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New sufficient conditions for Hamiltonian, pancyclic and edge-Hamilton graphs

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ABSTRACT: The decycling number $\nabla(G)$ of a graph G is the smallest number of vertices whose deletion yields a forest. Bau and Beineke proved that $\kappa(G) \leqslant \nabla(G)+1$ for every graph G, where $\kappa(G)$ is the connectivity of G (Australas J Combin, 25:285-298, 2002). In this paper, we consider graphs with $\kappa(G) = \nabla(G)+1$ and establish sufficient conditions for such graphs to be Hamiltonian, pancyclic and edge-Hamilton, respectively. To our knowledge, this is the first result studying Hamilton problem in terms of decycling number. It is well-known that determining the decycling number of a graph is equivalent to finding the greatest order of an induced forest and some sufficient conditions for Hamiltonian graphs are also sufficient for the existence of completely independent spanning trees. This paper may provide a new condition implying completely independent spanning trees.

KEYWORDS: Hamilton cycle, pancyclic, edge-Hamilton, decycling number, connectivity

MSC2020: 05C38 05C45

INTRODUCTION

Graphs considered in this paper are finite, simple and connected. For general theoretic notations, we follow Bondy and Murty [1]. Throughout the paper, the letter G denotes a graph. $\kappa(G)$ and $\alpha(G)$ denote the connectivity and independence number of G, respectively.

A cycle passing through all the vertices of a graph is called a *Hamilton cycle*. A graph is said to be *Hamiltonian* if it has a Hamilton cycle. We say that a graph is *pancyclic* if it contains cycles of all possible length from three up to the order of the graph. A graph is called an *edge-Hamilton* graph if every edge of the graph lies in a Hamilton cycle. Edge-Hamilton graphs and pancyclic graphs are generalizations of Hamiltonian graphs.

The decision problems that whether a graph contains a Hamilton cycle is one of the most famous NP-complete problems, and so it is unlikely that there exist good characterizations of such graphs. Although the Hamilton problem has been widely studied, researchers only went an initial step towards the sufficient and necessary conditions which ensure the existence of a Hamilton cycle. For this reason, it is natural to ask for sufficient or necessary conditions. The first sufficient condition for a graph to be Hamiltonian is due to Dirac in 1952 [2].

Theorem 1 *Every graph with $n \geqslant 3$ vertices and minimum degree at least $n/2$ is Hamiltonian.*

In 1971, Bondy [3] raised a sufficient condition for a graph to be pancyclic.

Theorem 2 *Let G be a Hamiltonian graph on n vertices and m edges. If $m \geqslant n^2/4$, then G is either pancyclic or else is $K_{\frac{n}{2},\frac{n}{2}}$.*

Since then, many other interesting sufficient or necessary conditions for a graph to be Hamiltonian have been obtained, see [4,5]. In particular, Chvátal and Erdös [6] proved that every graph G on at least three vertices and $\alpha(G) \leqslant \kappa(G)$ has a Hamilton cycle. In other words, forbidding small connectivity admits a Hamilton cycle. It is interesting that many sufficient conditions for Hamiltonicities on classical graph properties can be naturally extended to the random graphs, see [7,8]. Shang gave a sufficient condition for subgraphs random bipartite graph [9]. Furthermore, He studied the bipancyclicity of random bipartite graphs [10].

If $S \subseteq V(G)$ and $G-S$ is acyclic, then S is said to be a *decycling set* of G (also known as *feedback vertex set*). The smallest size of a decycling set of G is said to be *decycling number* of G and is denoted by $\nabla(G)$. A decycling set of this cardinality is called a ∇-set. In theory, determining the decycling number $\nabla(G)$ of graph is equivalent to finding the order of a greatest induced forest. The decycling number problem has a long and rich history and classical question concern its computation. However, it has been shown that determining the decycling number of graphs is NP-hard [11]. Indeed, only a few of graphs are available, such as cubic graphs [12]. It is worth noting that Bau and Beineke [13] considered the relation between decycling number and connectivity of graphs.

Theorem 3 *For every graph G, $\kappa(G) \leqslant \nabla(G)+1$.*

Motivated by the results above, in this paper we will characterize the Hamiltonian, pancyclic and edge-Hamilton properties in graphs with $\kappa(G) = \nabla(G) + 1$. Here, the purpose that $\kappa(G) = \nabla(G) + 1$ is to forbid small connectivity. Note that $G - S$ is a tree for every ∇-set S in such graphs. From now on, we use t to denote the number of leaves in $G - S$.

The main results of this paper are presented as follows.

Theorem 4 *Let G be a graph with $\kappa(G) = \nabla(G) + 1$. If there exists a ∇-set S of G such that $\nabla(G) \geqslant t - 1$, then G is Hamiltonian.*

The condition that $\nabla(G) \geqslant t - 1$ could not be weaken, due to the well-known 1-tough property [14].

Proposition 1 *If a graph G has a Hamilton cycle, then for each nonempty set $S \subseteq V(G)$, the graph $G - S$ has at most $|S|$ components.*

For example, the graph G as shown in Fig. 1 satisfies $\kappa(G) = 2$, $\nabla(G) = 1$ and $t = 3$ ($\nabla(G) < t - 1$). However, $G - \{s, u\}$ has 3 components, i.e., it fails the necessary condition of Proposition 1. Hence it is not Hamiltonian.

