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Maximum principles and Liouville theorems for fractional Kirchhoff equations

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ABSTRACT: In this paper, we consider the following nonlinear fractional Kirchhoff equation

$$\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s u(x)=f(u(x)),$$

where $0<s<1$, $a>0$ and $b\geqslant 0$. We first establish a maximum principle for anti-asymmetric functions on any half space, and then obtain a Liouville theorem to the above nonlinear fractional Kirchhoff equations in the whole space. In particular, we derive key ingredients for proving the symmetry and monotonicity of positive solutions to the nonlinear fractional Kirchhoff equations, which indicate that fractional Kirchhoff De Giorgi conjecture is valid under some conditions. We believe that the results obtained here can be conveniently applied to study a variety of properties for solutions to fractional Kirchhoff equations.

KEYWORDS: fractional Kirchhoff equations, Liouville theorems, maximum principle for anti-asymmetric functions

MSC2020: 35B09 35B50 35J60

INTRODUCTION

In this paper, we investigate the following nonlinear fractional Kirchhoff equation

$$\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s u(x)=f(u(x)), \quad (1)$$

with $a>0$, $b\geqslant 0$, $s\in(0,1)$ are real valued constants, and

$$\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x. \quad (2)$$

Here the fractional Laplacian is defined as

$$(-\Delta)^s u(x)=C_{n,s}\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y, \quad (3)$$

where P.V. stands for the Cauchy principal value and constant $C_{n,s}=2^{1-\frac{n}{2}+s}\pi^{-\frac{n}{2}}\frac{s(1-s)}{\Gamma(2-s)}>0$. Let

$$\mathscr{L}_{2s}=\left\{u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\,\middle|\,\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+2s}}\,\mathrm{d}x<\infty\right\},$$

and

$$H^s(\mathbb{R}^n)=\left\{u\in L^2(\mathbb{R}^n)\,\middle|\,\frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}}\in L^2(\mathbb{R}^n\times\mathbb{R}^n)\right\}.$$

One can easily verify that for any $u\in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)\cap\mathscr{L}_{2s}(\mathbb{R}^n)\cap H^s(\mathbb{R}^n)$, the integral on the right hand side of the definition (2) and (3) are well-defined.

We call Eq. (1) a nonlinear fractional Kirchhoff equation because of the appearance of the term $b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x$. Indeed, if choosing $s=1$ and $n=3$, then (1) transforms to the following classical Kirchhoff-type equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u(x)=f(x,u), \quad (4)$$

which is degenerate if $b=0$ and non-degenerate otherwise. If we replace $\mathbb{R}^3$ by a bounded domain $\Omega\subset\mathbb{R}^3$ in (4), then we get the Kirchhoff Dirichlet problem

$$-\left(a+b\int_{\Omega}|\nabla u|^2\,\mathrm{d}x\right)\Delta u(x)=f(x,u),$$

which is related to the stationary analog of the following equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=0 \quad (5)$$

proposed by Kirchhoff in [1] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. In particular, Eq. (5) received much attention after Lions in [2] proposed an abstract framework to the problem. In recent years, Kirchhoff equations especially those involving fractional and nonlocal operators, have been studied by more and more scholars and a series of results have been obtained, such as [3] for existence and multiplicity of solutions of fractional Kirchhoff superlinear equations, [4] for multiplicity and asymptotic behavior of solutions to fractional-p Kirchhoff-type equations, [5] for ground state solutions for a class of fractional Kirchhoff equations, [6] for existence and multiplicity of nontrivial non-negative entire (weak) solutions of a stationary nonlocal Kirchhoff eigenvalue problem, [7] for a Hopf lemma and the symmetry of solutions for fractional Kirchhoff equations, [8] for existence of sign-changing solutions for nonlinear fractional Kirchhoff equations, [9, 10] for non-degeneracy of positive solutions for fractional Kirchhoff problems.

