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Normal spacelike developable surfaces on Minkowski 3-space $\mathbb{R}^3_1$

Ibrahim Al-Dayel[a,*], **Emad Solouma**[a,b]

[a] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Saudi Arabia
[b] Department of Mathematics and Information Science, Faculty of Science, Beni-Suef University, Egypt

*Corresponding author, e-mail: iaaldayel@imamu.edu.sa

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ABSTRACT: In this paper, we introduce a normal spacelike developable surface that is normal to a surface Ω along a spacelike curve α in Minkowski 3-space $\mathbb{R}^3_1$. We study the existence and singularities of normal spacelike developable surface through two invariants of the spacelike curves on a surface. Furthermore, we will be interested in the case when the spacelike curve is a geodesic curve and when it lies on a surface of revolution.

KEYWORDS: normal spacelike developable surface, Darboux frame, Minkowski 3-space

MSC2020: 53A04 58Kxx

INTRODUCTION

The continuous moving of a straight line in the space along space curve (directrix) generates a surface which are called a ruled surface. A developable surface is a ruled surface which any generatrix is stationary, i.e., such that the tangent plane of the surface is the same at any point of the generatrix. Recently, developable surfaces have been by some authors [1–6].

This paper introduces a normal spacelike developable surface normal to a surface Ω along a spacelike curve α in Minkowski 3-space $\mathbb{R}^3_1$. We give the basic conception of Minkowski 3-space $\mathbb{R}^3_1$ and the Lorentziant Darboux frame, and classify the singularities of the normal spacelike developable surface along a curve on a surface. Then, we consider the existence and the uniqueness of the normal spacelike developable surface as well as a special curve on surfaces (geodesic curve) and the case when the curve lies on a surface of revolution. Additionally, we give an example when the curve α is a geodesic.

BASIC CONCEPTS

Let $\mathbb{R}^3_1$ be 3-dimensional Minkowski space rectangular coordinate system $(\varsigma_1, \varsigma_2, \varsigma_3)$ and with the Lorentzian inner product

$$L = -d\varsigma_1^2 + d\varsigma_2^2 + d\varsigma_3^2,$$

where $\varsigma_1, \varsigma_2, \varsigma_3 \in \mathbb{R}$.

Definition 1 Let u be any arbitrary vector in $\mathbb{R}^3_1$. Then, u is said to be:

1. spacelike if $L(u,u) > 0$ or u is a zero vector;
2. timelike if $L(u,u) < 0$;
3. null (lightlike) if $L(u,u) = 0$ and u is a nonzero vector.

A curve α parametrized by $\alpha = \alpha(s)\colon I \subset \mathbb{R} \to \mathbb{R}^3_1$ is said to be timelike, spacelike curve or null (lightlike) if for each $s \in I$, the curve $\alpha'(s)$ is timelike, spacelike or null (lightlike), respectively [7, 8].

Assume α is a regular spacelike curve with timelike principal normal vector in $\mathbb{R}^3_1$, then the moving Frenet frame $\{T, N, B\}$ of α satisfies:

$$\begin{pmatrix} T'(s) \\ B'(s) \\ N'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \qquad (1)$$

where $L(T,T) = L(B,B) = -L(N,N) = 1$ and $L(T,N) = L(N,B) = 0$.

Let $\bar{\alpha}\colon I \subset \mathbb{R} \to V$ and $\varphi\colon V \subset \mathbb{R}^2 \to \mathbb{R}^3_1$. Let $\varphi(V) = M$ be a regular curve and a spacelike embedding, respectively. Define a curve $\alpha\colon I \to M$ by $\alpha(s) = \varphi(\bar{\alpha}(s))$, then the vector field [9]:

$$\eta = \frac{\varphi_x \times \varphi_y}{\|\varphi_x \times \varphi_y\|} \qquad (2)$$

is a unit timelike vector field normal to $\varphi(V) = M$ and the vector $\zeta = T \times \eta$ is a spacelike vector.

Note that the frame $\{T, \eta, \zeta\}$ is a pseudo-orthonormal frame which is called the Lorentzian Darboux frame along α and the corresponding Frenet formulae of α:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} T \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 & \kappa_n & \kappa_g \\ \kappa_n & 0 & \tau_g \\ -\kappa_g & \tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ \eta \\ \zeta \end{pmatrix}, \qquad (3)$$

where $\kappa_g(s) = L\big(T'(s), \zeta(s)\big)$ is the asymptotic curvature of α, $\kappa_n(s) = -L\big(T'(s), \eta(s)\big)$ is the geodesic curvature of α, $\tau_g(s) = -L\big(\zeta'(s), \eta(s)\big)$ is the principal curvature of α , and s is arc-length parameter of α.

Recall that:

$$T\times\eta=\zeta,\quad \eta\times\zeta=-T,\quad \zeta\times T=\eta. \tag{4}$$

Also, it is well known that:

- α is an asymptotic curve if and only if $\kappa_n\equiv 0$;
- α is a geodesic curve if and only if $\kappa_g\equiv 0$;
- α is a principal curve if and only if $\tau_g\equiv 0$.

Now, consider the vector field $D_\rho(s)$ along α which is defined by

$$D_\rho(s)=\tau_g(s)\,T(s)-\kappa_g(s)\,\eta(s).$$

Recall that $D_\rho(s)$ rectifies spacelike Darboux vector along α. So, if $\tau_g^2(s)>\kappa_g^2(s)$, we define the pesudo-spherical rectifying spacelike Darboux image by

$$\bar{D}_\rho(s)=\frac{\tau_g(s)\,T(s)-\kappa_g(s)\,\eta(s)}{\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}. \tag{5}$$

Let $\alpha\colon I\to\mathbb{R}^3_1$ and $\psi\colon I\to\mathbb{R}^3_1\backslash\{0\}$ be two smooth curves such that $\|\psi(t)\|=1$. Then, we use these two smooth curves to define a ruled surface $\mathfrak{F}_{(\alpha,\psi)}\colon I\times\mathbb{R}\to\mathbb{R}^3_1$ by

$$\mathfrak{F}_{(\alpha,\psi)}(t,\upsilon)=\alpha(t)+\upsilon\,\psi(t). \tag{6}$$