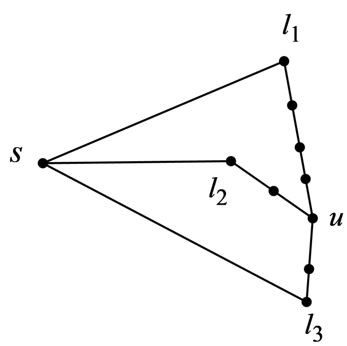

Fig. 1 $|S| = 1$, $t = 3$.

In 1971, Bondy [3] proposed his meta-conjecture: Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic. The following theorem supports his meta-conjecture in some sense.

Theorem 5 *Let G be a graph with $\kappa(G) = \nabla(G) + 1$. If there exists a ∇-set S of G such that $\nabla(G) \geqslant t$, then G is pancyclic.*

Our next result shows that the condition in Theorem 5 also ensures edge-Hamilton.

Theorem 6 *Let G be a graph with $\kappa(G) = \nabla(G) + 1$. If there exists a ∇-set S of G such that $\nabla(G) \geqslant t$, then G is edge-Hamilton.*

In the rest of this paper, we will prove our main results. A brief word about our notation. For $W \subseteq V(G)$, by $G - W$ and $G[W]$ we mean the subgraphs induced by $V(G) - W$ and $G[W]$, respectively. For a vertex $v \in V(G)$, we denote by $d(v)$ the degree of v, and by $N(v)$ the neighborhood of v. Given a subgraph H of G, we let $N_H(v) = N(v) \cap V(H)$, and $d_H(v) = |N_H(v)|$. If $S \subseteq V(G)$, we define $N_H(v) = N(v) \cap V(H)$ and $d_S(v) = |N_S(v)|$. We call a vertex of degree i a *i-vertex*. Given a tree T and a path $P = v_1 v_2 \cdots v_q$ ($q \geqslant 1$) on T, where v_1 is a leaf of T, then P is called a *pendant path* of T if it is a maximal path with no vertex of degree $\geqslant 3$ (see Fig. 2).

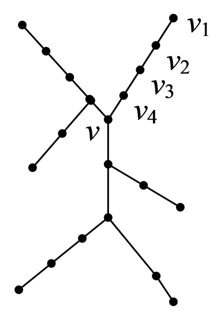

Fig. 2 $v_1 v_2 \cdots v_4$ is a pendant path.

PROOF OF THEOREMS

In order to prove Theorem 4, we need the following lemmas.

Lemma 1 *If G has a bipartition $V(G) = S + T$ such that: (a) $G[T]$ is a tree, (b) every i-vertex of $G[T]$ is adjacent to at least $|S| + 1 - i$ vertices of S, and (c) $|T| - 2 \geqslant |S| \geqslant t - 1$, where t is number of leaves of $G[T]$, then G is Hamiltonian.*

Lemma 2 *Let G be a graph with $\kappa(G) = \nabla(G) + 1$. If $\nabla(G) \geqslant |V(G)| - \nabla(G) - 1$, then G is Hamiltonian.*

Proof: If $\nabla(G) \geqslant |V(G)| - \nabla(G) - 1$, then $\nabla(G) + 1 \geqslant |V(G)|/2$. It follows that,

$$\delta(G) \geqslant \kappa(G) = \nabla(G) + 1 \geqslant \frac{|V(G)|}{2}.$$

According to Theorem 1, G is Hamiltonian. □

Combining Lemma 1 and Lemma 2, one can easily prove Theorem 4.

Proof of Theorem 4

Let S be a ∇-set of G such that $|S| \geqslant t - 1$. Define $T = G - S$. Then every i-vertex of T is adjacent to at least $|S| + 1 - i$ vertices of S, since $\delta(G) \geqslant \kappa(G) = |S| + 1$. If $|T| - 2 \geqslant |S|$, then the theorem is true according to Lemma 1. Otherwise, $|S| \geqslant |T| - 1$, then $\nabla(G) \geqslant$

$|V(G)| - \nabla(G) - 1$. So, the theorem holds in this case, due to Lemma 2.

Now, our goal is to prove Lemma 1. We first give a result which is a weaker version of Lemma 1.

Lemma 3 *If G has a bipartition $V(G) = S + T$ such that: (1) $G[T]$ is a tree, (2) every i-vertex of $G[T]$ is adjacent to at least $|S|+1-i$ vertices of S, and (3) $|S| = t-1$, where t is number of leaves of $G[T]$, then G is Hamiltonian.*

Proof: We will write T rather than the more customary $G[T]$. It suffices to prove the lemma in the case that every i-vertex of T is adjacent to $|S|+1-i$ vertices of S, i.e, every vertex of T has degree $|S|+1$ in G. Note that every leaf of T is adjacent to all vertices of S.