When $a = 1, b = 0$, then (1) can be reduced to the following fractional equation

$$(-\Delta)^s u(x) = f(u(x)).$$

There have seen a series of results on fractional differential equations during the last decade (see [11–18] and references therein). The nonlocality of fractional operators bring many new difficulties comparing with the Laplacian. To treat fractional operators, the extension method [19] turns the nonlocal problem involving the fractional Laplacian into a local one in higher dimensions; the method of moving planes in integral forms [20] investigates fractional equations by showing that they are equivalent to corresponding integral equations; a direct method of moving planes [21] is even valid for fully nonlinear nonlocal operators and the fractional p-Laplacians (see [22–25]); the sliding method [26–28] lies in comparing values of the solution for the equation at two different points, between which one point is obtained from the other by sliding the domain in a given direction; an asymptotic method of moving planes [29, 30] investigates qualitative properties of positive solutions for fractional parabolic equations. Finally, we also encourage readers to read [31]. Gu et al [31] established fast numerical approaches to solve a class of initial-boundary problem of time-space diffusion equation involving the fractional Laplacian, which are worthy of our attention.

It is well known that the classical Liouville theorem states: *Any harmonic function on $\mathbb{R}^n$ bounded from below or from above is constant*. Liouville theorems are very important in studying elliptic equations and systems. For example, they played an essential role in deriving a priori bounds for solutions in [32, 33] and were used to obtain uniqueness of solutions in [34]. Liouville theorems have also been used to prove the equivalence between fractional equations and the corresponding integral equations, thus one can employ integral equations methods, such as method of moving planes in integral forms to study qualitative properties of the solutions for the original fractional differential equations. Recently, Wu and Chen [35] derived a Liouville theorem for fractional p-harmonic functions by a maximum principle on any half space. Later, Dai et al [36] proved a Liouville theorem for pseudo-relativistic Schrödinger equations; He et al [37] obtained a Liouville theorem for fully nonlinear nonlocal harmonic equations.

Inspired by the papers [35–37], we are interested in a Liouville theorem and monotonicity for solutions to the following nonlinear fractional Kirchhoff equation

$$\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s u(x)=f(u(x)),\ x\in\mathbb{R}^n. \quad (6)$$

For this problem, Eq. (6) contains not only the nonlocal operator $(-\Delta)^s u$ but also nonlocal term $\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x$, which makes the research of this problem more interesting. The key is to establish a maximum principle for anti-asymmetric functions in an unbounded region. The main difficulty is the construction of antisymmetric auxiliary functions when establishing the maximum principle of antisymmetric functions in unbounded regions.

Theorem 1 (Liouville theorem) *Suppose that $u(x) \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)\cap\mathscr{L}_{2s}(\mathbb{R}^n)\cap H^s(\mathbb{R}^n)$ is a bounded solution to Eq. (6) and f is non-increasing with respect to u.*

Then for any $x \in \mathbb{R}^n$, we have $u(x) \equiv C$ with C satisfying $f(C) = 0$.

Remark 1 When $a = 1$, $b = 0$ and $f \equiv 0$, Theorem 1 will reduce to [35, Theorem 1]. For unbounded solutions, the theorem may not true. For instance, when $f(u) = 0$, if $u(x) = x_i$, $i = 1, 2, \dots, n$, then one can easily check that $\left(a + b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s u(x) = 0$.

In order to prove Theorem 1, we need a maximum principle for anti-asymmetric functions on a half space, in an unbounded region, without assuming that the function vanishes near infinity.

Before stating the maximum principle, we introduce some notation. Choose any direction to be the x_1 direction. For $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^n | x_1 = \lambda\}$$

be the moving plane,

$$\Sigma_\lambda = \{x \in \mathbb{R}^n | x_1 > \lambda\}$$

be a region to one side of the plane T_λ,

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$$

be the reflection of x about T_λ. Denote

$$u_\lambda(x) = u(x^\lambda) \text{ and } w_\lambda(x) = u_\lambda(x) - u(x).$$

Denote $I(u) = a + b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x$, from (2), $a > 0$ and $b \geqslant 0$, by a simple computation, one has

$$I(u) = I(u_\lambda) \text{ and } I(u) > 0.$$

Theorem 2 ***(Maximum principle for anti-symmetric functions)*** *Assume that* $w_\lambda \in C^{1,1}_{loc}(\mathbb{R}^n) \cap \mathscr{L}_{2s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ *is bounded from above. If* $c_\lambda(x)$ *is nonnegative in* Σ_λ *and*

$$\begin{cases} \left(a+b\displaystyle\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s w_\lambda(x)+c_\lambda(x)w_\lambda(x)\leqslant 0, \\ \qquad\qquad \text{at points } x\in\Sigma_\lambda \text{ where } w_\lambda(x)>0, \\ w_\lambda(x^\lambda)=-w_\lambda(x), \quad x\in\Sigma_\lambda, \end{cases} \tag{7}$$

