RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2022.091

ScienceAsia 48 (2022): **681–686**

Fermat type functional equations, several complex variables and Euler operator

Liu Yang[a,*], **Li-Xiong Shi**[b], **Sheng-Yao Zhou**[a]

[a] School of Mathematics and Physics, Anhui University of Technology, Maanshan 243032 China
[b] College of Science, Yunnan Agricultural University, Kunming 650201 China

*Corresponding author, e-mail: yangliu6@ahut.edu.cn

Received 12 Dec 2020, Accepted 7 May 2022
Available online 6 Jul 2022

ABSTRACT: We describe the entire solutions for Fermat type functional equations with functional coefficients in $\mathbb{C}^n$, i.e., $hf^p + kg^q = 1$, where $p, q \geqslant 2$ are two integers. We then apply the result to obtain that entire function solutions f, g of $f^2 + g^2 = 1$ in $\mathbb{C}^n$ are constant if $Df^{-1}(0) \subseteq Dg^{-1}(0)$ with ignoring multiplicities, where $D := \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ is the Euler operator. Meromorphic function solutions of $f^3 + g^3 = 1$ in $\mathbb{C}^n$ and applications to nonlinear (ordinary and partial) differential equations are also discussed.

KEYWORDS: entire function, meromorphic function, functional equation, Euler operator

MSC2020: 32A20 35F20 32A22

INTRODUCTION

The well-known "Fermat's Last Theorem", which was proved by Wiles [1] and Taylor and Wiles [2], states that there do not exist non-zero rational numbers x and y and an integer $m > 2$, such that $x^m + y^m = 1$. Analogous to this result of number theory, there have been similar function theory investigations. For example, it has been determined for which positive integers m, the Fermat type functional equation

$$f^m + g^m = 1 \tag{1}$$

has non-constant entire or meromorphic solutions f, g. It is known that Eq. (1) does not admit nonconstant global entire solutions in the complex plane when $m \geqslant 3$ and does not admit nonconstant meromorphic solutions when $m \geqslant 4$. But it does admit entire solutions when $m = 2$ and admit nonconstant meromorphic solutions when $m = 3$. For $m = 2$, Eq. (1) obviously has non-constant entire solutions $f = \sin h$, $g = \cos h$, where h is a non-constant entire function. For $m = 3$, Eq. (1) obviously has non-constant meromorphic solutions

$$\begin{aligned} f(z) &= \Big\{\frac{1}{2} + \frac{\wp'(z)}{\sqrt{12}}\Big\}/\wp(z), \\ g(z) &= \Big\{\frac{1}{2} - \frac{\wp'(z)}{\sqrt{12}}\Big\}/\wp(z), \end{aligned} \tag{2}$$

where $\wp$ is the Weierstrass elliptic function satisfying $(\wp')^2 = 4\wp - 1$ after appropriately choosing its periods (see [3–6] and references therein).

On the other hand, studies of the functional equations in several complex variables is natural. Throughout this paper we use the basic results and notation of Nevanlinna theory, such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, etc. (cf. [7, 8]). In 2005, Li [9] proved the following characterization for entire solutions of functional equations in $\mathbb{C}^2$ with functional coefficients in $\mathbb{C}$.

Theorem A *Let h and k be two non-zero meromorphic functions in $\mathbb{C}$, and $p, q \geqslant 2$ two integers. Then any entire solutions f and g of the functional equation*

$$h(z_1)\, f^p \; + \; k(z_2)\, g^q = 1$$

in $\mathbb{C}^2$ must satisfy that

$$T(r, f) + T(r, g) = O\{T(r, h) + T(r, k)\}$$

outside a set of r of finite Lebesgue measure, provided that f_{z_2} and g_{z_1} have the same zeros (counting multiplicities).

In the same paper [9], as an application of Theorem A, the author determined when entire solutions of the functional equation $f^2 + g^2 = 1$ in $\mathbb{C}^2$, or equivalently, holomorphic maps (f, g) to the surface $x^2 + y^2 = 1$, reduce to constant.