We called $\alpha(t)$ the base curve of $\mathfrak{F}$ and $\psi(t)$ the director curve of $\mathfrak{F}$. Now, take the partial derivative with respect to t and υ:

$$\frac{\partial\mathfrak{F}_{(\alpha,\psi)}}{\partial t}(t,\upsilon)=\dot{\alpha}(t)+\upsilon\,\dot{\psi}(t),\quad \frac{\partial\mathfrak{F}_{(\alpha,\psi)}}{\partial \upsilon}(t,\upsilon)=\psi(t),$$

where $\left(\cdot=\dfrac{\mathrm{d}}{\mathrm{d}t}\right)$. So that the unit pseudo-normal vector at a regular point (t,υ) is

$$n(t,\upsilon)=\frac{1}{\ell}\Big(\big[\dot{\alpha}(t)+\upsilon\,\dot{\psi}(t)\big]\times\psi(t)\Big), \tag{7}$$

where $\ell=\left\|\frac{\partial\mathfrak{F}_{(\alpha,\psi)}}{\partial t}(t,\upsilon)\times\frac{\partial\mathfrak{F}_{(\alpha,\psi)}}{\partial \upsilon}(t,\upsilon)\right\|$. If $n(t,\upsilon)$ is orthogonal to $\dot{\alpha}(t)$ for any (t,υ), then we say that $\mathfrak{F}_{(\alpha,\psi)}$ is a developable surface. Note that $\mathfrak{F}_{(\alpha,\psi)}$ is a developable surface if and only if $\det\big(\dot{\alpha}(t),\psi(t),\dot{\psi}(t)\big)=0$. Note that $\mathfrak{F}_{(\alpha,\psi)}$ is defined to be a spacelike developable surface if $n(t,\upsilon)$ is timelike.

Let $\Omega\subset\mathbb{R}^3_1$ be a spacelike surface. If $\mathfrak{F}\cap\Omega\neq\phi$ and $T_p\mathfrak{F}$ and $T_p\Omega$ are orthogonal at any point $p\in\mathfrak{F}\cap\Omega$ then the spacelike developable surface $\mathfrak{F}$ is called a normal spacelike developable surface of Ω (see [1]) and the intersection $\mathfrak{F}\cap\Omega$ is a regular spacelike curve. However if $\mathfrak{F}$ is a spacelike cylinder, then $\mathfrak{F}$ is called a spacelike normal cylinder of Ω and the intersection $\mathfrak{F}\cap\Omega$ is a spacelike normal cylindrical slice. Also, $\mathfrak{F}$ is called a spacelike normal cone of Ω if $\mathfrak{F}$ is a spacelike cone and the intersection $\mathfrak{F}\cap\Omega$ is a spacelike normal conical slice.

NORMAL SPACELIKE DEVELOPABLE SURFACES IN $\mathbb{R}^3_1$

Let $\Omega\subset\mathbb{R}^3_1$ be a spacelike surface and $\alpha\colon I\to\Omega\subset\mathbb{R}^3_1$ be a regular spacelike curve on Ω with $\tau_g^2(s)>\kappa_g^2(s)$. Define a map $ND_\alpha\colon I\times\mathbb{R}\to\mathbb{R}^3_1$ by

$$\begin{aligned}ND_\alpha(s,\upsilon)&=\alpha(s)+\upsilon\,\bar{D}_\rho(s)\\ &=\alpha(s)+\upsilon\left(\frac{\tau_g(s)\,T(s)-\kappa_g(s)\,\eta(s)}{\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}\right),\end{aligned}$$

which is a spacelike ruled surface. Note that

$$\bar{D}_\rho'=\left(\kappa_n+\frac{\kappa_g\tau_g'-\kappa_g'\tau_g}{\tau_g^2-\kappa_g^2}\right)\left(\frac{-\kappa_g\,T+\tau_g\,\eta}{\sqrt{\tau_g^2-\kappa_g^2}}\right).$$

Thus,

$$\begin{aligned}\det\big(\alpha',\bar{D}_\rho,\bar{D}_\rho'\big)=\det\Bigg\{&T,\left(\frac{\tau_g\,T-\kappa_g\,\eta}{\sqrt{\tau_g^2-\kappa_g^2}}\right),\\ &\left(\kappa_n+\frac{\kappa_g\tau_g'-\kappa_g'\tau_g}{\tau_g^2-\kappa_g^2}\right)\left(\frac{-\kappa_g\,T+\tau_g\,\eta}{\sqrt{\tau_g^2-\kappa_g^2}}\right)\Bigg\}=0,\end{aligned}$$

which implies that ND_α is a spacelike developable surface. Here we recall ND_α a normal spacelike developable surface of Ω along α. Also, note that the invariants $\delta_\rho(s)$ and $\sigma_\rho(s)$ of Ω along α are given by:

$$\delta_\rho(s)=\kappa_n(s)+\frac{\kappa_g(s)\tau_g'(s)-\kappa_g'(s)\tau_g(s)}{\tau_g^2(s)-\kappa_g^2(s)},$$

$$\sigma_\rho(s)=\frac{\tau_g(s)}{\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}+\left(\frac{\kappa_g(s)}{\delta_\rho(s)\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}\right)',$$

when $\delta_\rho(s)\neq 0$. As conclusion of the above computation, $\delta_\rho(s)=0$ if and only if $\bar{D}_\rho'(s)=0$. Also, we have

$$\frac{\partial ND_\alpha}{\partial s}\times\frac{\partial ND_\alpha}{\partial \upsilon}=-\left(\upsilon\,\delta_\rho(s)+\frac{\kappa_g(s)}{\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}\right)\zeta.$$

Therefore, $\delta_\rho(s_0)\neq 0$ if and only if (s_0,υ_0) is a singular point of ND_α and

$$\upsilon_0=\frac{-\kappa_g(s_0)}{\delta_\rho(s_0)\sqrt{\tau_g^2(s_0)-\kappa_g^2(s_0)}}.$$

If $\kappa_g(s_0)\neq 0$ this implies that $(s_0,0)$ is a regular point, then the timelike normal vector of ND_α at $ND_\alpha(s_0)=\alpha(s_0)$ is orthogonal to the timelike normal vector of Ω at $\alpha(s_0)$. So we recall ND_α the normal spacelike developable surface of Ω along α.