then

$$w_\lambda(x)\leqslant 0, \qquad x\in\Sigma_\lambda. \tag{8}$$

Furthermore, assume that

$$\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s w_\lambda(x)+c_\lambda(x)w_\lambda(x)\leqslant 0 \quad \text{at points } x\in\Sigma_\lambda \text{ where } w_\lambda(x)=0, \tag{9}$$

then

$$\text{either } w_\lambda<0 \text{ in } \Sigma_\lambda \text{ or } w_\lambda=0 \text{ in } \mathbb{R}^n. \tag{10}$$

This maximum principle will be powerful in carrying out the method of moving planes on unbounded domains. Now we introduce the results about De Giorgi conjecture related to the nonlinear fractional Kirchhoff equation. The well-known De Giorgi conjecture [38] may be stated as: If u is a solution of equation

$$-\Delta u(x)=u(x)-u^3(x), q \quad x\in\mathbb{R}^n$$

such that $|u(x)|\leqslant 1$, $\lim_{x_1\to\pm\infty}u(x_1,x')=\pm 1$, for all $x'\in\mathbb{R}^{n-1}$ and $\frac{\partial u}{\partial x_1}>0$. Then there exists a vector $\mathbf{a}\in\mathbb{R}^{n-1}$ and a function $u_1:\mathbb{R}\to\mathbb{R}$ such that

$$u(x_1,x')=u_1(x_1+\mathbf{a}\cdot x'), \qquad \forall\ x\in\mathbb{R}^n.$$

We derive

Theorem 3 *Suppose that* $u\in C^{1,1}_{loc}(\mathbb{R}^n)\cap\mathscr{L}_{2s}(\mathbb{R}^n)\cap H^s(\mathbb{R}^n)$ *is a solution to*

$$\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s u(x)=f(u(x)),\ x\in\mathbb{R}^n, \tag{11}$$

and satisfies that

$$|u(x)|\leqslant 1, \qquad \forall\ x\in\mathbb{R}^n,$$

$$u(x_1,x')\to\pm 1 \text{ uniformly in } x'=(x_2,\dots,x_n)\in\mathbb{R}^{n-1}, \quad \text{as } x_1\to\pm\infty, \tag{12}$$

and

$$f(z) \text{ is non-increasing for } |z| \text{ sufficiently close to } 1. \tag{13}$$

Then there exists $A>0$ *such that*

$$\frac{\partial u}{\partial x_1}\geqslant 0 \quad \text{for all } x \text{ with } |x_1|\geqslant A. \tag{14}$$

Remark 2

(i) Here the condition on f is satisfied for $f(u)=u-u^3$, as given in the De Giorgi Conjecture.

(ii) Conclusion (14) implies that

$$w_\lambda(x)\leqslant 0,\ \forall\ x\in\Sigma_\lambda \text{ for all sufficiently large } \lambda,$$

which actually provides a starting point to move the plane T_λ in studying the symmetry and monotonicity of solutions for the nonlinear fractional Kirchhoff equation (11).

PROOF OF MAXIMUM PRINCIPLE FOR ANTI-SYMMETRY FUNCTIONS

Before proving the maximum principle for anti-symmetry functions, we first prove an important Lemma.

Lemma 1 *Assume that* $\varphi\in C_0^\infty(\mathbb{R}^n)$ *and* $\epsilon>0$, *then for all small* $\delta>0$, *there holds*

$$|I(u)(-\Delta)^s(\epsilon\varphi)|\leqslant \epsilon C_\delta I(u)+C\delta^{2-2s}I(u),$$

where $I(u)=a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x$, *the constant* C *is independent of* ϵ, *while* C_δ *may dependent on* δ.