Theorem B *Entire function solutions f, g of $f^2 + g^2 = 1$ in $\mathbb{C}^2$ are constant if and only if f_{z_2} and g_{z_1} have the same zeros (counting multiplicities).*

Later, Li and Ye [10] considered the similar problem for the functional equation $f^3 + g^3 = 1$ in $\mathbb{C}^2$ with different method.

Theorem C *Meromorphic function solutions f, g of $f^3 + g^3 = 1$ in $\mathbb{C}^2$ are constant if and only if f_{z_2} and g_{z_1} have the same zeros (counting multiplicities).*

Recently, Lü and Li [11] strengthened Theorem C by improving the condition so that f_{z_2} and g_{z_1} have the same zeros with ignoring multiplicities.

Theorem D *Meromorphic function solutions f,g of $f^3+g^3=1$ in $\mathbb{C}^2$ are constant if and only if f_{z_2} and g_{z_1} have the same zeros (ignoring multiplicities).*

More recently, Li and Lü [12] improved the functional coefficients $h(z_1)$ and $k(z_2)$ to meromorphic functions $h(z_1,z_2)$ and $k(z_1,z_2)$ in $\mathbb{C}^2$.

Theorem E *Let h and k be two non-zero meromorphic functions in $\mathbb{C}^2$, and f and g be non-zero entire solutions of the functional equation*

$$h(z_1,z_2)f^2+k(z_1,z_2)g^2=1$$

in $\mathbb{C}^2$. Suppose that a zero of $\left(hf^2\right)_{z_2}/(hf)$ is also a zero of $\left(kg^2\right)_{z_1}/(kg)$ (ignoring multiplicities) and $\left(kg^2\right)_{z_1}\not\equiv 0$. Then f,g must be constant if h,k are constant and f,g must be polynomials if h,k are rational functions.

We point out that the conclusion in Theorem A holds also for Theorem E (see [12]). In addition, Theorem E immediately yields Theorem B with the condition "counting multiplicities " removed.

Theorem F *Entire function solutions f,g of $f^2+g^2=1$ in $\mathbb{C}^2$ are constant if and only if f_{z_2} and g_{z_1} have the same zeros (ignoring multiplicities).*

Furthermore, Li [13] studied the functional equation $f^2+g^2=1$ in $\mathbb{C}$ and derived the following conclusion.

Theorem G *Entire function solutions f,g of $f^2+g^2=1$ in $\mathbb{C}$ are constant if and only if $(f')^{-1}(0)\subseteq(g')^{-1}(0)$ (ignoring multiplicities).*

MAIN RESULTS

As we saw, all of the results mentioned in previous section are related to Fermat type functional equations in $\mathbb{C}$ or $\mathbb{C}^2$. It is natural to ask the following question: *how to describe entire/meromorphic solutions for the above Fermat type functional equations in $\mathbb{C}^n$?* In this paper, we first describe entire solutions for the Fermat type functional equations with functional coefficients $hf^p+kg^q=1$ in $\mathbb{C}^n$. Our basic tool is the Euler operator on meromorphic functions of several complex variables. We refer to [14] for some applications of Euler operator.

Definition 1 Let f be a meromorphic function on $\mathbb{C}^n$, the Euler operator on f is defined by

$$Df(z)=\sum_{j=1}^{n}z_jf_{z_j}(z), \tag{3}$$

where $z=(z_1,z_2,\dots,z_n)\in\mathbb{C}^n$, and f_{z_j} is the partial derivative of f with respect to z_j $(j=1,2,\dots,n)$. For any positive integer k, the k-th order total derivative D^kf of f is defined inductively by $D^{k+1}f=D(D^kf)$.

We note that the Euler operator on f is also called the total derivatives (see [15] and [16]) and the radial derivative of f (see for example [17] and [18]).

Our first result is stated as follows.

