \max\{\sigma, \lambda_f\} < n-1. \tag{15}$$

For $i \neq j$, we have $\deg(s(z+i)-s(z+j)) = m-1 \geqslant 1$. By (15), we have

$$T\left(r, c_j \frac{I_1(z+j)}{I_2(z+j)} + l_j \phi_j(z+j)\right) = o(T(r, e^{s(z+i)-s(z+j)})), \tag{16}$$

$$T(r, d_k \phi_k(z)) = o(T(r, e^{s(z+i)-s(z+j)})). \tag{17}$$

By Lemma 1, (16) and (17), we have $c=0$, a contradiction. So $\sigma \geqslant \sigma(f)-1$.

(ii): If $\sigma < \lambda_f$, using the similar way as (i), we have $\lambda_f \geqslant \sigma(f)-1$

PROOF OF Theorem 3

Substituting $f(z) = w(z)+d$ into (3), we have

$$\sum_{j=0}^{n} c_j w(z+j) + \sum_{j=0}^{n} l_j w^{(j)}(z+j) + \sum_{k=0}^{m} d_k w^{(k)}(z) + d\sum_{j=0}^{n} c_j + d_0 d + c = 0. \tag{18}$$

Let

$$W(z) = \sum_{j=0}^{n} c_j w(z+j) + \sum_{j=0}^{n} l_j w^{(j)}(z+j) + \sum_{j=0}^{m} d_k w^{(k)}(z).$$

$$m\left(r, \tfrac{1}{f-d}\right) = m\left(r, \tfrac{1}{w}\right), \tag{19}$$

By Lemma 2 and Lemma 3, we have

$$\begin{aligned} m\left(r, \frac{W(z)}{w(z)}\right) &= m\left(r, \sum_{j=0}^{n} (c_j \frac{w(z+j)}{w(z)} + l_j \frac{w^{(j)}(z+j)}{w(z)}) + \sum_{k=0}^{m} d_k \frac{w^{(k)}(z)}{w(z)}\right) \\ &\leqslant \sum_{j=0}^{n} m\left(r, c_j \frac{w(z+j)}{w(z)}\right) + \sum_{j=0}^{n} m\left(r, l_j \frac{w^{(j)}(z+j)}{w(z)}\right) \\ &\quad + \sum_{k=0}^{m} m\left(r, d_k \frac{w^{(k)}(z)}{w(z)}\right) = S(r,w). \end{aligned} \tag{20}$$

By (18), (20) and $d\sum\limits_{j=0}^{n} c_j + d_0 d + c \neq 0$, we have

$$\begin{aligned} m\left(r, \tfrac{1}{f-d}\right) &= m\left(r, \tfrac{1}{w}\right) \\ &= m\left(r, \frac{d\sum_{j=0}^{n} c_j + d_0 d + c}{w}\right) + m\left(r, \frac{1}{d\sum_{j=0}^{n} c_j + d_0 d + c}\right) \\ &= m\left(r, \tfrac{W}{w}\right) + S(r,w) = S(r,w) = S(r,f). \end{aligned}$$

So $\lambda(f-d) = \sigma(f)$.

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