Proof: For simplicity, we assume $C_{n,s}=1$. For any δ and $x\in\mathbb{R}^n$, we have

$$\begin{aligned} I(u)(-\Delta)^s(\epsilon\varphi)(x) &= I(u)\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{\epsilon(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}y \\ &= I(u)\,\mathrm{P.V.}\Bigg[\int_{B_\delta(x)}\frac{\epsilon(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}y \\ &\quad +\int_{B^c_\delta(x)}\frac{\epsilon(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}y\Bigg]. \end{aligned}$$

By $\varphi\in C_0^\infty(\mathbb{R}^n)$, we derive

$$\begin{aligned} \left|\int_{B^c_\delta(x)}\frac{\epsilon(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}y\right| &\leqslant \epsilon C\int_{B^c_\delta(x)}\frac{1}{|x-y|^{n+2s}}\,\mathrm{d}y \\ &\leqslant \epsilon C_\delta. \end{aligned}$$

On the other hand, by Taylor expansion, for any fixed x, we obtain

$$\epsilon\varphi(x)-\epsilon\varphi(y)=\epsilon\nabla\varphi(x)\cdot(x-y)+O(|x-y|^2).$$

Since the anti-symmetry of $\epsilon\nabla\varphi(x)\cdot(x-y)$ for $y\in B_\delta(x)$, we have

$$\mathrm{P.V.}\int_{B_\delta(x)}\frac{\epsilon\nabla\varphi(x)\cdot(x-y)}{|x-y|^{n+2s}}\,\mathrm{d}y=0.$$

Then

$$\begin{aligned} &\left|\mathrm{P.V.}\int_{B_\delta(x)}\frac{\epsilon(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}y\right| \\ &\quad=\left|\mathrm{P.V.}\int_{B_\delta(x)}\frac{O(|x-y|^2)}{|x-y|^{n+2s}}\,\mathrm{d}y\right|\leqslant C\delta^{2-2s}. \end{aligned}$$

Hence,

$$|I(u)(-\Delta)^s(\epsilon\varphi)| \leqslant \epsilon C_\delta I(u) + C\delta^{2-2s} I(u),$$

which completes the proof. □

Next we will prove maximum principle for anti-symmetric functions in unbounded domains (Theorem 2).

Proof of Theorem 2

We use the contradiction arguments. Suppose that (8) is false, we have

$$M := \sup_{\Sigma_\lambda} w_\lambda(x) > 0.$$

Since $w_\lambda(x)$ is bounded from above on $\mathbb{R}^n$, then we have immediately $0 < M < +\infty$. We will discuss the following two cases; *Case (i)*: the supremum M may be attained in Σ_λ and *Case (ii)*: the supremum M may not be attained because the domain Σ_λ is unbounded. In both cases, we obtain contradictions and then proved (8) is true.

Case (i): If this supremum M can be attained in Σ_λ, say at the point $\tilde{x}$, then we have

$$\begin{aligned}
&\left(a + b\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x\right)(-\Delta)^s w_\lambda(\tilde{x}) \\
&= I(u) C_{n,s}\, \text{P.V.} \int_{\mathbb{R}^n} \frac{w_\lambda(\tilde{x}) - w_\lambda(y)}{|\tilde{x} - y|^{n+2s}} \, \mathrm{d}y \\
&= I(u) C_{n,s}\, \text{P.V.} \int_{\Sigma_\lambda} \frac{w_\lambda(\tilde{x}) - w_\lambda(y)}{|\tilde{x} - y|^{n+2s}} + \frac{w_\lambda(\tilde{x}) + w_\lambda(y)}{|\tilde{x} - y^\lambda|^{n+2s}} \, \mathrm{d}y \\
&\geqslant I(u) C_{n,s} 2 w_\lambda(\tilde{x}) \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y^\lambda|^{n+2s}} \, \mathrm{d}y > 0, \qquad (15)
\end{aligned}$$

where the second inequality from bottom is due to $|\tilde{x} - y| < |\tilde{x} - y^\lambda|$ and the last inequality is due to $w_\lambda(\tilde{x}) > 0$ and $I(u) > 0$. Therefore, by non-negative property of c_λ, we obtain

$$\left(a + b\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x\right)(-\Delta)^s w_\lambda(\tilde{x}) + c_\lambda(\tilde{x}) w_\lambda(\tilde{x}) > 0.$$

This contradicts the first inequality of (7). Thus (8) is right.