Theorem 1 *If $\alpha\colon I\to\Omega\subset\mathbb{R}^3_1$ is a unit speed spacelike curve on Ω with $\tau_g^2(s)>\kappa_g^2(s)$. Then, we have:*

(1) *the following statements are equivalent:*

(i) ND_α *is a spacelike cylinder,*

(ii) $\delta_\rho(s) \equiv 0$,

(iii) α *is the slice of* Ω *with a spacelike pseudo-normal cylinder;*

(2) *if* $\delta_\rho(s) \neq 0$, *then the following statements are equivalent*

(i) ND_α *is a spacelike cone,*

(ii) $\sigma_\rho(s) \equiv 0$,

(iii) α *is the slice of* Ω *with a spacelike pseudo-normal conical.*

Proof: (1) From the definition, ND_α is a spacelike cylinder if and only if $\bar{D}_\rho(s)$ is a constant. Then,

$$\bar{D}'_\rho(s) = \delta_\rho(s)\left(\frac{-\kappa_g(s)\,T(s)+\tau_g(s)\,\eta(s)}{\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}\right),$$

so, $\bar{D}_\rho(s)$ is a constant if and only if $\delta_\rho(s) = 0$ this implies that (i) is equivalent to (ii). Now, suppose that α is the slice of Ω with a spacelike pseudo-normal cylinder, then there is a vector $v \in S_1^2$ such that $L\big(\zeta(s), v\big) = 0$ where v is the director of the spacelike normal cylinder. So, we can write $v = \lambda\,T(s) + \beta\,\eta(s)$ for some $\alpha, \beta \in \mathbb{R}$. Thus $-\lambda\,\kappa_g(s)+\beta\,\tau_g(s) = 0$ because $L\big(\zeta'(s), v\big) = 0$. So $v = \bar{D}'_\rho(s)$ which implies that condition (i) holds. It clear that condition (i) implies condition (iii).

(2) Note that, ND_α is a spacelike cone, which means that the singular value of ND_α is a constant vector. Let us consider the function $g(s)$ defined as

$$g(s) = \alpha(s) + \left(\frac{\kappa_g(s)}{\delta_\rho(s)\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}\right)\bar{D}_\rho(s).$$

So, condition (i) is equivalent to the condition $g'(s) = 0$. But

$$\begin{aligned} g' &= T + \left(\frac{\kappa_g}{\delta_\rho\sqrt{\tau_g^2-\kappa_g^2}}\right)'\bar{D}_\rho + \left(\frac{\kappa_g}{\delta_\rho\sqrt{\tau_g^2-\kappa_g^2}}\right)\bar{D}'_\rho \\ &= T + \left(\frac{\kappa_g}{\delta_\rho\sqrt{\tau_g^2-\kappa_g^2}}\right)'\bar{D}_\rho + \left(\frac{\kappa_g}{\sqrt{\tau_g^2-\kappa_g^2}}\right)\left(\frac{-\kappa_g T+\tau_g\eta}{\sqrt{\tau_g^2-\kappa_g^2}}\right) \\ &= \left[\left(\frac{\tau_g}{\sqrt{\tau_g^2-\kappa_g^2}}\right) + \left(\frac{\kappa_g}{\delta_\rho\sqrt{\tau_g^2-\kappa_g^2}}\right)'\right]\bar{D}_\rho = \sigma_\rho\bar{D}_\rho. \end{aligned}$$

It follows that (i) is equivalent to (ii). From the definition of the spacelike conical slice, condition (iii) implies that there exists $\theta \in \mathbb{R}_1^3$ such that $L\big(\alpha(s) - \theta, \zeta(s)\big) = 0$. If (i) holds, then the vector valued function $g(s)$ is constant. Now, for the constant point $\theta = g(s) \in \mathbb{R}_1^3$, we have

$$\begin{aligned} L\big(\alpha(s)-\theta, \zeta(s)\big) &= L\big(\alpha(s)-g(s), \zeta(s)\big) \\ &= \left(\left(\frac{-\kappa_g(s)}{\delta_\rho(s)\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}\right)\bar{D}_\rho(s), \zeta(s)\right) = 0. \end{aligned}$$

This means that (iii) holds. Conversely, by condition (iii), there exist a point $\theta \in \mathbb{R}_1^3$ such that $L\big(\alpha(s) - \theta, \zeta(s)\big) = 0$. Differentiating both sides, we have

$$\begin{aligned} &L\big(\alpha(s)-\theta, \zeta(s)\big)' \\ &\quad = L\big(\alpha(s)-\theta, -\kappa_g(s)\,T(s)+\tau_g(s)\,\eta(s)\big) = 0. \end{aligned}$$

Then there exists $\varepsilon \in \mathbb{R}$ such that $\alpha(s) - \theta = \varepsilon\bar{D}_\rho(s)$. Taking the derivative, we have

$$\begin{aligned} 0 &= L\big(T(s), -\kappa_g(s)\,T(s)+\tau_g(s)\big) \\ &\quad + L\big(\alpha(s)-\theta, (-\kappa_g(s)\,T(s)+\tau_g(s))'\big) \\ &= -\kappa_g(s) + \varepsilon\,\delta_\rho(s)\sqrt{\tau_g^2-\kappa_g^2}. \end{aligned}$$

It follows that

$$\theta = \alpha(s) - \varepsilon\,\bar{D}_\rho(s) = \alpha(s) + \left(\frac{\kappa_g}{\delta_\rho\sqrt{\tau_g^2-\kappa_g^2}}\right)\bar{D}_\rho(s) = g(s).$$

which implies that $g(s)$ is constant, so condition (i) holds. □

Let $\alpha\colon I \to \Omega \subset \mathbb{R}_1^3$ be a unit speed spacelike curve on Ω. Define a function $F\colon I \times \mathbb{R}_1^3 \to \mathbb{R}$ by $F(s, y) = \mathscr{L}\big(y - \alpha(s), \zeta(s)\big)$. Recall that F is a support function on α with respect to ζ. We will write $f_{y_0}(s) = F(s, y_0)$ for any $s \in I$ and $y_0 \in \mathbb{R}_1^3$.