We apply induction on t. For $t = 2$, put $S = \{s_1\}$. In this case, T is a path. Suppose that $T = v_1v_2\cdots v_n$, $n \geqslant 2$. Then $s_1v_1v_2\cdots v_ns_1$ is a Hamilton cycle. Now, assume that $t = k+1$ with $k \geqslant 2$ and set $S = \{s_1, s_2, \ldots, s_k\}$. Choose a pendant path $P = u_1u_2\cdots u_q$ with $q \geqslant 1$, where u_1 is a leaf of T. Since u_q is a 2-vertex of T, u_q is adjacent to $|S|-1$ vertices of S. Without loss of generality, let $N_S(u_q) = \{s_1, s_2, \ldots, s_{k-1}\}$ and let u be a vertex (other than u_{q-1}) adjacent to u_q on T. Then we have $d_S(u) \leqslant k-2$, since $d_T(u) \geqslant 3$. So, we assume that $s_1 \notin N_S(u)$. Let $G_1 = G - (P \cup \{s_1\})$, $T_1 = T - V(P)$ and $S_1 = S - \{s_1\}$. Then, G_1 has a bipartition $V(G_1) = T_1 + S_1$ satisfies (1), (2), and (3). Based on the induction hypothesis, G_1 is Hamiltonian. For any leaf l_1 of T_1 (in G_1), there exists a Hamilton cycle C_1 of G_1 passes through l_1. Since l_1 has only one neighbour in T_1, C_1 passes through an edge s_il_1, where, s_{i_1} is a vertex of S_1. We get a Hamilton cycle of G by replacing the edge $s_{i_1}l_1$ of C_1 with the path $s_{i_1}u_1u_2\cdots u_qs_1l_1$. The proof is completed. □

For example, Fig. 3 shows a construction of a Hamilton cycle.

Remark 1 Remark that we restrict the number of edges between S and T (every i-vertex of T is adjacent to $|S|+1-i$ vertices of S) in the proof of Lemma 3. In the proofs of Lemma 1, Lemma 4, Lemma 6, Lemma 7 and Lemma 9, we will keep this restriction.

By refining slightly the proof of Lemma 3, one can obtain Lemma 1.

Proof of Lemma 1

We finish this lemma by applying double induction on t and $|T|$.

For $t = 2$, we prove it by using induction on $|T|$. If $|T| = 3$, then $|S| = 1$. Based on Lemma 3, the statement is true. Assume that $T = v_1v_2\cdots v_k$, $k \geqslant 4$ and set $S = \{s_1, s_2, \ldots, s_i\}$, $2 \leqslant i \leqslant k-2$ (note that the case $|S| = 1$ could be treated by Lemma 3). Without loss of generality, we may assume that $N_S(v_2) = \{s_1, s_2, \ldots, s_{i-1}\}$. define $G_0 = G - \{v_1, s_i\}$, $T_0 = T - \{v_1\}$ and $S_0 = S - \{s_i\}$. Then, G_0 has a bipartition $V(G_0) = T_0 + S_0$ satisfies (a), (b), and (c). By the induction hypothesis on $|T|$, G_0 has a Hamilton cycle, say C_0. It follows that there exists a vertex $s_{i_0} \in S - \{s_i\}$ such that C_0 passes through the edge $s_{i_0}v_k$. Replacing the edge $s_{i_0}v_k$ by the path $s_{i_0}v_1s_iv_k$ turns into a Hamilton cycle of G.

Now, assume that $t = m \geqslant 3$. If $|T| = m+1$, then $|S| = m-1$. Based on Lemma 3, the statement is true. Now, assume that $|T| = k$ with $k \geqslant m+2$ and let $S = \{s_1, s_2, \ldots, s_i\}$, where $m \leqslant i \leqslant k-2$. We distinguish two cases.

Case 1. There is a pendant path P_1 on T with $|P_1| = 1$.

In this case, suppose that $P_1 = u_1$. Let u be a vertex adjacent to u_1 on T and assume that $N_S(u) \subseteq \{s_1, s_2, \ldots, s_{i-2}\}$. Denote $G_1 = G - \{u_1, s_i\}$. It is easy to check that G_1 satisfies (a), (b), and (c). By the induction hypothesis on t, G_1 has a Hamilton cycle, say C_1. For a leaf l_1 of $T - \{u_1\}$, there exists a vertex $s_{i_1} \in S - \{s_i\}$ such that C_1 passes through the edge $s_{i_1}l_1$. Replacing the edge $s_{i_1}l_1$ by the path $s_{i_1}u_1s_il_1$ makes up a Hamilton cycle of G.

Case 2. Every pendant path on T contains at least two vertices.

Under this case, we use induction on $|T|$. First, choose a pendant path $P_2 = w_1w_2\cdots w_q$ of T, $q \geqslant 2$. Assume, without loss of generality, that $N_S(w_2) \subseteq \{s_1, s_2, \ldots, s_{i-1}\}$. Let $G_2 = G - \{w_1, s_i\}$. Then G_2 satisfies (a), (b), and (c). By the induction hypothesis on $|T|$,

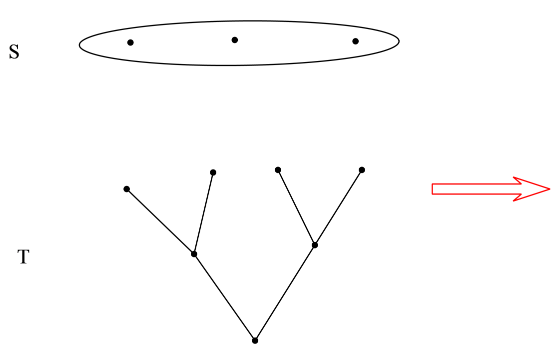

Fig. 3 A construction of a Hamilton cycle.