Case (ii): If this supremum M cannot be attained, by the definition of supremum, there exist sequences $x^k \in \Sigma_\lambda$ and $0 < \beta_k < 1$ with $\beta_k \to 1$ as $k \to \infty$ such that

$$w_\lambda(x^k) \geqslant \beta_k M. \qquad (16)$$

Denote $d_k := \frac{1}{2}\text{dist}(x^k, T_\lambda)$. Set

$$\Psi(x) = \begin{cases} \mathrm{e}^{\frac{|x|^2}{|x|^2 - 1}}, & |x| < 1, \\ 0, & |x| \geqslant 1. \end{cases}$$

It is well known that $\Psi(x) \in C_0^\infty(\mathbb{R}^n)$, thus $|(-\Delta)^s \Psi(x)| \leqslant C$ for all $x \in \mathbb{R}^n$. Obviously, $\Psi(0) = \max_{\mathbb{R}^n} \Psi(x) = 1$. Let

$$\Psi_k(x) = \Psi\left(\frac{x - (x^k)^\lambda}{d_k}\right) \text{ and } \tilde{\Psi}_k(x) = \Psi_k(x^\lambda) = \Psi\left(\frac{x - x^k}{d_k}\right).$$

Then $\tilde{\Psi}_k(x) - \Psi_k(x)$ is anti-symmetry with respect to T_λ. Taking $\varepsilon_k = (1 - \beta_k)M$, we obtain

$$w_\lambda(x^k) + \varepsilon_k[\tilde{\Psi}_k - \Psi_k](x^k) \geqslant M.$$

Denote

$$w_k(x) := w_\lambda(x) + \varepsilon_k[\tilde{\Psi}_k - \Psi_k](x).$$

So $w_k(x)$ is also anti-symmetric with respect to T_λ.

Since for any $x \in \Sigma_\lambda \backslash B_{d_k}(x^k)$, $w_\lambda(x) \leqslant M$ and $\tilde{\Psi}_k(x) = \Psi_k(x) = 0$, we have

$$w_k(x^k) \geqslant w_k(x), \qquad \forall\, x \in \Sigma_\lambda \backslash B_{d_k}(x^k).$$

Hence the supremum of $w_k(x)$ in Σ_λ is achieved in $\overline{B_{d_k}(x^k)}$. Thus, there exists a point $\bar{x}^k \in \overline{B_{d_k}(x^k)}$ such that

$$w_k(\bar{x}^k) = \sup_{x \in \Sigma_\lambda} w_k(x) \geqslant M. \qquad (17)$$

By the choice of ε_k, it is easy to verify that $w_\lambda(\bar{x}^k) \geqslant \beta_k M > 0$.

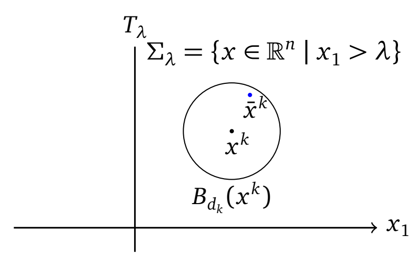

Fig. 1 The maximum point.

Now we will evaluate the upper bound and the lower bound of

$$\left(a + b\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x\right)(-\Delta)^s w_k(\bar{x}^k), \qquad (18)$$

respectively, then to derive a contradiction.

As a consequence of the first inequality to (7), $c_\lambda(x) \geqslant 0$ and Lemma 1, we obtain the upper bound

$$\begin{aligned}
&\left(a + b\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x\right)(-\Delta)^s w_k(\bar{x}^k) \\
&\quad \leqslant -c_\lambda(\bar{x}^k) w_\lambda(\bar{x}^k) + \epsilon_k C_\delta I(u) + C\delta^{2-2s} I(u) \\
&\quad \leqslant \epsilon_k C_\delta I(u) + C\delta^{2-2s} I(u). \qquad (19)
\end{aligned}$$