Proposition 1 *Let* $\alpha\colon I \to \Omega \subset \mathbb{R}_1^3$ *be a unit speed spacelike curve on* Ω *with* $\tau_g^2(s) > \kappa_g^2(s)$. *Assume that* $\delta_\rho(s_0) \neq 0$, *then we have the following statements:*

(1) $f_{y_0}(s_0) = 0$ *if and only if there are* $u, v \in \mathbb{R}$ *such that*

$$y_0 - \alpha(s_0) = u\,T(s_0) + v\,\eta(s_0).$$

(2) $f_{y_0}(s_0) = f'_{y_0}(s_0) = 0$ *if and only if there exists* $u \in \mathbb{R}$ *such that*

$$y_0 - \alpha(s_0) = u\left(\frac{\tau_g(s_0)\,T(s_0) - \kappa_g(s_0)\,\eta(s_0)}{\sqrt{\tau_g^2(s_0)-\kappa_g^2(s_0)}}\right).$$

(3) $f_{y_0}(s_0) = f'_{y_0}(s_0) = f''_{y_0}(s_0) = 0$ *if and only if one of the following is satisfied:*

i.

$$\begin{aligned} y_0 - \alpha(s_0) &= \left(\frac{-\kappa_g(s_0)}{\delta_\rho(s_0)\sqrt{\tau_g^2(s_0)-\kappa_g^2(s_0)}}\right) \\ &\quad \times \left(\frac{\tau_g(s_0)\,T(s_0) - \kappa_g(s_0)\,\eta(s_0)}{\sqrt{\tau_g^2(s_0)-\kappa_g^2(s_0)}}\right), \end{aligned} \tag{8}$$

ii. $\kappa_g(s_0) = 0$, $\kappa'_g(s_0) = -\kappa_n(s_0)\tau_g(s_0)$ *and there exists* $u \in \mathbb{R}$ *such that* $y_0 - \alpha(s_0) = u\,T(s_0)$.

(4) $f_{y_0}(s_0) = f'_{y_0}(s_0) = f''_{y_0}(s_0) = f^{(3)}_{y_0}(s_0) = 0$ *if and only*

i. if $\sigma_\rho(s_0) = 0$ *and* (8) *holds;*

ii. if one of the following conditions satisfies

(a) $\delta'_\rho(s_0) \neq 0$, $\kappa_g(s_0) = 0$, *i.e.,*

$$\kappa_g(s_0) = 0, \quad \kappa'_g(s_0) = -\kappa_n(s_0)\tau_g(s_0),$$
$$2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) - \kappa''_g(s_0) \neq 0,$$
$$y_0 - \alpha(s_0) = \frac{3\kappa'_g(s_0)}{2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) - \kappa''_g(s_0)}.$$

(b) $\delta'_\rho(s_0) = 0$, $\kappa_g(s_0) = \kappa'_g(s_0) = 0$, *i.e.,* $\kappa_g(s_0) = \kappa'_g(s_0) = \kappa_n(s_0) = 0$, $\kappa''_g(s_0) = \kappa'_n(s_0)\tau_g(s_0)$, *and there is* $u \in \mathbb{R}$ *such that* $y_0 - \alpha(s_0) = u\,T(s_0)$.

(5) $f_{y_0}(s_0) = f'_{y_0}(s_0) = f''_{y_0}(s_0) = f^{(3)}_{y_0}(s_0) = f^{(4)}_{y_0}(s_0) = 0$ *if and only if* $\sigma_\rho(s_0) = \sigma'_\rho(s_0) = 0$ *and* (8) *holds.*

Proof: Since

$$f_{y_0}(s_0) = L\big(y_0 - \alpha(s_0), \zeta(s_0)\big). \tag{9}$$

Then, we have

$$f'_{y_0}(s_0) = L\big(y_0 - \alpha, -\kappa_g\,T + \tau_g\,\eta\big), \tag{10}$$

$$f''_{y_0}(s_0) = \kappa_g + L\Big(y_0 - \alpha, \big[-\kappa'_g + \kappa_n\tau_g\big]T + \big[\tau_g - \kappa_g\kappa_n\big]\eta + \big[\tau_g^2 - \kappa_g^2\big]\zeta\Big), \tag{11}$$

$$\begin{aligned} f^{(3)}_{y_0}(s_0) &= 2\kappa'_g - \kappa_n\tau_g \\ &+ L\Big(y_0 - \alpha, \big[2\kappa_n\tau'_g + \kappa'_n\tau_g - \kappa''_g - \kappa_g(\kappa_n^2 - \kappa_g^2 + \tau_g^2)\big]T \\ &+ \big[\tau''_g + \tau_g(\kappa_n^2 - \kappa_g^2 + \tau_g^2) - \kappa_g\kappa'_n - 2\kappa_n\kappa'_g\big]\eta \\ &+ 3\big[\tau'_g\tau_g - \kappa'_g\kappa_g\big]\zeta\Big), \end{aligned} \tag{12}$$

$$\begin{aligned} f^{(4)}_{y_0}(s_0) &= 3\kappa''_g - 3\kappa_n\tau'_g - 2\kappa'_n\tau_g + \kappa_g(\kappa_n^2 - \kappa_g^2 + \tau_g^2) \\ &+ L\Big(y_0 - \alpha, \big[\kappa''_n\tau_g + \kappa'_n(2\tau_g + \tau'_g) + \kappa_n(2\kappa_n\kappa'_g - 3\kappa'_n\kappa_g + 2\tau'_g + \tau''_g) \\ &+ (\kappa_n\tau_g - \kappa'_g)(\kappa_n^2 - \kappa_g^2 + \tau_g^2) - 5\kappa_g(\tau_g\tau'_g - \kappa_g\kappa'_g) - \kappa'''_g\big]T \\ &+ \big[\tau'''_g - 3\kappa'_g\kappa'_n - \kappa_g\kappa''_n + 5\tau_g(\tau_g\tau'_g - \kappa_g\kappa'_g) \\ &+ (\tau'_g - \kappa_g\kappa_n)(\kappa_n^2 - \kappa_g^2 + \tau_g^2) + \kappa_n(3\kappa'_n\tau_g + 2\kappa_n\tau'_g - 3\kappa''_g)\big]\eta \\ &+ \big[(\tau_g^2 - \kappa_g^2)(\kappa_n^2 - \kappa_g^2 + \tau_g^2) + 4(\tau_g\tau''_g - \kappa_g\kappa''_g) \\ &+ 3(\tau'^2_g - \kappa'^2_g) + \kappa_g(2\kappa_n\tau'_g + \kappa'_n\tau_g) - \tau_g(\kappa_g\kappa'_n + 2\kappa_n\kappa'_g)\big]\zeta\Big). \end{aligned} \tag{13}$$

By (9) and by the definition, condition **(1)** holds. Also, by (10), we have $f_{y_0}(s_0) = f'_{y_0}(s_0) = 0$ if and only if $y_0 - \alpha(s_0) = u\,T(s_0) + v\,\eta(s_0) = 0$ this yields to $u\,\kappa_g(s_0) = v\,\tau_g(s_0)$. So, if $\kappa_g(s_0) \neq 0$, $\tau_g(s_0) \neq 0$, then we have

$$u = v\left(\frac{\tau_g(s_0)}{\kappa_g(s_0)}\right), \qquad v = u\left(\frac{\kappa_g(s_0)}{\tau_g(s_0)}\right).$$

