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Computation of conditional expectation related to pricing American options with localization function under multidimensional J-process

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ABSTRACT: Our basic objective is to introduce a new methodology using the localization function to compute the conditional expectation $E(V_t(X_t)|(X_s))$ for $s \leqslant t$, where the asset price is generated by the multi-dimensional J-process.

KEYWORDS: localization function, Malliavin derivatives, multidimensional J-process, American option pricing

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INTRODUCTION AND PRELIMINARY

Over the last years, numerous papers have proven the importance of applying Malliavin calculus in financial engineering, see e.g., [1–4]. The results developed in [5] correspond to the background basis for the ones which were published later. Malliavin calculus is an important tool for calculating the conditional expectation to resolve multiple financial engineering problems. For instance, it is was used by Bally et al [2], Abbas and Lapeyre [1] and lastly by Kharrat [4]. Using Malliavin calculus to assess the American option problem, these authors have elaborated formulas for the conditional expectation, under both cases, constant and stochastic volatility.

At any time s where $s \leqslant t$, the value of the American put option is equal to

$$V_s(X_s) = \max\left((K - X_s)^+, \mathrm{e}^{-r(t-s)} E(V_t(X_t)|(X_s))\right) \quad (1)$$

where X_s, X_t are respectively the asset price at times s and t, K is the strike price at the maturity and r is the interest rate. In this study, our contribution resides in elaborating a method, with localization function, in order to compute this conditional expectation $E(V_t(X_t)|(X_s))$ for all $s \leqslant t$ where X_t follows the J-process [6] using the Malliavin calculus. In the paper [7], Jerbi and Kharrat reduced the problem of pricing American option under two stochastic processes into an equivalent stochastic process. They identified a model for pricing American option using Malliavin calculus without considering the effect of the localization function. As an extension of [7], Kharrat [8] developed a new formula for pricing American options generated by the multidimensional J-process. Using the J-process instead of a Brownian motion for the underlying asset process, Jerbi and Kharrat provided as far as the work of Bally et al [2], to be considered with the kurtosis and the skewness effects. The above referred to effects are displayed in the distribution density of J-law, thus, in the J-process. In his study [6], Jerbi has proven that the parameter θ influences the kurtosis and the skewness. As far as our work is concerned, we display the problem and the hypothesis under the multidimensional J-process, we elaborate and compute the Malliavin weights through considering the localization function in order to establish the already mentioned conditional expectation [3, 8].

In the following, we introduce the J-law as well as the J-process [6, 9].

Definition 1 Consider Y a random variable which follows the standard J-law:

$$Y \sim J(\mu, \theta),$$

where its distribution is written in the following form:

$$h(\nu, \mu, \theta) = \frac{1}{\mathrm{Jer}(\mu, \theta)\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\nu^2} N(\mu\nu + \theta), \quad (2)$$

with θ and μ are two constants and $N(\cdot)$ is the cumulative function of the Gaussian distribution, and $\mathrm{Jer}(\mu, \theta)$ is written as:

$$\mathrm{Jer}(\mu, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}\nu^2} N(\mu\nu + \theta)\,\mathrm{d}\nu. \quad (3)$$

When $\mu = 0$, the J-law becomes the Gaussian distribution.

Jerbi defined a new method as an extension of the Brownian motion relying on the J-law [6]. Subsequently, Kharrat rectified Jerbi's definition (see [9]).

Definition 2 Consider $(\Omega, \mathscr{F}, P, (\mathscr{F}_t))$ a filtered probability space. A stochastic process $(X_t)_{t \geqslant 0}$ follows a J-process, if:

- the continuous stochastic process X_t is $\mathscr{F}_t$-adapted,

- for $s < t$, $X_t - X_s$ follows Q_{t-s},

- $\mathrm{d}X_t = m(X_t, t)\,\mathrm{d}t + n(X_t, t)\,\mathrm{d}Q_t$,

where m and n are two functions of X_t and the time t. Q_t is a random variable, which can be indicated as $Q_t = U\sqrt{t}$, where U follows the J-law: $U = (Y - E(Y))/\sigma(Y)$, where $Y \sim J(\mu, \theta)$, $E(Y) = \mu Z(\mu, \theta)$, $\sigma_Y^2 = 1 - \frac{\mu^2\theta}{1+\mu^2} Z(\mu, \theta) - \mu^2 Z^2(\mu, \theta)$, and

$$Z(\mu, \theta) = \frac{\mathrm{e}^{-\frac{\theta^2}{2(1-\mu^2)}}}{\mathrm{Jer}(\mu, \theta)\sqrt{2\pi(1-\mu^2)}}.$$