Next, we estimate the lower bound of (18) by direct calculations. We have

$$\begin{aligned}
&\Big(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)(-\Delta)^s w_k(\bar{x}^k)=I(u)(-\Delta)^s w_k(\bar{x}^k)\\
&=I(u)(-\Delta)^s[w_\lambda(\bar{x}^k)+\varepsilon_k(\tilde{\Psi}-\Psi)(\bar{x}^k)]\\
&=I(u)C_{n,s}\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{w_\lambda(\bar{x}^k)+\varepsilon_k(\tilde{\Psi}-\Psi)(\bar{x}^k)-w_\lambda(y)-\varepsilon_k(\tilde{\Psi}-\Psi)(y)}{|\bar{x}^k-y|^{n+2s}}\,\mathrm{d}y\\
&=I(u)C_{n,s}\mathrm{P.V.}\Big[\int_{\Sigma_\lambda}\frac{w_\lambda(\bar{x}^k)+\varepsilon_k(\tilde{\Psi}-\Psi)(\bar{x}^k)-w_\lambda(y)-\varepsilon_k(\tilde{\Psi}-\Psi)(y)}{|\bar{x}^k-y|^{n+2s}}\,\mathrm{d}y\\
&\quad+\int_{\Sigma_\lambda}\frac{w_\lambda(\bar{x}^k)+\varepsilon_k(\tilde{\Psi}-\Psi)(\bar{x}^k)+w_\lambda(y)+\varepsilon_k(\tilde{\Psi}-\Psi)(y)}{|\bar{x}^k-y^\lambda|^{n+2s}}\,\mathrm{d}y\Big]\\
&=I(u)C_{n,s}\mathrm{P.V.}\int_{\Sigma_\lambda}\Big(\frac{1}{|\bar{x}^k-y|^{n+2s}}-\frac{1}{|\bar{x}^k-y^\lambda|^{n+2s}}\Big)\\
&\quad\times\big(w_\lambda(\bar{x}^k)+\varepsilon_k(\tilde{\Psi}-\Psi)(\bar{x}^k)-w_\lambda(y)-\varepsilon_k(\tilde{\Psi}-\Psi)(y)\big)\,\mathrm{d}y\\
&\quad+I(u)C_{n,s}2\big(w_\lambda(\bar{x}^k)+\varepsilon_k(\tilde{\Psi}-\Psi)(\bar{x}^k)\big)\int_{\Sigma_\lambda}\frac{1}{|\bar{x}^k-y^\lambda|^{n+2s}}\,\mathrm{d}y\\
&:=J_1+J_2. \qquad (20)
\end{aligned}$$

We first estimate J_1. Since $\frac{1}{|\bar{x}^k-y|^{n+2s}}-\frac{1}{|\bar{x}^k-y^\lambda|^{n+2s}}>0$, for all $y\in\Sigma_\lambda$, we have

$$J_1\geqslant 0, \qquad (21)$$

due to (17). Then we derive

$$\begin{aligned}
J_2&=I(u)2C_{n,s}\big(w_\lambda(\bar{x}^k)+\varepsilon_k(\tilde{\Psi}-\Psi)(\bar{x}^k)\big)\int_{\Sigma_\lambda}\frac{1}{|\bar{x}^k-y^\lambda|^{n+2s}}\,\mathrm{d}y\\
&\geqslant CMI(u)\int_{\Sigma_\lambda}\frac{1}{|\bar{x}^k-y^\lambda|^{n+2s}}\,\mathrm{d}y\\
&\geqslant CMI(u)\int_{\Sigma_\lambda}\frac{1}{|x^k-y^\lambda|^{n+2s}}\,\mathrm{d}y, \qquad (22)
\end{aligned}$$

where the last inequality we have used the fact

$$|\bar{x}^k-y|\leqslant|\bar{x}^k-x^k|+|x^k-y|\leqslant\frac{3}{2}|x^k-y|.$$

Let $E=\{y\mid 2<y_1-x_1^k<3,\ |y'-(x^k)'|<1\}$, $t=y_1-x_1^k$, $\rho=|y'-(x^k)'|$ and ω_{n-2} denotes the area of unit sphere in $\mathbb{R}^{n-1}$. Now we estimate the last integral in (22) as

$$\begin{aligned}
\int_{\Sigma_\lambda}\frac{1}{|x^k-y^\lambda|^{n+2s}}\,\mathrm{d}y\geqslant\int_E\frac{1}{|x^k-y|^{n+2s}}\,\mathrm{d}y
&=\int_2^3\int_0^1\frac{\omega_{n-2}\rho^{n-2}}{(t^2+\rho^2)^{\frac{n+2s}{2}}}\,\mathrm{d}\rho\,\mathrm{d}t\\
&=\int_2^3\int_0^{\frac{1}{t}}\frac{\omega_{n-2}(tl)^{n-2}t}{t^{n+2s}(1+l^2)^{\frac{n+2s}{2}}}\,\mathrm{d}l\,\mathrm{d}t\\
&\geqslant\int_2^3\frac{1}{t^{1+2s}}\int_0^{\frac{1}{3}}\frac{\omega_{n-2}l^{n-2}}{(1+l^2)^{\frac{n+2s}{2}}}\,\mathrm{d}l\,\mathrm{d}t\\
&\geqslant c_1\int_2^3\frac{1}{t^{1+2s}}\,\mathrm{d}t=c_2>0. \qquad (23)
\end{aligned}$$