Theorem 1 *Let h and k be two nonzero meromorphic functions in $\mathbb{C}^n$, and $p,q\geqslant 2$ two integers. If a zero of $D\left(hf^p\right)/\left(hf^{p-1}\right)$ is also a zero of $D\left(kg^q\right)/\left(kg^{q-1}\right)$ (ignoring multiplicities) and $D\left(kg^q\right)\not\equiv 0$, then any entire solutions f and g of the functional equation*

$$hf^p+kg^q=1 \tag{4}$$

in $\mathbb{C}^n$ must satisfy that

$$T(r,f)+T(r,g)=O\{T(r,h)+T(r,k)+\log r\}$$

outside a set of r of finite Lebesgue measure.

In order to prove Theorem 1, we require the logarithmic derivative lemma concerning Euler operator. Since the Euler operator is a linear combination of partial derivatives with polynomial coefficients, the following lemma is a direct consequence of the well-known logarithmic derivative lemma in several complex variables due to Vitter [19].

Lemma 1 ([20], Lemma 2.2) *Let f be a transcendental meromorphic function in $\mathbb{C}^n$. Then for any positive integer k,*

$$m\left(r,\tfrac{D^kf}{f}\right)=O\{\log rT(r,f)\}.$$

holds for all $r>0$ outside a set E with finite Lebesgue measure.

Inspired by some ideas used in [9, 11, 12], we next give the proof of our main results, i.e., Theorem 1.

Proof: Suppose that f and g are entire solutions of the given functional equation (4) in Theorem 1. Let $F=hf^p$ and $G=kg^q$. Then

$$F+G=1. \tag{5}$$

Taking total derivatives on both sides of (5) we have

$$DF+DG=0. \tag{6}$$

It is easily seen that

$$DF=f^{p-1}F^*,\quad DG=g^{q-1}G^*, \tag{7}$$

where $F^*=(Dh)f+ph(Df)$ and $G^*=(Dk)g+qk(Dg)$. Set

$$\psi=\frac{(F^*)^p(G^*)^q}{FG}. \tag{8}$$

Thus (7) gives

$$\begin{aligned}\psi&=\frac{(DF)^p(DG)^q}{f^{p(p-1)}g^{q(q-1)}FG}\\&=h^{p-1}k^{q-1}\Big(\frac{DF}{F}\Big)^p\Big(\frac{DG}{G}\Big)^q\end{aligned} \tag{9}$$

If $\psi \equiv 0$, it follows from (9) that DF or DG must be identically zero. Combing this fact with (6) shows $DF \equiv 0$ and $DG \equiv 0$, which is a contradiction with the assumption that $D(kg^q) \not\equiv 0$. Therefore, we may now assume that $\psi \not\equiv 0$.

Note that, for any rational function R in $\mathbb{C}^n$, we have $m(r,R) = O(\log r)$. Hence, by (9) and Lemma 1, we deduce

$$\begin{aligned} m(r,\psi) &\leqslant (p-1)T(r,h)+(q-1)T(r,k) \\ &\quad + O\{\log r\, T(r,f)T(r,g)\} \\ &= O\{T(r,h)+T(r,k)\}+O\{\log r\, T(r,f)T(r,g)\}. \quad (10) \end{aligned}$$

Claim 1 *The function ψ is holomorphic at each zero of f or g which is not a zero or pole of h,k.*

To see this, suppose that z_0 in $\mathbb{C}^n$ such that $f(z_0) = 0$ with $\mathrm{div}_f(z_0) = \tau$ and $h(z_0), k(z_0) \neq 0, \infty$. Here, we will use $\mathrm{div}_f(z_0)$ to denote the multiplicity of a zero z_0 of f, which is the degree of the first homogeneous polynomial in the Taylor series expansion of f at z_0. Then, we have $\mathrm{div}_F(z_0) = p\tau$. It follows from (6) that $\mathrm{div}_{DF}(z_0) = \mathrm{div}_{DG}(z_0) \geqslant p\tau - 1$. Note that $g(z_0) \neq 0$, we get $\mathrm{div}_{G^*}(z_0) \geqslant p\tau - 1$ and $\mathrm{div}_{F^*}(z_0) \geqslant p\tau - 1 - (p-1)\tau = \tau - 1$ in view of (7). Hence, the above discussion, together with (8), yields that