Then, there exists $\varepsilon \in \mathbb{R}$ such that

$$y_0 - \alpha(s_0) = \varepsilon\left(\frac{\tau_g(s_0)\,T(s_0) - \kappa_g(s_0)\,\eta(s_0)}{\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}}\right).$$

Suppose that $\kappa_g(s_0) = 0$, then $\tau_g(s_0) \neq 0$ and $v\,\tau_g(s_0) = 0$. Therefore, we obtain

$$y_0 - \alpha(s_0) = u\,T(s_0) = \pm u\left(\frac{\tau_g(s_0)\,T(s_0) - \kappa_g(s_0)\,\eta(s_0)}{\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}}\right).$$

If $\tau_g(s_0) = 0$, then we have $y_0 - \alpha(s_0) = v\,\eta(s_0)$ which implies that condition **(2)** holds.

By (11), we have $f_{y_0}(s_0) = f'_{y_0}(s_0) = f''_{y_0}(s_0) = 0$ if and only if

$$y_0 - \alpha(s_0) = \varepsilon\left(\frac{\tau_g(s_0)\,T(s_0) - \kappa_g(s_0)\,\eta(s_0)}{\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}}\right),$$

and

$$\kappa_g(s_0) + \varepsilon\left(\frac{\tau_g(s_0)\big(\kappa'_g(s_0) - \kappa_n(s_0)\kappa_g(s_0)\big)}{\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}} - \frac{\kappa_g(s_0)\big(\kappa_g(s_0)\kappa_n(s_0) - \tau'_g(s_0)\big)}{\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}}\right) = 0.$$

It follows that

$$\frac{\kappa_g(s_0)}{\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}} + \varepsilon\left(\kappa_n(s_0) + \frac{\tau'_g(s_0)\kappa_g(s_0) - \kappa'_g(s_0)\tau_g(s_0)}{\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}}\right) = 0.$$

Then, we have

$$\delta_\rho(s_0) = \kappa_n(s_0) + \frac{\tau'_g(s_0)\kappa_g(s_0) - \kappa'_g(s_0)\tau_g(s_0)}{\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}} \neq 0$$

$$\text{and} \quad \varepsilon = \frac{-\kappa_g(s_0)}{\delta_\rho(s_0)\sqrt{\tau_g^2(s_0) - \kappa_g^2(s_0)}}.$$

Also, if $\kappa_g(s_0) = \delta_\rho(s_0) = 0$ then, condition **(3)** hold.

Now, suppose that $\delta_\rho(s_0)\neq 0$. By (12), we have $f_{y_0}(s_0)=f'_{y_0}(s_0)=f''_{y_0}(s_0)=f^{(3)}_{y_0}(s_0)=0$ if and only if

$$2\kappa'_g-\kappa_n\tau_g-\left(\frac{\kappa_g}{\delta_\rho\sqrt{\tau_g^2-\kappa_g^2}}\right)\times\left\{\left(2\kappa_n\tau'_g+\kappa'_n\tau_g-\kappa''_g-\kappa_g\left(\kappa_n^2-\kappa_g^2+\tau_g^2\right)\right)\left(\frac{\tau_g}{\sqrt{\tau_g^2-\kappa_g^2}}\right)+\left(\tau''_g-\kappa'_n\kappa_g+2\kappa_n\kappa'_g+\tau_g\left(\kappa_n^2-\kappa_g^2+\tau_g^2\right)\right)\left(\frac{\kappa_g}{\sqrt{\tau_g^2-\kappa_g^2}}\right)\right\}=0.$$

For $s=s_0$, we get

$$2\kappa'_g(s_0)-\kappa_n(s_0)\tau_g(s_0)-\left(\frac{\kappa_g(s_0)}{\delta_\rho(s_0)}\right)\times\left\{\kappa'_n(s_0)+2\kappa_n(s_0)\left(\frac{\tau'_g(s_0)\kappa_g(s_0)-\kappa'_g(s_0)\tau_g(s_0)}{\tau_g^2(s_0)-\kappa_g^2(s_0)}\right)-\frac{\tau_g(s_0)\kappa''_g(s_0)-\kappa_g(s_0)\tau''_g(s_0)}{\tau_g^2(s_0)-\kappa_g^2(s_0)}\right\}=0.$$

Since

$$\delta'_\rho=\kappa'_n+\frac{(\kappa_g\tau'_g-\kappa'_g\tau_g)(\tau_g\tau'_g+\kappa_g\kappa'_g)}{(\tau_g^2-\kappa_g^2)^2}-\frac{\tau_g\kappa''_g-\kappa_g\tau''_g}{\tau_g^2-\kappa_g^2},$$

we have

$$2\kappa'_g(s_0)-\kappa_n(s_0)\tau_g(s_0)-\kappa_g(s_0)\left(\frac{\delta'_\rho(s_0)}{\delta_\rho(s_0)}\right)+\kappa_g(s_0)\left(\frac{\tau'_g(s_0)\kappa_g(s_0)-\kappa'_g(s_0)\tau_g(s_0)}{\tau_g^2(s_0)-\kappa_g^2(s_0)}\right)=0.$$

Moreover, we use the relation

$$\left(\frac{\kappa_g}{\sqrt{\tau_g^2-\kappa_g^2}}\right)'=\left(\frac{\tau_g}{\sqrt{\tau_g^2-\kappa_g^2}}\right)\left(\frac{\kappa_g\tau'_g-\kappa'_g\tau_g}{\tau_g^2-\kappa_g^2}\right)=\left(\kappa_n-\delta_\rho\right)\left(\frac{\tau_g}{\sqrt{\tau_g^2-\kappa_g^2}}\right),$$

then we have

$$\delta_\rho(s_0)\sqrt{\tau_g^2(s_0)-\kappa_g^2(s_0)}\left\{\frac{\tau_g(s_0)}{\sqrt{\tau_g^2(s_0)-\kappa_g^2(s_0)}}+\left(\frac{\kappa_g(s_0)}{\delta_\rho(s_0)\sqrt{\tau_g^2(s_0)-\kappa_g^2(s_0)}}\right)'\right\}=\delta_\rho(s_0)\sigma_\rho(s_0)\sqrt{\tau_g^2(s_0)-\kappa_g^2(s_0)}=0.$$

So, we have $\sigma_\rho(s_0)=0$ which implies that the proof of condition **(4i)** is complete.