G_2 contains a Hamilton cycle C_2. For a leaf l_2 of $T-P_2$, there exists a vertex $s_{i_2} \in S-\{s_i\}$ such that C_2 passes through the edge $s_{i_2}l_2$. Replacing the edge $s_{i_2}l_2$ by the path $s_{i_2}w_1s_il_2$ turns into a Hamilton cycle of G.

We are now ready to prove Theorem 5 employing the ideas in Theorem 4. First, we also give two lemmas.

Lemma 4 *If G has a bipartition $V(G)=S+T$ such that: (a) $G[T]$ is a tree, (b) every i-vertex of $G[T]$ is adjacent to at least $|S|+1-i$ vertices of S, and (c) $|T|-2 \geqslant |S| \geqslant t$, where t is number of leaves of $G[T]$, then G is pancyclic.*

Lemma 5 *Let G be a graph with $\kappa(G)=\nabla(G)+1$. If $\nabla(G) \geqslant |V(G)|-\nabla(G)-1$, then G is pancyclic.*

Proof: If $\nabla(G) \geqslant |V(G)|-\nabla(G)-1$, then $\nabla(G)+1 > |V(G)|/2$. Thereby, $\delta(G) > |V(G)|/2$. Based on Theorem 2, G is a pancyclic. □

Proof of Theorem 5

Let S be a ∇-set of G such that $|S| \geqslant t$. Define $T=G-S$. Then every i-vertex of T is adjacent to at least $|S|+1-i$ vertices of S, since $\delta(G) \geqslant |S|+1$. If $|T|-2 \geqslant |S| \geqslant t$, then our theorem is true according to Lemma 4. Otherwise, $|S| \geqslant |T|-1$, i.e., $\nabla(G) \geqslant |V(G)|-\nabla(G)-1$. By Lemma 5, the theorem holds.

To prove Lemma 4, we raise the following result.

Lemma 6 *If G has a bipartition $V(G)=S+T$ such that: (1) $G[T]$ is a tree, (2) every i-vertex of $G[T]$ is adjacent to at least $|S|+1-i$ vertices of S, and (3) $|S|=t$, where t is the number of leaves of $G[T]$, then G is pancyclic.*

Proof: Let n denote the number of vertices in G. We establish the lemma in the same way as we did in Lemma 3. We use induction on t.

For $t=2$, let $S=\{s_1,s_2\}$. For $|T|=2$, there is nothing to prove, so assume that $T=v_1v_2\cdots v_m$, where $m \geqslant 3$ and $m+2=n$. Notice that for a given integer i with $1 \leqslant i \leqslant m$, v_i is adjacent to at least one vertex of S. Without loss of generality, assume that v_i is adjacent to s_1. Then $s_1v_1v_2\cdots v_is_1$ is a cycle of length $i+1$. In addition, by Lemma 3, G has a Hamilton cycle, i.e., a cycle of length n. Thereby, the lemma holds for $t=2$.

Let us now consider the case $t=k \geqslant 3$. We choose a pendant path $P=u_1u_2\cdots u_q$ with $q \geqslant 1$ on T and a vertex u adjacent to u_q. We may as well suppose that $N_S(u) \subseteq \{s_1,s_2,\dots,s_{k-2}\}$.

These cycles can be constructed as follows.

Let $G_1=G-(V(P)\cup\{s_k\})$. Then G_1 satisfies (1), (2) and (3). It follows that G_1 is pancyclic, in other words, G_1 has cycles of length from 3 up to $n-q-1$. Pick a cycle C_1 of length $n-q-1$ in G_1 and a leaf l_1 of $T-P$. Then there exists a vertex $s_{i_0} \in S-\{s_k\}$ such that C_1 passes through the edge $s_{i_0}l_1$. For each $1 \leqslant j \leqslant q$, if u_j is adjacent to s_k, then we complete a cycle of length $n-q+j$ by replacing the edge $s_{i_0}l_1$ with path $s_{i_0}u_1\cdots u_js_kl_1$; otherwise, u_j is adjacent to every one of $S-\{s_k\}$, then we complete a cycle of length $n-q+j$ by replacing the edge $s_{i_0}l_1$ with path $s_{i_0}u_ju_{j-1}\cdots u_1s_kl_1$. In addition, G_1 also has a cycle C_2 of length $n-q-2$. What's more, the cycle C_2 must contain a leaf l_2 of $T-P$, since $T-P$ has at least two leaves. Thereby, there exists a vertex $s_{i_1} \in S-\{s_k\}$ such that C_2 passes through the edge $s_{i_1}l_2$. We get a cycle of length $n-q$ by replacing $s_{i_1}l_2$ with path $s_{i_1}u_1s_kl_2$. This builds the lemma. □

Proof of Lemma 4

Let n denote the number of vertices in G. We achieve the lemma by applying induction on $|S|$. According to Lemma 6, the lemma holds for $|S|=t$. Assume that $|S|=k$, where $|T|-2 \geqslant k > t$. Put $S=\{s_1,s_2,\dots,s_k\}$. Let $T_1=T$, $S_1=S-\{s_k\}$ and $G_1=G-\{s_k\}$. Since each i-vertex u in T satisfies $d_{S_1}(u) \geqslant d_S(u)-1=|S|-i+1-1=|S_1|-i+1$, G_1 has a bipartition $V(G_1)=T_1+S_1$ satisfies (a), (b), and (c). By the induction hypothesis, G_1 is a pancyclic graph, i.e., G_1 has cycles of length j for all $3 \leqslant j \leqslant n-1$. Therefore, G has cycles of length j for all $3 \leqslant j \leqslant n-1$. Further, according to Lemma 1, G is Hamiltonian. That is to say G has a cycle of length n. It follows that G is a pancyclic graph.