$$\begin{aligned} \mathrm{div}_\psi(z_0) &\geqslant p\,\mathrm{div}_{F^*}(z_0) + q\,\mathrm{div}_{G^*}(z_0) - \mathrm{div}_F(z_0) \\ &\geqslant p(\tau-1) + q(p\tau-1) - p\tau \\ &= pq\tau - p - q \\ &\geqslant pq - p - q \\ &\geqslant 2\max(p,q) - p - q \\ &\geqslant 0. \end{aligned}$$

This means that the function ψ is holomorphic at z_0. By symmetry, ψ is also holomorphic at each zero of g which is not a zero or pole of h,k. Thus, Claim 1 is proved.

Therefore, we have by Claim 1.

$$\begin{aligned} N(r,\psi) &= O\left\{N(r,h) + N\left(r,\tfrac{1}{h}\right) + N(r,k) + N\left(r,\tfrac{1}{k}\right)\right\} \\ &= O\{T(r,h) + T(r,k)\}. \quad (11) \end{aligned}$$

By combining with (10) and (11), it yields

$$T(r,\psi) = O\{T(r,h)+T(r,k)+\log r\, T(r,f)T(r,g)\}. \quad (12)$$

On the other hand, we deduce from (9) that

$$\frac{\psi}{G} = h^{p-1}k^{q-1}\left(\frac{DF}{F}\right)^{p-1}\left(\frac{DG}{G}\right)^{q}\left(\frac{DF}{F-1} - \frac{DF}{F}\right)$$

This, together with Lemma 1 and (12), implies that

$$\begin{aligned} m\left(r,\tfrac{1}{g}\right) &= \frac{1}{q} m\left(r,\tfrac{1}{g^q}\right) \\ &\leqslant \frac{1}{q}\left(m\left(r,\tfrac{1}{G}\right) + m(r,k)\right) \\ &\leqslant \frac{1}{q}\left(m\left(r,\tfrac{\psi}{G}\right) + m\left(r,\tfrac{1}{\psi}\right) + m(r,k)\right) \\ &\leqslant \frac{1}{q}\left(m\left(r,\tfrac{\psi}{G}\right) + T(r,\psi) + T(r,k)\right) \\ &= O\{T(r,h)+T(r,k)+\log r\, T(r,f)T(r,g)\}. \quad (13) \end{aligned}$$

Claim 2 *The function ψ vanishes at each zero of g which is not a zero or pole of h,k with a higher multiplicity.*

Suppose that w_0 in $\mathbb{C}^n$ such that $g(w_0) = 0$ with $\mathrm{div}_g(w_0) = \upsilon$ and $h(w_0), k(w_0) \neq 0, \infty$. Use the same argument as in the proof of Claim 1, we have

$$\begin{aligned} \mathrm{div}_\psi(w_0) &\geqslant p\,\mathrm{div}_{F^*}(w_0) + q\,\mathrm{div}_{G^*}(w_0) - \mathrm{div}_G(w_0) \\ &\geqslant p(q\upsilon - 1) + q(\upsilon - 1) - q\upsilon \\ &= pq\upsilon - p - q. \end{aligned}$$

Since $p,q \geqslant 2$, $\frac{p+q}{pq-1} \leqslant 2$. Hence we have that $\upsilon \geqslant \frac{p+q}{pq-1}$, provide $\upsilon \geqslant 2$. Then $pq\upsilon - p - q \geqslant \upsilon$ holds for $\upsilon \geqslant 2$. What is left is to consider the case of $\upsilon = 1$. By (8), we have $DG(w_0) = 0$. Then it follows from (7) that $DF(w_0) = 0$. Note the hypotheses that a zero of $D(hf^p)/\left(hf^{p-1}\right)$ is also a zero of $D(kg^q)/\left(kg^{q-1}\right)$, we have $D(kg^q)/\left(kg^{q-1}\right)(w_0) = 0$. So (8) gives $G^*(w_0) = 0$. By the above discussion, we have

$$\begin{aligned} \mathrm{div}_\psi(w_0) &\geqslant p\,\mathrm{div}_{F^*}(w_0) + q\,\mathrm{div}_{G^*}(w_0) - \mathrm{div}_G(w_0) \\ &\geqslant p + q - q = p > \upsilon. \end{aligned}$$