Suppose that $\delta_\rho(s_0)=0$, then from (13), $f_{y_0}(s_0)=f'_{y_0}(s_0)=f''_{y_0}(s_0)=f^{(3)}_{y_0}(s_0)=0$ if and only if $\kappa_g(s_0)=0$ and $\kappa'_g(s_0)=-\kappa_n(s_0)\tau_g(s_0)$. Then there is $u\in\mathbb{R}$ such that

$$y_0-\alpha(s_0)=u\,T(s_0),$$

and

$$2\kappa'_g(s_0)-\kappa_n(s_0)\tau_g(s_0)-u\left[2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)-\kappa''_g(s_0)\right]=0.$$

Since $\delta_\rho(s_0)=0$ and $\kappa_g(s_0)=0$, we have $\kappa'_g(s_0)=\kappa_n(s_0)\tau_g(s_0)$ so

$$3\kappa'_g(s_0)-u\left[2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)-\kappa''_g(s_0)\right]=0.$$

It follows that $2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)-\kappa''_g(s_0)\neq 0$ and

$$u=\frac{3\kappa'_g(s_0)}{2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)-\kappa''_g(s_0)},$$

or $2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)-\kappa''_g(s_0)=0$ and $\kappa'_g(s_0)=0$. Therefore, condition **(4ii)** holds. Additionally, we can obtain the proof of condition **(5)** by similar arguments of those above. □

EXISTENCES OF NORMAL SPACELIKE DEVELOPABLE SURFACES

Let $\Omega\subset\mathbb{R}^3_1$ be a spacelike surface and $\alpha\colon I\to\Omega\subset\mathbb{R}^3_1$ be a spacelike curve on Ω with $\tau_g^2(s)>\kappa_g^2(s)$. In this section, we will investigate the existence and uniqueness of spacelike developable surface that is normal to Ω along α.

Theorem 2 *Let Ω be a spacelike surface and $\alpha\colon I\to\Omega\subset\mathbb{R}^3_1$ be a unit speed spacelike curve with $\tau_g^2(s)>\kappa_g^2(s)$. Then there is a unique spacelike developable surface that is normal to Ω along α.*

Proof: Consider a normal spacelike developable surface ND_α along α. Now, let N_α be a spacelike developable surface that is normal to Ω along α. Since N_α is a spacelike ruled surface, we assume that

$$N_\alpha(s,u)=\alpha(s)+u\,\Upsilon(s),$$

and we can write

$$\Upsilon(s)=\lambda(s)\,T(s)+\mu(s)\,\eta(s)+\gamma(s)\,\zeta(s).$$

Then

$$\Upsilon'=(\lambda'+\mu\kappa_n-\gamma\kappa_g)T+(\mu'+\lambda\kappa_n+\gamma\tau_g)\eta+(\gamma'+\lambda\kappa_g+\mu\tau_g)\zeta.$$

Since N_α is a spacelike developable surface, thus $\det\left(\alpha',\Upsilon,\Upsilon'\right)=0$ or equivalently

$$\gamma(\mu'+\lambda\kappa_n+\gamma\tau_g)-\mu(\gamma'+\lambda\kappa_g+\mu\tau_g)=0. \qquad (14)$$

Furthermore, N_α is a spacelike developable surface that is normal to Ω along α. We have

$$\frac{\partial N_\alpha}{\partial s}(s,u)\times\frac{\partial N_\alpha}{\partial u}(s,u)=\vartheta(s,u)\zeta(s). \qquad (15)$$

Now, suppose that N_α is non-singular at $(s,0)$, then $\vartheta(s,0)\neq 0$. Using straightforward computation, we get

$$\begin{aligned}\frac{\partial N_\alpha}{\partial s}&=\big(1+u(\lambda'+\mu\kappa_n-\gamma\kappa_g)\big)T\\&\quad+u\big(\mu'+\lambda\kappa_n+\gamma\tau_g\big)\eta+u\big(\gamma'+\lambda\kappa_g+\mu\tau_g\big)\zeta,\\\frac{\partial N_\alpha}{\partial u}&=\lambda T+\mu\eta+\gamma\zeta.\end{aligned}$$

So

$$\begin{aligned}&\frac{\partial N_\alpha}{\partial s}(s,u)\times\frac{\partial N_\alpha}{\partial u}(s,u)\\&=u\big[\mu(\gamma'+\lambda\kappa_g+\mu\tau_g)-\gamma(\mu'+\lambda\kappa_n+\gamma\tau_g)\big]T(s)\\&+\big[u\lambda(\gamma'+\lambda\kappa_g+\mu\tau_g)-\mu\big(1+u(\lambda'+\mu\kappa_n-\gamma\kappa_g)\big)\big]\eta(s)\\&+\big[\mu\big(1+u(\lambda'+\mu\kappa_n-\gamma\kappa_g)\big)-u\lambda(\mu'+\lambda\kappa_n+\gamma\tau_g)\big]\zeta(s).\end{aligned}$$

If we substitute $u=0$, we have

$$\frac{\partial N_\alpha}{\partial s}(s,0)\times\frac{\partial N_\alpha}{\partial u}(s,0)=-\gamma\,\eta(s)+\mu\,\zeta(s).$$

From (15), we have $\vartheta(s,0)=\mu(s)$, $\gamma(s)=0$. By (14), we have

$$\mu(s)\Big(\lambda(s)\kappa_g(s)+\mu(s)\tau_g(s)\Big)=0.$$

Suppose that N_α is non-singular along α, then $\vartheta(s,0)\neq 0$, thus $\mu(s)\neq 0$. This implies that $\lambda(s)\kappa_g(s)+\mu(s)\tau_g(s)=0$. If $\kappa_g(s)\neq 0$, then

$$\lambda(s)=-\left(\frac{\tau_g(s)}{\kappa_g(s)}\right)\mu(s).$$

Therefore

$$\begin{aligned}\Upsilon(s)&=-\left(\frac{\tau_g(s)}{\kappa_g(s)}\right)\mu(s)\,T(s)+\mu(s)\,\eta(s)\\&=-\mu(s)\left(\frac{\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}{\kappa_g(s)}\right)\left(\frac{\tau_g(s)T(s)-\kappa_g(s)\eta(s)}{\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}\right)\\&=-\mu(s)\bar{D}_\rho(s)\left(\frac{\sqrt{\tau_g^2(s)-\kappa_g^2(s)}}{\kappa_g(s)}\right).\end{aligned}$$

This implies that $\Upsilon(s)$ is in the opposite direction of $\bar{D}_\rho(s)$. If $\tau_g(s)\neq 0$, then $\Upsilon(s)$ and $\bar{D}_\rho(s)$ have the same direction.