Remark 1 Proceeding in this way, we express the J-process by the following process:

$$\mathrm{d}X_t = m(X_t, t)\,\mathrm{d}t + n(X_t, t) U \sqrt{\mathrm{d}t}. \tag{4}$$

Definition 3 Consider $(\Omega, \mathscr{F}, V, (\mathscr{F}_t))$ a filtered probability space. A multi-dimensional J-process $X = (X_t)_{t\in[0,\infty)}$ in $\mathbb{R}^m$ is written as follows:

- The continuous stochastic process X is $\mathscr{F}_t$-adapted;

- for $s < t$, $X_t^i - X_s^i$ follows J-law, i.e. follows Q_{t-s}, and is independent of $\mathscr{F}_s$.

Following a multi-dimensional J-process, X is denoted as follows:

$$\mathrm{d}X_t = r \circ X_t\,\mathrm{d}t + \sigma X_t Q_t \tag{5}$$
$$X_0 = x, \tag{6}$$

where $\circ$ denotes the element-wise product, $x \in \mathbb{R}_+^m$, $r \in \mathbb{R}_+^m$, with $r_i = r_0$ for all $i = 1, \dots, m$, and r_0 is the interest rate at t_0 supposed to be constant, σ is the $m \times m$ volatility matrix supposed to be non-degenerate and a sub-triangle matrix, and Q is an m-dimensional J-process.

All components of X_t can be defined, for $i = 1, \dots, m$,

$$X_t^i = x_i \exp\left[t\left(r_i - \frac{1}{2}\sum_{j=1}^{i} \sigma_{ij}^2\right) + \sum_{j=1}^{i} \sigma_{ij} Q_t^j\right]. \tag{7}$$

To assess the American option, we shall compute this conditional expectation

$$E\left(V_t(X_t) | X_s = \alpha\right),$$

where $0 \leqslant s \leqslant t$, $\alpha \in \mathbb{R}_+^m$, and V_t is the American option price at the time t which stands for an $\mathbb{R}^m$ measurable function.

THEORETICAL FRAMEWORK

Let $l_t = (l_t^1, \dots, l_t^m)$ be a fixed C^1 function. In addition, let us specify, for $i = 1, \dots, m$,

$$\widetilde{X}_t^i = x_i \exp\left[t\left(r_i - \frac{1}{2}\sum_{j=1}^{i} \sigma_{ij}^2\right) + l_t^i + \sigma_{ii} Q_t^i\right]. \tag{8}$$

Now, we shall examine the alteration so as to change process $\widetilde{X}$ instead of X.

Proposition 1 *For any time $t \geqslant 0$, there exists a function $P_t(\cdot) : \mathbb{R}_+^m \to \mathbb{R}_+^m$ where P_t is invertible, and*

$$X_t = P_t(\widetilde{X}_t) \tag{9}$$
$$\widetilde{X}_t = P_t^{-1}(X_t). \tag{10}$$

The proof of the proposition is detailed in [8].

Since $\widetilde{\sigma}$ is a triangular matrix, it's easy to determine $\widetilde{\sigma}^{-1}$. Likewise, $\widetilde{\sigma}^{-1}$ is triangular and $(\widetilde{\sigma}^{-1})_{ii} = 1$ for any i. From this perspective, the function P_t and its inverse $G_t = P_t^{-1}$ (thit is why we have $X_t = P_t(\widetilde{X}_t)$ and $\widetilde{X}_t = G_t(X_t)$) are, respectively, expressed by, for $i = 1, \dots, m$ and $y, z \in \mathbb{R}_+^m$,

$$P_t^i(y) = y_i \left(\exp\left(-\sum_{j=1}^{i} \widetilde{\sigma}_{ij} l_t^j\right)\right) \prod_{j=1}^{i-1} \left(\frac{y_i\,\mathrm{e}^{-(r_j - \frac{1}{2}\sum_{j=1}^{i}\sigma_{jj}^2)t}}{x_j}\right)^{\widetilde{\sigma}_{ij}}, \tag{11}$$

and

$$G_t^i(z) = z_i \exp(l_t^i) \prod_{j=1}^{i-1} \left(\frac{z_i e^{-(r_j - \frac{1}{2}\sum_{j=1}^{i}\sigma_{jj}^2)t}}{x_j}\right)^{\widetilde{\sigma}_{ij}^{-1}}. \tag{12}$$

Owing to the fact that all components of the process $\widetilde{X}$ are independent, it is straightforward to obtain the one-dimensional value for the conditional expectation, using the results found in [7].