Thus, we always have $\mathrm{div}_\psi(w_0) \geqslant \mathrm{div}_g(w_0)$ for each positive integer number υ. This completes the proof of Claim 2.

From Claim 2 and (12), we have that

$$\begin{aligned} &N\left(r,\tfrac{1}{g}\right) \\ &\leqslant N\left(r,\tfrac{1}{\psi}\right) + O\left\{N(r,h) + N\left(r,\tfrac{1}{h}\right) + N(r,k) + N\left(r,\tfrac{1}{k}\right)\right\} \\ &\leqslant T(r,\psi) + O\{T(r,h) + T(r,k)\} \\ &= O\{T(r,h) + T(r,k) + \log r\, T(r,f)T(r,g)\}. \quad (14) \end{aligned}$$

Using Nevanlinna first fundamental theorem, (13) and (14), we deduce that

$$\begin{aligned} T(r,g) &= T\left(r,\tfrac{1}{g}\right) + O(1) \\ &= O\{T(r,h) + T(r,k) + \log r\, T(r,f)T(r,g)\}. \end{aligned}$$

This, together with the hypotheses that $hf^p + kg^q = 1$, implies $T(r,g) = O\{T(r,h)+T(r,k)+\log r\}$ and $T(r,f) = O\{T(r,h)+T(r,k)+\log r\}$. Hence, we have $T(r,f)+T(r,g) = O\{T(r,h)+T(r,k)+\log r\}$. This completes the proof of the theorem. □

Under the conditions of Theorem 1, the following holds.

Corollary 1 *Any entire solutions f and g of (4) must be polynomials, provided that h and k are rational functions.*

Proof: Note that, for any rational function R in $\mathbb{C}^n$, we have $T(r,R) = O(\log r)$. Hence, by Theorem 1, we have $T(r,f)+T(r,g) = O\{T(r,h)+T(r,k)+\log r\} = O(\log r)$. Therefore, we have f and g are polynomials. □

APPLICATIONS

As applications of our main result, we shall describe entire/meromorphic solutions in $\mathbb{C}^n$ to functional equations of the form $f^2+g^2=1$, the form $f^3+g^3=1$, and nonlinear partial differential equations $f^2+\varphi^2\left(z_1f_{z_1}+\cdots+z_nf_{z_n}\right)^{2p}=1$, $f^3+\varphi^3\left(z_1f_{z_1}+\cdots+z_nf_{z_n}\right)^{3q}=1$, where φ is an arbitrary entire function in $\mathbb{C}^n$ and $p,q\geqslant 2$ are integers.

Entire function solutions of $f^2+g^2=1$ in $\mathbb{C}^n$

We shall apply Theorem 1 to obtain the condition such that entire function solutions f,g of $f^2+g^2=1$ in $\mathbb{C}^n$ are constant.

Theorem 2 *Entire solutions f,g of $f^2+g^2=1$ in $\mathbb{C}^n$ are constant if and only if $Df^{-1}(0)\subseteq Dg^{-1}(0)$ (ignoring multiplicities).*