In the remainder of this paper, we will finish the proof of Theorem 6. The actual proof will be preceded by two lemmas.

Lemma 7 *If G has a bipartition $V(G)=S+T$ such that: (a) $G[T]$ is a tree, (b) every i-vertex of $G[T]$ is adjacent to at least $|S|+1-i$ vertices of S, and (c) $|T|-2 \geqslant |S| \geqslant t$, where t is the number of leaves of $G[T]$, then G is an edge-Hamilton graph*

Lemma 8 *Let G be a graph with $\kappa(G)=\nabla(G)+1$. If $\nabla(G) \geqslant |V(G)|-\nabla(G)-1$, then G is Edge-Hamilton.*

In order to prove Lemma 8, we should introduce anther concept. A graph G is called a *Hamilton-connected* graph if every two vertices of G are connected by a Hamilton path. Surely, all Hamilton-connected graphs are edge-Hamilton. Benhocine and Wojda [15] have shown the following result.

Theorem 7 *Let G be a 2-connected graph on $n \geqslant 3$ vertices. If*

$$d_G(u,v)=2 \implies \max\{d_G(u),d_G(v)\} \geqslant \frac{n+1}{2}$$

for every pair of vertices u and v in G, then G is Hamilton-connected.

Proof of Lemma 8

Combining the conditions that $\kappa(G)=\nabla(G)+1$ and $\nabla(G) \geqslant |V(G)|-\nabla(G)-1$, we deduce that G is 2-connected and $2\nabla(G)+2 \geqslant |V(G)|+1$. Hence

$$\delta(G) \geqslant \kappa(G)=\nabla(G)+1 \geqslant \frac{|V(G)|+1}{2}.$$

Based on Theorem 7, G is Hamilton-connected. It follows that G is edge-Hamilton.

Theorem 6 follows from Lemma 7 and Lemma 8. Its proof is similar to that of Theorem 5. So, we omit it here.

In the following, we will prove Lemma 7. First, we also provide a weaker version.

Lemma 9 *If G has a bipartition $V(G)=S+T$ such that: (1) $G[T]$ is a tree, (2) every i-vertex of $G[T]$ is adjacent to at least $|S|+1-i$ vertices of S, and (3) $|S|=t$, where t is the number of leaves of $G[T]$, then G is edge-Hamilton.*

Proof: We complete this lemma by applying induction on t as well. For $t=2$, let $S=\{s_1,s_2\}$. It is easy to check that the lemma holds for $|T|=2$, so suppose that $T=v_1v_2\cdots v_m$ with $m\geqslant 3$. When considering an edge sv, where $s\in S$ and $v\in T$, we only refer to the edge s_1v_2, since the other cases resemble it. If v_3 is adjacent to s_1, then $s_1v_2v_1s_2v_mv_{m-1}\cdots v_3s_1$ is a Hamilton cycle with the edge s_1v_2. Otherwise, v_3 is adjacent to s_2, then $v_2s_1v_mv_{m-1}\cdots v_3s_2v_1v_2$ is a Hamilton cycle with s_1v_2. Notice that from procedure of finding Hamilton cycle passing through s_1v_2, it is easy to find a Hamilton cycle passing through any given edge on T. In addition, $v_1v_2\cdots v_ms_1s_2v_1$ is a Hamilton cycle containing the edge s_1s_2. Hence, the statement is true for $t=2$.

Let us consider the case $t=k\geqslant 3$ and set $S=\{s_1,s_2,\dots,s_k\}$.

(I). Edge xy on T.

We first choose a pendant path $P=v_1v_2\cdots v_q$, $q\geqslant 1$, such that $x,y\notin P$. Let further v (other than v_{q-1}) be a vertex adjacent to v_q on T. We suppose that $N_S(v)\subseteq\{s_1,s_2,\dots,s_{k-2}\}$. Let $G_1=G-(V(P)\cup\{s_k\})$. Then G_1 satisfies (1), (2), and (3), which implies that G_1 has a Hamilton cycle C_1 containing xy. For a leaf l_1 of $T-P$, there exists a vertex $s_{i_1}\in S-\{s_k\}$ such that C_1 passes through the edge $s_{i_1}l_1$. If v_q is adjacent to s_k, then we make up a Hamilton cycle of G containing the edge xy by replacing $s_{i_1}l_1$ with the path $s_{i_1}v_1v_2\cdots v_qs_kl_1$. Otherwise, v_q is adjacent to every vertex of $S-\{s_k\}$, then we complete a Hamilton cycle of G containing the edge xy by replacing $s_{i_1}l_1$ with the path $s_{i_1}v_qv_{q-1}\cdots v_1s_kl_1$.

(II). Edges sz, where $s\in S$ and $z\in T$.

Here, we only refer to the edge s_1z, since the other cases resemble it. Pick a pendant path $P=u_1u_2\cdots u_q$ with $q\geqslant 1$, such that z belongs to $T-P$. Choose a vertex u (other than u_{q-1}) adjacent to u_q on T.