Hence, from (22) and (23), we obtain

$$J_2\geqslant C_1MI(u). \qquad (24)$$

Combining (20), (21) and (24), we deduce

$$\Big(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)(-\Delta)^s w_k(\bar{x}^k)\geqslant C_1I(u)M. \qquad (25)$$

Then combining (19) and (25), we derive

$$C_1I(u)M\leqslant\varepsilon_kC_\delta I(u)+C\delta^{2-2s}I(u).$$

It follows from $I(u)>0$ that

$$C_1M\leqslant\varepsilon_kC_\delta+C\delta^{2-2s}.$$

Now we choose δ small such that

$$C\delta^{2-2s}\leqslant\frac{C_1}{3}M,$$

then for such δ, let $k\to\infty$, thus $\varepsilon_k=(1-\beta_k)M$ is small such that

$$\varepsilon_kC_\delta\leqslant\frac{C_1}{3}=M,$$

which contradicts with $M>0$. Hence (8) is true.

Next, we will prove (10). Since we have proved that $w_\lambda(x)\leqslant 0$, $x\in\Sigma_\lambda$, if there exists a point $x_0\in\Sigma_\lambda$ such that

$$w_\lambda(x_0)=\min_{x\in\Sigma_\lambda}w_\lambda(x)=0.$$

Then by

$$\begin{aligned}
&\Big(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)(-\Delta)^s w_\lambda(x_0)\\
&=I(u)C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{-w_\lambda(y)}{|x_0-y|^{n+2s}}\,\mathrm{d}y\\
&=I(u)C_{n,s}\,\mathrm{P.V.}\,w_\lambda(y)\int_{\Sigma_\lambda}\Big[\frac{1}{|x_0-y^\lambda|^{n+2s}}-\frac{1}{|x_0-y|^{n+2s}}\Big]\mathrm{d}y\\
&\geqslant 0.
\end{aligned}$$

Thus from (9) and $I(u)>0$, we have $w_\lambda=0$ $a.e.$ in Σ_λ and hence $w_\lambda=0$ $a.e.$ in $\mathbb{R}^n$.

This completes the proof of Theorem 2. □

LIOUVILLE THEOREM

In this section, let us prove Theorem 1 by the maximum principle for anti-symmetry functions (Theorem 2).

Proof of Theorem 1

We show that $u(x)$ is symmetric with respect to any hyper plane. Let x_1 be any given direction in $\mathbb{R}^n$,

$$T_\lambda=\{x\in\mathbb{R}^n\mid x_1=\lambda\text{ for }\lambda\in\mathbb{R}^n\}$$

be a plane perpendicular to x_1 axis, and Σ_λ be a region to one side of the plane T_λ.

We have

$$\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s w_\lambda(x) = f(u_\lambda(x))-f(u(x)) = -c_\lambda(x)w_\lambda(x),$$

here $c_\lambda(x)=\frac{f(u_\lambda(x))-f(u(x))}{u(x)-u_\lambda(x)}\geqslant 0$ at points $x\in\Sigma_\lambda$ where $u_\lambda>u$ due to nonincreasing properties of f with respect to u.

Applying Theorem 2, we derive

$$w_\lambda(x)\leqslant 0,\qquad x\in\Sigma_\lambda.$$

Through the same discussion, we obtain

$$w_\lambda(x)\geqslant 0,\qquad x\in\Sigma_\lambda.$$

Hence, we have

$$w_\lambda(x)\equiv 0,\qquad x\in\Sigma_\lambda. \tag{26}$$

Therefore, this implies that $u(x)$ is symmetric with respect to plane T_λ for any $\lambda\in\mathbb{R}$.

Since the x_1-direction can be chosen arbitrarily, (26) implies u is radially symmetric about any point. Hence

$$u(x)\equiv C.$$

Here $u(x)\equiv C$ satisfies fractional Kirchhoff equation (6) due to $f(C)=0$. This completes the proof of Theorem 1. □

Proof of Theorem 3

In this section, we prove Theorem 3.