Proof: First,we have by Theorem 1 that $T(r,f)+T(r,g) = O(\log r)$ which means that f and g are polynomials. Suppose, to the contrary, that f and g are non-constant polynomials in $\mathbb{C}^n$. Set

$$f = P_0+P_1+\cdots+P_k, \quad g = Q_0+Q_1+\cdots+Q_l,$$

where P_j, Q_j are either identically zero or homogeneous polynomials of degree j. If $f=P_0$ or $g=Q_0$, we immediately have the entire solutions f,g of $f^2+g^2=1$ in $\mathbb{C}^n$ are constant. This is a contradiction. Since $f^2+g^2=1$, we have $k=l$. If we write

$$R_f = P_1+\cdots+P_k, \quad R_g = Q_1+\cdots+Q_l,$$

then $(P_0+R_f)^2+(Q_0+R_g)^2=1$ which gives

$$P_0^2+Q_0^2=1, \tag{15}$$

$$P_0R_f+Q_0R_g=0, \tag{16}$$

and

$$R_f^2+R_g^2=0. \tag{17}$$

Via (17), we have $R_f=\pm iR_g$. Then, by (16), one has $P_0/Q_0=-R_f/R_g=\mp i$. So $P_0^2+Q_0^2=0$ which contradicts (15). Thus we finish the proof. □

If we take $n=1$ in Theorem 2, we are able to obtain Theorem G directly. To compare with Theorem B and Theorem F, we give the following result which is the particular case of Theorem 2 with $n=2$.

Corollary 2 *Entire solutions f,g of $f^2+g^2=1$ in $\mathbb{C}^2$ are constant if and only if $z_1f_{z_1}+z_2f_{z_2}=0$ implies $z_1g_{z_1}+z_2g_{z_2}=0$ (ignoring multiplicities).*

Meromorphic function solutions of $f^3+g^3=1$ in $\mathbb{C}^n$

In view of Theorem 2, it is natural to ask whether this theorem is valid for meromorphic solutions of $f^3+g^3=1$ in $\mathbb{C}^n$. The following example shows that the answer is negative. Let $f(z)$ and $g(z)$ be the same as in (2). Then $f(z_1/z_2)$ and $g(z_1/z_2)$ are meromorphic solutions of $f^3+g^3=1$ in $\mathbb{C}^2$. Since $Df\equiv 0$ and $Dg\equiv 0$, the condition $Df^{-1}(0)\subseteq Dg^{-1}(0)$ is satisfied. But $f(z_1/z_2)$ and $g(z_1/z_2)$ are not constant. For meromorphic solutions of $f^3+g^3=1$ in $\mathbb{C}^n$, we have the following result.

Theorem 3 *Let f and g be nonzero meromorphic solutions of $f^3+g^3=1$ in $\mathbb{C}^n$. If $Df^{-1}(0)\subseteq Dg^{-1}(0)$ (ignoring multiplicities), then $Df\equiv 0$ and $Dg\equiv 0$.*

Proof: Suppose that f and g are meromorphic solutions of the given functional equation in Theorem 3. That is

$$f^3+g^3=1 \tag{18}$$

in $\mathbb{C}^n$. Next we discuss two cases.

Case 1: f is a rational function in $\mathbb{C}^n$. Then g is a rational function by (18) and we can assume that $f=P/H$, $g=Q/H$, where P,Q,H are three polynomials in $\mathbb{C}^n$. In view of [3, Theorem 4.1(b)], we have that P,Q,H are three constant numbers. Then, f and g are constant.

Case 2: f is a transcendental meromorphic function in $\mathbb{C}^n$. Taking total derivative in (18), we obtain that

$$f^2Df+g^2Dg=0. \tag{19}$$

Suppose that $Df\not\equiv 0$, by (19), $Dg\not\equiv 0$.

Let $\omega_j=\mathrm{e}^{\frac{2\pi j}{3}\mathrm{i}}$, $j=0,1,2$, then ω_1, ω_2 and ω_3 are the three complex unitary roots of the equation $z^3=1$. Suppose that $f(z_0)=\omega_j$. Then (18) yields $g(z_0)=0$. It follows from (19) that $Df(z_0)=0$. Further, by the hypothesis of the Theorem 3, we have that $Dg(z_0)=0$ and $\mathrm{div}_{Df}(z_0)\geqslant 3$. Then, for any $z\in\mathbb{C}^n$

$$\mathrm{div}_{Df}(z)\geqslant 3\sum_{j=1}^{3}\min\{\mathrm{div}_{f-\omega_j}(z),1\},$$

which implies

$$N\left(r,\frac{1}{Df}\right)\geqslant 3\sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{f-\omega_j}\right). \tag{20}$$