Now, we can draw and establish the following result.

Theorem 1 *Let $X_t = X_s B$ where $0 \leqslant s \leqslant t$, with $B = \mathrm{e}^{(r-\frac{1}{2}\sigma^2)(t-s)+\sigma(Q_t-Q_s)}$. Let B be independent of X_s and let its distribution function be $\Gamma(\gamma(B))$. Let $g : \mathbb{R} \to \mathbb{R}$ with polynomial growth. Let $\alpha > 0$ be fixed and let $(X_t)_{t\geqslant 0}$ follows the J-process. Let $\Psi \in C_b^1(\mathbb{R})$ such that $\Psi|_{D_\varepsilon(\alpha)=(\alpha-\varepsilon,\alpha+\varepsilon)} = 1$ with $\varepsilon > 0$. For any $\mathbb{R}^m$-measurable function V_t, and for $\alpha \in \mathbb{R}_+^m$, we get:*

$$E\left(V_t(X_t) | X_s = \alpha\right) = \frac{E\left(\Xi_s[V_t](X_t) H(X_s - \alpha)\right)}{E\left(\Xi_s[1](X_t) H(X_s - \alpha)\right)}, \tag{13}$$

where

$$\Xi_s[g](X_t) = \prod_{i=1}^{d} \frac{\sigma_{Y_s}\Psi(X_s) g(X_t)}{\sigma X_s \sqrt{t-s}} \Bigg[Y_s + \frac{\sigma\sqrt{t-s}}{\sigma_{Y_s}} - \mu \frac{N'(\mu Y_s + \theta)}{N(\mu Y_s + \theta)} + \frac{\sigma}{\sigma_{Y_s}} \left[1 + B \frac{\gamma'(B)\Gamma'(\gamma(B))}{\Gamma(\gamma(B))}\right]\Bigg]. \tag{14}$$

Notice that $N(\cdot)$ is the cumulative function distribution of the standard Gaussian, and for all $x \in \mathbb{R}$, $H(x) = 1_{x \geqslant 0}$, with $\widetilde{X}_s = G_s(X_s)$ and $\widetilde{\alpha} = G_s(\alpha)$.

Proof: In $D_\varepsilon(w)$, we have $\varphi' \Psi = \varphi'$. Therefore,

$$E\left(\varphi'(X_s)g(X_t)\right) = E\left(\varphi'(X_s)\Psi(X_s)g(X_t)\right) \tag{15}$$

Let us set $\widetilde{V}_t(y) = V_t \circ P_t(y)$; $y \in \mathbb{R}_+^m$; P_t being defined in (11). Since $X_t P_t(\widetilde{X}_t)$ for any t, then

$$E(V_t(X_t)|X_s = \alpha) = E\left(\widetilde{V}_t(\widetilde{X}_t)|\widetilde{X}_s = G_s(\alpha)\right). \tag{16}$$

Hence, by defining $\widetilde{\alpha} = G_s(\alpha)$, it is sufficient to prove that

$$E\left(\widetilde{V}_t(\widetilde{X}_t)|\widetilde{X}_s = \widetilde{\alpha}\right) = \frac{E\left(\widetilde{\Xi}_s[\widetilde{V}_t](\widetilde{X}_t)H(\widetilde{X}_s - \widetilde{\alpha})\right)}{E\left(\widetilde{\Xi}_s[1](\widetilde{X}_t)H(\widetilde{X}_s - \widetilde{\alpha})\right)}. \tag{17}$$

Let $\widetilde{V}_t(\widetilde{X}_t) = \widetilde{V}_t^{\,1}(\widetilde{X}_t^1)\cdot\widetilde{V}_t^{\,2}(\widetilde{X}_t^2)\cdots\widetilde{V}_t^{\,m}(\widetilde{X}_t^m)$, i.e. $\widetilde{V}_t$ can be represented in terms of the product of m-measurable functions. In such case, we obviously have:

$$E\left(\widetilde{V}_t(\widetilde{X}_t)|\widetilde{X}_s = \widetilde{\alpha}\right) = \prod_{i=1}^{m} E\left(\widetilde{V}_t^i(\widetilde{X}_t^i)|\widetilde{X}_s^i = \widetilde{\alpha}_i\right). \tag{18}$$