Proof: Let T_λ, Σ_λ, x^λ, u_λ, w_λ and $I(u)$ be defined as before. In order to prove (14), we only need to prove

$$w_\lambda(x)=u_\lambda(x)-u(x)\leqslant 0 \text{ for sufficiently large } \lambda. \tag{27}$$

Indeed, for sufficiently large λ, (12) implies that u_λ and u are close to 1. At the same time, by (13), we derive

$$\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s w_\lambda(x)=f(u_\lambda)-f(u)\leqslant 0$$

at the points $x\in\Sigma_\lambda$ where

$$u_\lambda(x)>u(x).$$

Then from Theorem 2, we obtain (27). Thus, a standard arguments will lead to (14) for all $|x_1|\geqslant A$.

Now, we prove (27) by the contradiction arguments. Suppose, by contradiction that

$$\sup_{\Sigma_\lambda} w_\lambda(x):=B>0. \tag{28}$$

Then for any $\sigma\in(0,1)$, there exists $x_0\in\Sigma_\lambda$ such that

$$w_\lambda(x_0)\geqslant\sigma B.$$

By rescaling, we may suppose that $\mathrm{dist}(x_0,T_\lambda)=2$. Define

$$\gamma(x)=\begin{cases}\mathrm{e}^{\frac{|x|^2}{|x|^2-1}}, & |x|<1,\\ 0, & |x|\geqslant 1.\end{cases}$$

Obviously, $\gamma(0)=\max_{\mathbb{R}^n}\gamma(x)=1$. Set

$$\Phi(x)=\gamma(x-x_0^\lambda) \text{ and } \Phi_\lambda(x)=\gamma(x-x_0).$$

Then $\Phi_\lambda(x)-\Phi(x)$ is an anti-symmetric function with respect to the plane T_λ.

Next, take $\varepsilon=(1-\sigma)B>0$ such that

$$w_\lambda(x_0)+\varepsilon(\Phi_\lambda(x_0)-\Phi(x_0))\geqslant B.$$

Similar to the proof in Theorem 2, there exists a point $\bar{x}\in\overline{B_1(x_0)}$ such that

$$\begin{aligned}&w_\lambda(\bar{x})+\varepsilon(\Phi_\lambda(\bar{x})-\Phi(\bar{x}))\\ &\quad=\max_{\Sigma_\lambda}[w_\lambda(x)+\varepsilon(\Phi_\lambda(x)-\Phi(x))]\geqslant B.\end{aligned} \tag{29}$$

Then we estimate the upper lower bounds of

$$\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s w_\lambda \tag{30}$$

at the maximum point $\bar{x}$.

On the one hand, as in the proof of Theorem 2, we can obtain the lower bound of (30) as

$$\begin{aligned}&\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s[w_\lambda(\bar{x})+\varepsilon(\Phi_\lambda(\bar{x})-\Phi(\bar{x}))]\\ &\qquad\geqslant C_1 I(u)B.\end{aligned} \tag{31}$$

On the other hand, since

$$w_\lambda(\bar{x})+\varepsilon(\Phi_\lambda(\bar{x})-\Phi(\bar{x}))>w_\lambda(x_0)+\varepsilon(\Phi_\lambda(x_0)-\Phi(x_0)),$$

$\Phi(\bar{x})=\Phi(x_0)=0$ and $\Phi_\lambda(x_0)=1\geqslant\Phi_\lambda(\bar{x})$, we obtain

$$w_\lambda(\bar{x})\geqslant w_\lambda(x_0)>0.$$

It yields that

$$u_\lambda(\bar{x})>u(\bar{x}).$$

By the monotonicity of f and Lemma 1, we arrive at

$$\begin{aligned}&\left(a+b\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s[w_\lambda(\bar{x})+\varepsilon(\Phi_\lambda(\bar{x})-\Phi(\bar{x}))]\\ &\quad\leqslant f(u_\lambda(\bar{x}))-f(u(\bar{x}))+I(u)\varepsilon C_\delta+CI(u)\delta^{2-2s}\\ &\quad\leqslant I(u)\varepsilon C_\delta+CI(u)\delta^{2-2s}.\end{aligned}$$

Hence, this contradicts (31) when δ is small and σ is sufficiently close to 1. This completes the proof. □

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