Furthermore, by Nevanlinna second fundamental theorem and Lemma 1, we deduce that

$$\begin{aligned}
T(r,f) &\leqslant \sum_{j=1}^{3} \overline{N}\left(r, \tfrac{1}{f-\omega_j}\right) + O(\log rT(r,f)) \\
&\leqslant \tfrac{1}{3} N\left(r, \tfrac{1}{Df}\right) + O(\log rT(r,f)) \\
&\leqslant \tfrac{1}{3} T(r, Df) + O(\log rT(r,f)) \\
&= \tfrac{1}{3}\{m(r,Df) + N(r,Df)\} + O(\log rT(r,f)) \\
&\leqslant \tfrac{1}{3}\left\{m\left(r, \tfrac{Df}{f}\right) + m(r,f) + 2N(r,f)\right\} + O(\log rT(r,f)) \\
&\leqslant \tfrac{1}{3}\{m(r,f) + 2N(r,f)\} + O(\log rT(r,f)) \\
&\leqslant \tfrac{2}{3} T(r,f) + O(\log rT(r,f)).
\end{aligned}$$

Hence, $T(r,f) = O(\log rT(r,f))$, a contradiction. Thus, $Df \equiv 0$, and by (19), $Dg \equiv 0$. This completes the proof of Theorem 3. □

In the particular case of $n = 1$, by Theorem 3, we have

Corollary 3 *Meromorphic solutions f, g of $f^3 + g^3 = 1$ in $\mathbb{C}$ are constant if and only if $(f')^{-1}(0) \subseteq (g')^{-1}(0)$ (ignoring multiplicities).*

Nonlinear differential equations in $\mathbb{C}^n$

Using our results above, we shall character complex analytic solutions of some nonlinear (ordinary and partial) differential equations in $\mathbb{C}^n$.

Theorem 4

(i) *Entire solutions of*

$$f^2 + \varphi^2\left(z_1 f_{z_1} + \cdots + z_n f_{z_n}\right)^{2p} = 1 \tag{21}$$

in $\mathbb{C}^n$ are exactly $f = \pm 1$, where φ is an arbitrary entire function in $\mathbb{C}^n$ and $p \geqslant 2$ is an integer.

(ii) *Meromorphic solutions of*

$$f^3 + \varphi^3\left(z_1 f_{z_1} + \cdots + z_n f_{z_n}\right)^{3q} = 1 \tag{22}$$

in $\mathbb{C}^n$ are exactly $f = \mathrm{e}^{\frac{2k\pi}{3}\mathrm{i}}$, $k = 0, 1, 2$, where φ is an arbitrary entire function in $\mathbb{C}^n$ and $q \geqslant 2$ is an integer.

Proof: (i): If we set $g = \varphi\left(z_1 f_{z_1} + \cdots + z_n f_{z_n}\right)^p = \varphi(Df)^p$, then $Dg = \left(D\varphi Df + p\varphi D^2 f\right)(Df)^{p-1}$. Obviously, Theorem 2 implies f is constant and the conclusion is valid.

(ii): Similarly, if we put $g = \varphi(Df)^q$, then $Dg = \left(D\varphi Df + q\varphi D^2 f\right)(Df)^{q-1}$. Thus, we get $Df \equiv 0$ by Theorem 3, which implies from (22) that $f^3 \equiv 1$. This completes the proof. □

In the particular case $n = 1$, Theorem 4 (i) was obtained in [13, Corollary 3.1] and Theorem 4 (ii) implies the following corollary.