Case 1. u is adjacent to s_1.

Assume that $N_S(u)\subseteq\{s_1,s_2,\dots,s_{k-2}\}$. Denote then $G_1=G-(V(P)\cup\{s_k\})$. Then G_1 satisfies (1), (2) and (3), which yields that G_1 has a Hamilton cycle C_1 passing through the edge s_1z. Since $T-P$ contains at least two leaves, there is an edge $s_{i_1}l_1$ on C_1, where $s_{i_1}\in S-\{s_k\}$ and l_1 is a leaf of $T-(V(P)\cup\{z\})$. If u_q is adjacent to s_k, then we form a Hamilton cycle of G passing through the edge s_1z by replacing $s_{i_1}l_1$ with path $s_{i_1}u_1u_2\cdots u_qs_kl_1$. Otherwise, u_q is adjacent to every vertex of $S-\{s_k\}$, then we complete a Hamilton cycle of G passing through the edge s_1z by replacing $s_{i_1}l_1$ with the path $s_{i_1}u_qu_{q-1}\cdots u_1s_kl_1$.

Case 2. u and s_1 are non-adjacent.

We may assume that $N_S(u)\subseteq\{s_3,s_4,\dots,s_k\}$, as well. Let $G_2=G-(V(P)\cup\{s_2\})$. Then we can find a Hamilton cycle C_2 of G_2 containing the edge s_1z. Let l_2 be a leaf of $T-(P\cup\{z\})$. Then there is a vertex $s_{i_2}\in S-\{s_2\}$ such that the edge $s_{i_2}l_2$ lies in C_2. If u_q is adjacent to s_2, then we make up a Hamilton cycle of G passing through the edge s_1z by replacing $s_{i_2}l_2$ with the path $s_{i_2}u_1u_2\cdots u_qs_2l_2$. Otherwise, u_q is adjacent to every vertex $s\in S-\{s_2\}$, then we finish a Hamilton cycle G with the edge s_1z by replacing $s_{i_2}l_2$ with the path $s_{i_2}u_qu_{q-1}\cdots u_1s_2l_2$.

(III). Edge s_is_j with $1\leqslant i<j\leqslant k$.

We only refer to the edge s_1s_2. Choose a pendant path $P=w_1w_2\cdots w_q$ with $q\geqslant 1$ and a vertex w (other than w_{q-1}) adjacent to w_q on T.

Case 1. w is adjacent to both s_1 and s_2.

Without loss of generality, suppose that $N_S(w)\subseteq\{s_1,s_2,\dots,s_{k-2}\}$. Denote $G_1=G-(V(P)\cup\{s_k\})$. Then G_1 has a Hamilton cycle C_1 passing through the edge s_1s_2. Furthermore there is an edge $s_{i_1}l_1$ in C_1, where $s_{i_1}\in S-\{s_k\}$ and l_1 is a leaf of $T-P$. If w_q is adjacent to s_k, then we complete a Hamilton cycle of G with the edge s_1s_2 by replacing $s_{i_1}l_1$ with path $s_{i_1}w_1w_2\cdots w_qs_kl_1$. Otherwise, w_q is adjacent to every vertex $S-\{s_k\}$, then we complete a Hamilton cycle G with the edge s_1s_2 by replacing $s_{i_1}l_1$ with path $s_{i_1}w_qw_{q-1}\cdots w_1s_kl_1$.

Case 2. w is adjacent to neither s_1 nor s_2.

Under this case, $N_S(w)\subseteq\{s_3,s_4,\dots,s_k\}$. Note that w_q is adjacent to at least one of $\{s_1,s_2\}$. Suppose that w_q is adjacent to s_1 and let $G_2=G-(V(P)\cup\{s_1,s_2\})$. By Lemma 4, G_2 has a Hamilton cycle C_2. Pick a leaf l_2 of $T-P$. Then there exists a vertex $s_{i_2}\in S-\{s_1,s_2\}$ such that C_2 passes through the edge $s_{i_2}l_2$. Consequently, we complete a Hamilton cycle of G with the edge s_1s_2 by replacing the edge $s_{i_2}l_2$ with path $s_{i_2}w_1\cdots w_qs_1s_2l_2$.

Case 3. w is adjacent to only one of $\{s_1,s_2\}$.

Assume that w is adjacent to s_2 and $N_S(w)\subseteq\{s_2,s_3,\dots,s_{k-1}\}$. Denote $G_3=G-(V(P)\cup\{s_k\})$. Then G_3 has a Hamilton cycle C_3 passing through the edge s_2w. We replace s_2w with $s_2s_1w_1w_2\cdots w_qw$, forming a Hamilton cycle of G with the edge s_1s_2. □

Proof of Lemma 7

Similarly, we use double induction method on t and T.

(I). Edge xy on T.