At this stage of analysis, it is quite easy to confirm that, for each $\widetilde{X}_t^i$, we can invest the result recorded by Kharrat in [8]. Therefore, the following result is obtained:

$$\begin{aligned} E\left(\widetilde{V}_t(\widetilde{X}_t)|\widetilde{X}_s = \widetilde{\alpha}\right) &= \prod_{i=1}^{m} E\left(\widetilde{V}_t^{\,i}(\widetilde{X}_t^i)|\widetilde{X}_s^i = \widetilde{\alpha}_i\right) \\ &= \prod_{i=1}^{m} \frac{E\left(\Xi_s^i[\widetilde{V}_t^i](\widetilde{X}_t^i)H(\widetilde{X}_s^i - \widetilde{\alpha}^i)\right)}{E\left(\Xi_s^i[1](\widetilde{X}_t^i)H(\widetilde{X}_s^i - \widetilde{\alpha}^i)\right)}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \Xi_s^i[\widetilde{V}_t^i](\widetilde{X}_t^i) = \frac{\Psi(X_s)\sigma_{Y_s}\widetilde{V}_t^i(\widetilde{X}_t^i)}{\sigma X_s\sqrt{t-s}}\Bigg[\frac{\sigma}{\sigma_{Y_s}}\Bigg[B\frac{\Gamma'(\gamma(B))\gamma'(B)}{\Gamma(\gamma(B))}+1\Bigg] \\ + Y_s - \mu\frac{N'(\mu Y_s+\theta)}{N(\mu Y_s+\theta)} + \frac{\sigma\sqrt{t-s}}{\sigma_{Y_s}}\Bigg], \end{aligned} \tag{20}$$

and

$$\begin{aligned} \Xi_s^i[1](\widetilde{X}_t^i) = \frac{\Psi(X_s)\sigma_{Y_s}}{\sigma X_s\sqrt{t-s}}\Bigg[\frac{\sigma}{\sigma_{Y_s}}\Bigg[B\frac{\Gamma'(\gamma(B))\gamma'(B)}{\Gamma(\gamma(B))}+1\Bigg] \\ + Y_s - \mu\frac{N'(\mu Y_s+\theta)}{N(\mu Y_s+\theta)} + \frac{\sigma\sqrt{t-s}}{\sigma_{Y_s}}\Bigg]. \end{aligned} \tag{21}$$

We deduce

$$\widetilde{\Xi}_s[\widetilde{V}_t](\widetilde{X}_t) = \prod_{i=1}^{m} \Xi_s^i[\widetilde{V}_t^i](\widetilde{X}_t^i), \tag{22}$$

and

$$\widetilde{\Xi}_s[1](\widetilde{X}_t) = \prod_{i=1}^{m} \Xi_s^i[1](\widetilde{X}_t^i). \tag{23}$$

□

Remark 2 In the previous theorem, we computed the conditional expectation related to pricing American option with localization function. Therefore, we can deduce:

- when $d = 1$, we obtain exactly the same result established in [10].
- For $\theta = 0$ and $\lambda = 1$, we rely upon the results from [2] for the multidimensional case.

NUMERICAL SIMULATIONS

In this part, as an application of the obtained results, we provide the price of the American put options on the geometric mean of three and five assets using the Monte Carlo simulation with 1000 iterations.

At first, we compute the American put option on the geometric mean of three assets with a payoff equal to $\max\left((K - \prod_{i=1}^{3} X_s^i)^{1/3}, 0\right)$. We assume that the initial values are equal, i.e. $X_0^1 = X_0^2 = X_0^3$, $K = 10$, the volatility is equal to 0.15 and the interest rate $r = 0.05$.

Afterwards, we compute the obtained results for five price assets with a payoff that is equal to $\max\left((K - \prod_{i=1}^{5} X_s^i)^{1/5}, 0\right)$.

In Table 1, the numerical results are displayed. Our obtained results are compared to the binomial model with 1000 steps, which will considered as a âĂŸâĂŸtrue" reference price, as well as the Malliavin calculus without localization function which are obtained in [8]. All results go in good accordance with the American option's theory.

Table 1 Pricing American put option with localization function for three and five price assets, respectively, compared to the classical binomial model (1000 time-steps), and the price without localization function ($K = 10$, $\sigma = 0.15$, $r = 0.05$, and $T = 1$).

	Three assets	Five assets
Binomial 1000	0.625	0.297
Without localization function	0.869	0.451
With localization function	0.673	0.329

CONCLUSION

In this study, we extended the results of [2] through considering the skewness and the kurtosis effects for pricing American options. Additionally, we built upon Kharrat's results in [8] taking into consideration the localization function. Eventually, we set forward an application of the obtained results respectively for three and five assets, which go in good agrement with the theory. As future perspectives, we will compute and investigate the Greeks of the American options in different cases (for the one- and multi-dimensional cases, under both cases with and without localization function).

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