Now suppose that N_α has a singular point at $(s_0,0)$. Then $\vartheta(s_0,0)=0$, which implies that $\mu(s_0)=\gamma(s_0)=0$. Thus, we have $\Upsilon(s_0)=\lambda(s_0)\,T(s_0)$. If the singular point $\alpha(s_0)$ is on the closure of A where A is the set of all points where the normal spacelike developable surface is regular on α, then there is a point s in any neighbourhood of s_0 such that at this point s the uniqueness of normal spacelike developable surface holds. Taking the limit as s approaches to s_0, then at s_0 the uniqueness of the normal spacelike developable surface holds. Suppose that there is an open interval $J\subset I$ such that for any $s\in J$, N_α is singular at $\alpha(s)$. Then for any $J\subset I$

$$N_\alpha(s)=\alpha(s)+u\,\lambda(s)\,T(s).$$

So, we have

$$\frac{\partial N_\alpha}{\partial s}(s,u)\times\frac{\partial N_\alpha}{\partial u}(s,u)=u\lambda^2(s)\big(\kappa_g(s)\eta(s)-\kappa_n(s)\zeta(s)\big).$$

This vector is directed to ζ, so for any $J\subset I$, $\kappa_g(s)=0$ and in this case $\bar{D}_\rho(s)=\pm T(s)$, which implies that the uniqueness holds. □

Proposition 2 *Let $\alpha\colon I\to\Omega$ be a regular spacelike curve on Ω with $\kappa_g(s)\equiv\tau_g(s)\equiv 0$. Then α is a normal slice of Ω if and only if $N_\alpha(s)$ is a normal spacelike developable surface along α.*

Proof: If α is a normal slice of Ω, then there is a plane $\mathscr{P}$ such that $\alpha(I)=\Omega\cap\mathscr{P}$ and for any $s\in I$, $T(s),\eta(s)\in\mathscr{P}$. Therefore, for any $s\in I$, $\mathscr{P}$ is orthogonal to $\zeta(s)$. Then $\mathscr{P}$ is a normal spacelike developable surface of Ω along α.

Conversely, suppose that $N_\alpha(s)$ is a normal spacelike developable surface along α. Note that the torsion of α is given by

$$\tau=\tau_g-\frac{\kappa_g'\kappa_n-\kappa_g\kappa_n'}{\kappa_g^2-\kappa_n^2}.$$

If $\kappa_g(s)\equiv\tau_g(s)\equiv 0$, then $\tau\equiv 0$, so α is a plane curve. Furthermore, we have $\zeta'=-\kappa_g\,T+\tau_g\,\eta\equiv=0$. So N_α is a plane normal to Ω. As α is the intersection of Ω and N_α, α is a normal slice of Ω. □

Corollary 1 *Let Ω be a spacelike surface and $\alpha\colon I\to\Omega\subset\mathbb{R}^3_1$ be a unit speed spacelike curve. If there are more than one normal spacelike developable surfaces of Ω along α, then α is a straight line.*

Proof: Suppose that $\tau_g^2(s)>\kappa_g^2(s)$, then by Theorem 2 there is a unique spacelike developable surface that is normal to Ω along α. If $\kappa_g(s)\equiv\tau_g(s)\equiv 0$, α is a spacelike normal slice and then a normal plane $\mathscr{P}$ of Ω at $\alpha(s_0)$ is a normal spacelike developable surface along α. Let N_α be another spacelike developable surface that is normal to Ω along α, then N_α is tangential to $\mathscr{P}$ along α and therefore $\mathscr{P}$ is a tangent plane of N_α. So $\mathscr{P}$ is a tangent to N_α along a ruling of N_α which is α and thus α is a straight line. □

CURVES ON A NORMAL SPACELIKE DEVELOPABLE SURFACES

Geodesics

Let $\Omega \subset \mathbb{R}_1^3$ be a spacelike surface and $\alpha: I \to \Omega$ be a unit speed spacelike curves. Then α is a geodesic on Ω if and only if $\kappa_g \equiv 0$. Thus, if $\tau_g \neq 0$, then

$$ND_\alpha(s,u) = \alpha(s) + u\,T(s),$$

which is the tangent surface of α. Then

$$\delta_\rho(s) = \kappa_n(s), \qquad \sigma_\rho(s) = \pm 1, \qquad \sigma'_\rho(s) = 0.$$

Example 1 Consider a spacelike ruled surface Ω with spacelike base curve $\alpha(t) = \Big(\frac{t}{10}\big(\sinh(2\ln t) - 2\cosh(2\ln t)\big), \frac{t}{10}\big(2\sinh(2\ln t) - \cosh(2\ln t)\big), t\Big)$ by

$$M(t,u) = \Big(\frac{t}{10}\big(\sinh(2\ln t) - 2\cosh(2\ln t)\big), \frac{t}{10}\big(2\sinh(2\ln t) - \cosh(2\ln t)\big), t\Big) + u\Big(\frac{-3\sinh(2\ln t)}{\sqrt{109}}, \frac{3\cos(2\ln t)}{\sqrt{109}}, \frac{10}{\sqrt{109}}\Big).$$

Thus α is a regular spacelike curve on the surface $\Omega = \operatorname{Im} M$. So, we have

$$\begin{cases} \dot\alpha(t) = \Big(\frac{-3}{10}\sinh(2\ln t), \frac{3}{10}\cosh(2\ln t), 1\Big), \\ \ddot\alpha(t) = \Big(\frac{-3}{5t}\cosh(2\ln t), \frac{3}{5t}\sinh(2\ln t), 0\Big), \\ T(t) = \Big(\frac{-3\sinh(2\ln t)}{\sqrt{109}}, \frac{3\cosh(2\ln t)}{\sqrt{109}}, \frac{10}{\sqrt{109}}\Big), \\ \eta = \dfrac{M_t \times M_u}{\|M_t \times M_u\|} = (-\cosh(2\ln t), \sinh(2\ln t), 0), \\ \zeta = T \times \eta = \frac{1}{\sqrt{109}}(10\sinh(2\ln t), 10\cosh(2\ln t), 3), \\ \kappa_g(t) = \dfrac{\det(\dot\alpha(t), \ddot\alpha(t), \eta(t))}{\|\dot\alpha(t)\|^3} = 0, \\ \tau_g(t) = \dfrac{\det(\dot\alpha(t), \eta(t), \dot\eta(t))}{\|\dot\alpha(t)\|^2} = \frac{-2}{109t}. \end{cases}$$

So α is a geodesic of M.