Corollary 4 *Meromorphic solutions of $f^3 + \varphi^3(f')^{3q} = 1$ in $\mathbb{C}$ are exactly $f = \mathrm{e}^{\frac{2k\pi}{3}\mathrm{i}}$, $k = 0, 1, 2$, where φ is an arbitrary entire function in $\mathbb{C}$ and $q \geqslant 2$ is an integer.*

In the above corollary, the function φ cannot be assumed to be a meromorphic function with $q = 1$. Here is a counterexample. Let $f(z)$ be the same as in (2) and $\varphi(z) = \left\{\frac{\wp(z)}{2} - \frac{\wp(z)\wp'(z)}{\sqrt{12}}\right\} / \left\{\frac{\wp(z)\wp''(z)}{\sqrt{12}} - \frac{\wp'(z)}{2} - \frac{(\wp'(z))^2}{\sqrt{12}}\right\}$. Then, by (2), we have $f^3 + \varphi^3(f')^3 = 1$. But it is clear that f is a transcendental meromorphic solution of the equation.

Acknowledgements: The first author would like to thank Bao-Qin Li for valuable discussions. This research was supported by NSF of China (No. 11701006), and by Natural Science Foundation of Anhui Province (No. 1808085QA02), China.

REFERENCES

1. Wiles A (1995) Modular elliptic curves and Fermat's Last Theorem. *Ann Math* **141**, 443–551.
2. Taylor R, Wiles A (1995) Ring-theoretic properties of certain Hecke algebras. *Ann Math* **141**, 553–572.
3. Gundersen GG, Hayman WK (2004) The strength of Cartans version of Nevanlinna theory. *Bull Lond Math Soc* **36**, 433–454.
4. Gross F (1966) On the equation $f^n + g^n = 1$. I. *Bull Amer Math Soc* **72**, 86–88.
5. Gross F (1968) On the equation $f^n + g^n = 1$. II. *Bull Amer Math Soc* **74**, 647–648.
6. Huang ZB, Lan ST, Zhang RR (2021) Meromorphic solutions of certain types of complex functional equations. *ScienceAsia* **47**, 520–529.
7. Steinmetz N (2017) *Nevanlinna Theory, Normal Families, and Algebraic Differential Equations*, Springer-Verlag, New York.
8. Yang L (1993) *Value Distribution Theory*, Springer-Verlag, Berlin.
9. Li BQ (2005) On certain functional and partial differential equations. *Forum Math* **17**, 77–86.
10. Li BQ, Ye Z (2008) On meromorphic solutions of $f^3 + g^3 = 1$. *Arch Math* **90**, 39–43.
11. Lü F, Li Z (2019) Meromorphic solutions of Fermat type partial differential equations. *J Math Anal Appl* **478**, 864–873.
12. Li BQ, Lü F (2020) On entire solutions of a Pythagorean functional equation and associated PDEs. *Complex Anal Oper Theory* **14**, 50.
13. Li BQ (2018) An equivalent form of Picard's theorem and beyond. *Can Math Bull* **61**, 142–148.
14. Henson CW, Rubel LA (1984) Some applications of Nevanlinna theory to mathematical logic: identities of exponential functions. *Trans Am Math Soc* **282**, 1–32.
15. Jin L (2003) Theorem of Picard type for entire functions of several complex variables. *Kodai Math J* **26**, 221–229.
16. Xu L, Cao TB (2020) A uniqueness theorem for meromorphic functions concerning total derivatives in several complex variables. *Bull Malays Math Sci Soc* **43**, 3923–3940.
17. Rudin W (1980) *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer, New York.

18. Zhu KH (2005) *Spaces of Holomorphic Functions in the Unit Ball*, GTM, vol 226, Springer, New York.
19. Vitter A (1977) The lemma of the logarithmic derivative in several complex variables. *Duke Math J* **44**, 89–104.
20. Lü F (2013) Theorems of Picard type for meromorphic function of several complex variables. *Complex Var Elliptic Equ* **58**, 1085–1092.
21. Chen W, Han Q, Liu JB (2019) On Fermat Diophantine functional equations, little Picard theorem and beyond. *Aequationes Math* **93**, 425–432.
22. Li BQ (2008) On meromorphic solutions of $f^2+g^2=1$. *Math Z* **258**, 763–771.