For $t=2$, we prove our statement by using induction on $|T|$. When $|T|=4$, we have $|S|=2$. As we discussed in Lemma 9, the statement is true. So, assume that $T=v_1v_2\cdots v_n$, $n\geqslant 5$ and let $S=\{s_1,s_2,\dots,s_i\}$, $2<i\leqslant n-2$. Without loss of generality, assume that $v_1\neq x,y$ and $N_S(v_2)=\{s_1,s_2,\dots,s_{i-1}\}$. Define $G_0=G-\{v_1,s_i\}$. Then G_0 satisfies (a), (b), and

(c). It follows that there is a Hamilton cycle C_0 of G_0 passing through the edge xy. Further, there exists a vertex s_{i_0} in $S-\{s_i\}$ such that the edge $s_{i_0}v_n$ belongs to C_0. Replacing the edge $s_{i_0}v_n$ by the path $s_{i_0}v_1s_iv_n$ turns into a Hamilton cycle of G containing the edge xy.

For $t=m\geqslant 3$, we prove that by using induction on $|T|$. If $|T|=m+2$, then $|S|=m$. As we proved in Lemma 9, the statement is true. Let $|T|=k\geqslant m+3$ and $S=\{s_1,s_2,\dots,s_i\}$, $m<i\leqslant k-2$. We deal with the following cases.

Case 1. T contains at least two pendant paths consisting of only one vertex.

In this case, we choose a pendant path $P_1=u_1$, such that $u_1\neq x,y$. Let u be a vertex adjacent to u_1 on T. We suppose that $N_S(u)\subseteq\{s_1,s_2,\dots,s_{i-2}\}$. Let $G_1=G-\{u_1,s_i\}$. It is easy to check that G_1 satisfies (a), (b), and (c). By the induction hypothesis on t, G_1 is edge-Hamilton, in other words there is a Hamilton cycle C_1 of G_1 passing through the edge xy. Pick a leaf l_1 of $T-\{u_1\}$. Then there exists $i_1\in\{1,2,\dots,i-1\}$ such that C_1 passes through the edge $s_{i_1}l_1$. Replacing the edge $s_{i_1}l_1$ by path $s_{i_1}u_1s_il_1$ turns into a Hamilton cycle of G containing the edge xy.

Case 2. T contains at most one pendant path consisting of only one vertex.

Under this case, we choose a pendant $P_2=w_1w_2\cdots w_q$, $q\geqslant 2$ on T such that $x,y\in T-P$. Suppose first that $N_S(w_2)=\{s_1,s_2,\dots,s_{i-1}\}$. Let now $G_2=G-\{w_1,s_i\}$. By induction hypothesis on $|T|$, there is a Hamilton cycle C_2 of G_2 passing through the edge xy. Pick a leaf l_2 of $T-P_2$. Then there exists $i_2\in\{1,2,\dots,i-1\}$ such that C_2 passes through the edge $s_{i_2}l_2$. Replacing the edge $s_{i_2}l_2$ by path $s_{i_2}w_1s_il_2$ turns into a Hamilton cycle of G containing the edge xy.

(II). Edges sz, where $s\in S$ and $z\in T$.

For $t=2$, set $T=v_1v_2\cdots v_n$. We only treat the edge s_1v_2, since we could solve the other cases analogously. As before, we prove our statement by using induction on n. When $n=4$, we have $|S|=2$. According to Lemma 9, the statement is true. Assume that $n\geqslant 5$, let $S=\{s_1,s_2,\dots,s_i\}$ with $2<i\leqslant n-2$.

Case 1. v_3 is adjacent to s_1.

Let $G_1=G-\{s_1\}$. By (I), G_1 has a Hamilton cycle C_1 passing through the edge v_2v_3. We replace the edge v_2v_3 with path $v_2s_1v_3$, forming a Hamilton cycle of G with the edge s_1v_2.

Case 2. v_3 is not adjacent to s_1.

Define $G_2=G-\{v_1,v_2,s_1\}$. Then G_2 satisfies (a), (b), and (c). By induction hypothesis, G_2 has a Hamilton cycle C_2 passing through s_2v_n. Replacing the edge s_2v_n by the path $s_2v_1v_2s_1v_n$ turns into a Hamilton cycle of G with the edge s_1v_2.

Suppose that the result is true for $t\leqslant m-1$. For $t=m\geqslant 3$, we prove it by induction on $|T|$ and. Choose a vertex of S and a vertex of T, say s_1 and z, respectively. If $|T|=m+2$, then $|S|=m$. According to Lemma 9, the edge s_1z lies in a Hamilton cycle. Assume that $|T|=k\geqslant m+3$ and let $S=\{s_1,s_2,\dots,s_i\}$ with $m<i\leqslant k-2$.

T has at least three pendant paths, we deal with the following cases.

Case 1. T contains at least two pendant paths consisting of only one vertex.