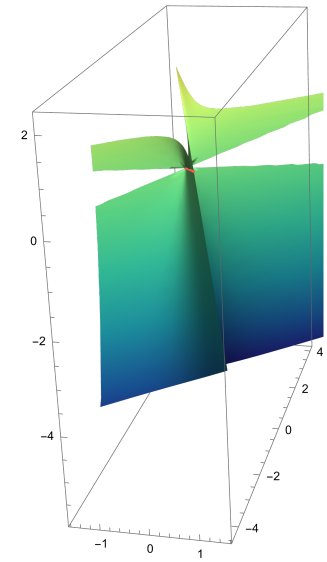

Fig. 1 M and α.

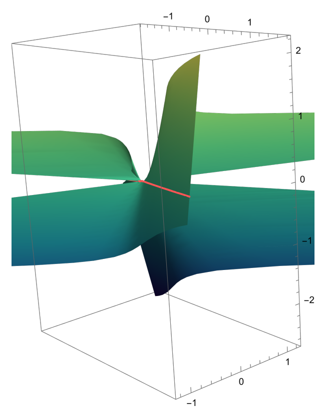

Fig. 2 ND_α and α.

Curves on a surface of revolution

Consider curves of a spacelike surface of revolution: $U \subset \mathbb{R}^2 \to \Omega \subset \mathbb{R}^3$ defined as

$$U(u,v) = (f(u)\cosh v, f(u)\sinh v, g(u)).$$

Assume that $f(u) \neq 0$. The unit timelike normal vector field along $\Omega = M(U)$ is

$$n_\Omega(u,v) = \left(\frac{g'(u)\cosh v}{\sqrt{\omega(t)}}, \frac{g'(u)\sinh v}{\sqrt{\omega(t)}}, \frac{f'(u)}{\sqrt{\omega(t)}}\right),$$

where $\omega(t) = g'^2(u(t)) - f'^2(u(t))$.

The Darboux frame of the spacelike curve $\alpha(t) = \big(f(u(t))\cosh v(t), f(u(t))\sinh v(t), g(u(t))\big)$ on Ω is given by

$$T(t) = \frac{1}{\sqrt{f^2\dot v^2 + \omega\dot u^2}}\Big(f'\dot u\cosh v(t) + f\dot v\sinh v(t), f'\dot u\sinh v(t) + f\dot v\cosh v(t), g'\dot u\Big),$$

$$\eta(t) = \left(\frac{g'\cosh v(t)}{\sqrt{\omega(t)}}, \frac{g'\sinh v(t)}{\sqrt{\omega(t)}}, \frac{f'}{\sqrt{\omega(t)}}\right),$$

$$\zeta(t) = \left(\frac{\omega\dot u\sinh v(t) - ff'\dot v\cosh v(t)}{\sqrt{\omega}\sqrt{f^2\dot v^2 + \omega\dot u^2}}, \frac{\omega\dot u\cosh v(t) - ff'\dot v\sinh v(t)}{\sqrt{\omega}\sqrt{f^2\dot v^2 + \omega\dot u^2}}, \frac{-fg'\dot v}{\sqrt{\omega}\sqrt{f^2\dot v^2 + \omega\dot u^2}}\right),$$

where $f' = \frac{df}{du}$, $g' = \frac{dg}{du}$, $\dot u(t) = \frac{du(t)}{dt}$, $\dot v(t) = \frac{dv(t)}{dt}$.

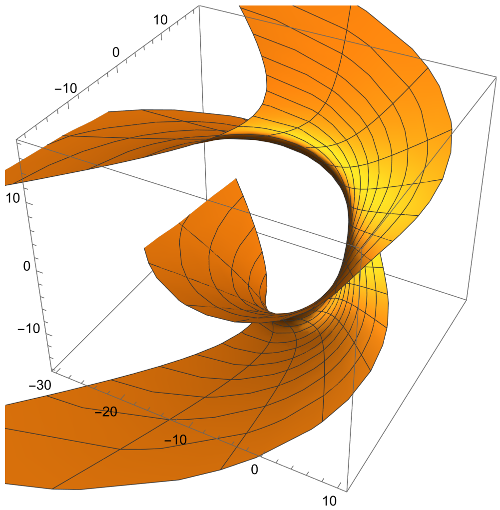

Fig. 3 $\Omega(u,v)=(\sin u\cosh v,\sin u\sinh v,u)$.

Then we have

$$\kappa_g(t)=\frac{f\dot{v}\big[\dot{u}^2(g'g''-f'f'')+\ddot{u}(f'^2+gg')+ff'\dot{v}\big]}{\sqrt{\omega}\big(f^2\dot{v}^2+\omega\dot{u}^2\big)^{\frac{3}{2}}}-\frac{\omega\dot{u}\big(f\ddot{v}+2f\dot{u}\dot{v}\big)}{\sqrt{\omega}\big(f^2\dot{v}^2+\omega\dot{u}^2\big)^{\frac{3}{2}}},$$

$$\kappa_n(t)=\frac{\dot{u}^2\big(f'g''-g'f''\big)+\ddot{u}\big(f'g-fg'\big)+fg'\dot{v}^2}{\sqrt{\omega}\big(f^2\dot{v}^2+\omega\dot{u}^2\big)},$$

$$\tau_g(t)=\frac{1}{\omega^3}\Big\{f\dot{v}\big[f'\dot{u}\big(g''\omega-gg'^2+ff'g'\big)-g'\big(f''\omega-f'(gg'-ff')\big)\big]+g'\omega\dot{u}\dot{v}(g'^2-f'^2)\Big\}.$$

So, for a meridian curve $\alpha(u)=\Omega(u,v_0)=\big(f(u)\cosh v_0,f(u)\sinh v_0,g(u)\big)$. Then, we have

$$\kappa_g(u)=0,\qquad \tau_g(u)=0,$$
$$\kappa_n(u)=\frac{\dot{u}^2\big(f'g''-g'f''\big)+\ddot{u}\big(f'g-fg'\big)}{\big(g'^2-f'^2\big)^{\frac{3}{2}}}$$

and thus the normal spacelike developable surface along α is a normal spacelike slice of Ω.

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