In this case, we choose a pendant which consist of only one vertex, say u_1, such that $u_1\neq z$. Let u be a vertex adjacent to u_1 on T, then $d_T(u)\geqslant 3$. Thus, u is adjacent to at most $i+1-3=i-2$ vertices of S. If u is adjacent to s_1, we suppose that $N_S(u)\subseteq\{s_1,s_2,\dots,s_{i-2}\}$. Let $G_3=G-\{u_1,s_i\}$. It is easy to check that G_3 has a bipartition $V(G_3)=(T-\{u_1\})+(S-\{s_i\})$ satisfies (a), (b), and (c). By the induction hypothesis on t, G_3 is edge-Hamilton, in other words G_3 has a Hamilton cycle C_3 containing the edge s_1z. Pick a leaf l_3 of $T-\{u_1,z\}$ (T has at least three leaves) such that C_3 passes through l_3. Since one neighbour of l_3 must belongs to $(S_1-\{s_i\})$. Then there exists $i_3\in\{1,2,\dots,i-1\}$ such that C_3 passes through the edge $s_{i_3}l_3$. Replacing the edge $s_{i_3}l_3$ by the path $s_{i_3}u_1s_il_3$ turns into a Hamilton cycle of G containing the edge s_1z. Otherwise, suppose that $N_S(u)\subseteq\{s_3,\cdots,s_i\}$. Let $G_3^{'}=G-\{u_1,s_2\}$. Then $G_3^{'}$ contains a Hamilton cycle $C_3^{'}$ passing through s_1z. Pick a leaf $l_3^{'}$ of $T-\{u_1,z\}$. Then there exists $i_3^{'}\in\{1,\dots,i-1,i\}$ such that $C_3^{'}$ passes through the edge $s_{i_3^{'}}l_3^{'}$. Replacing the edge $s_{i_3^{'}}l_3^{'}$ by path $s_{i_3^{'}}u_1s_il_3^{'}$ turns into a Hamilton cycle of G containing the edge s_1z.

Case 2. T contains at most one pendant path consisting of one vertex.

Under this case, T contains at least two pendant paths which consists of more than one vertex. We choose a pendant path $P_2=w_1w_2\cdots w_q$ on T such that $z\in T-P_2$. If w_2 is adjacent to s_1, we suppose that $N_S(w_2)=\{s_1,s_2,\dots,s_{i-1}\}$ ($d_S(w_2)=i+1-2=i-1$). Let $G_4=G-\{w_1,s_i\}$, then G_4 has a bipartition $V(G_4)=(T-\{w_1\})+(S-\{s_i\})$ satisfies (a), (b), and (c). By induction hypothesis on $|T|$, there is a Hamilton cycle C_4 of G_4 passing through s_1z. Pick a leaf l_4 of $T-(V(P_2)\cup\{z\})$. Then there exists $i_4\in\{1,2,\dots,i-1\}$ such that C_4 passes through the edge $s_{i_4}l_4$. Replacing the edge $s_{i_4}l_4$ by path $s_{i_4}w_1s_il_4$ turns into a Hamilton cycle of G containing s_1z. Otherwise, suppose that $N_S(w_2)=\{s_2,s_3,\dots,s_i\}$. We consider the following two subcases.

Subcase 2.1. There is a vertex $s^{'}$ in $S-\{s_1\}$, such that $s^{'}$ is adjacent to z.

Let $G_4^{'}=G-\{w_1,s_1\}$. Then $G_4^{'}$ has a bipartition $V(G_4^{'})=(T-\{w_1\})+(S-\{s_1\})$ satisfies (a), (b), and (c). By the induction hypothesis on $|T|$, $G_4^{'}$ contains a Hamilton cycle $C_4^{'}$ passing through $s^{'}z$. We replace $s^{'}z$ by the path $s^{'}w_1s_1z$, forming a Hamilton cycle containing s_1z.

Subcase 2.2. z is only adjacent s_1.

Since every j-vertex of T is adjacent to at least $|S|+1-j$ vertices of S, $d_T(z) \geqslant |S|$. Thereby, T has at least $|S| \geqslant 4$ pendant paths. Under the case, T contains at most one pendant path consisting of one vertex, this implies that there is at least tree pendant paths consisting of at least two vertices. Assume that $P_3 = a_1 a_2 \cdots a_q$, $q \geqslant 2$ is pendant path of T. Then $|T| \geqslant 1+|S|+1+1+q-1 \geqslant |S|+q+2$. Thereby, $|T-P_3| \geqslant |S|$. By repeating the procedure of II in Lemma 9, one can get a Hamilton cycle with the edge $s_1 z$.

(III). Edges $s_i s_j$, where $s_i, s_j \in S$.

Here, we only consider $s_1 s_2$. The proof follows by induction on $|S|$. If $|S| = t$, then our statement is true and assume that $|S| = k$ where $t < k \leqslant |T|-2$, let $G_1 = G-\{s_2\}$. According to part II, G_1 has a Hamilton cycle with the edge $s_1 l_1$, where l_1 is a leaf of T. Replacing $s_1 l_1$ with $s_1 s_2 l_1$ forms a Hamilton cycle of G containing the edge $s_1 s_2$.

CONCLUSION

In this paper, we discover the new applications of decycling number, that is giving sufficient conditions for a class of graphs to be Hamiltonian, pancyclic and edge-Hamilton, respectively. This opens a new perspective for the study of Hamilton problem. In the proofs, we mainly use the double induction on t and $|T|$. Here, the pendant path plays a huge role. The difficulty in the proof is the construction of edge-Hamilton graphs. In addition, we try to solve the Hamilton problem in graphs with $\kappa(G) = \nabla(G)$, the proof is still incomplete.

Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph G. For any two vertices u, v of G, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent spanning trees in G. By the definition, completely independent spanning trees are also edge-disjoint spanning trees. It is worth mentioning some sufficient conditions for Hamiltonicity also guarantees the existence of completely independent spanning trees [16]. It happened that our conditions is proposed in terms of decycing number. It is our hope that researchers alike will find in this work inspiration and ideas to further light on this fascinating topic, especially, explore a sufficient condition for the existence of completely independent spanning trees.

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