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Unicity of meromorphic functions concerning small functions and differences

Xiaohuang Huang, Mingliang Fang*

Department of Mathematics, Hangzhou Dianzi University, Hangzhou 310012 China

*Corresponding author, e-mail: mlfang@scau.edu.cn

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ABSTRACT: In this paper, we study the unicity of meromorphic functions concerning their small functions and differences. Our results improve and extend the existed results of Chen-Chen [Bull Malays Math Sci Soc **35** (2012):765–774] and Qi-Li-Yang [Comput Methods Funct Theory **18** (2018):567–582].

KEYWORDS: unicity, meromorphic functions, shifts, derivatives

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INTRODUCTION AND MAIN RESULTS

In this paper, we assume that the readers are familiar with the basic notions of Nevanlinna value distribution theory, see ([1–3]). In the following, a meromorphic function means meromorphic in the whole complex plane. By $S(r,f)$, we denote any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of an exceptional set E with finite logarithmic measure $\int_E \mathrm{d}r/r<\infty$. A meromorphic function a is said to be a small function of f if it satisfies $T(r,a)=o(T(r,f))$. We say that two non-constant meromorphic functions f and g share small function a IM(CM) if $f-a$ and $g-a$ have the same zeros ignoring multiplicities (counting multiplicities). Let f be a non-constant meromorphic function. We denote by $N_1(r,1/f)$ the counting function of simple zeros of f.

Let f be a non-constant meromorphic function. The order of f is defined by

$$\lambda=\overline{\lim_{r\to\infty}}\frac{\log^+ T(r,f)}{\log r}.$$

Let a be a small function of f and g and let $S(f=a=g)$ be the set of all common zeros of $f-a$ and $g-a$ counting multiplicities. We say that two non-constant meromorphic functions f and g share small function a CM almost if

$$N\Big(r,\tfrac{1}{f-a}\Big)+N\Big(r,\tfrac{1}{g-a}\Big)-2N(r,f=a=g)\\ =S(r,f)+S(r,g).$$

Let c be a nonzero finite complex constant, and let f be a meromorphic function, we define its shift by $f(z+c)$ and its difference operator by

$$\Delta_c f(z)=f(z+c)-f(z).$$

Nevanlinna [2] proved the following famous five-value theorem.

Theorem A *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let a_j ($j=1,2,3,4,5$) be five distinct values in the extended complex plane. If $f(z)$ and $g(z)$ share a_j ($j=1,2,3,4,5$) IM, then $f(z)\equiv g(z)$.*

In 2000, Li and Qiao [4] proved that Theorem A is still valid for five small functions, they proved

Theorem B *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a_j(z)$ ($j=1,2,3,4,5$) (one of them can be ∞) be five distinct small functions of $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a_j(z)$ ($j=1,2,3,4,5$) IM, then $f(z)\equiv g(z)$.*

Recently, value distribution in difference analogue of meromorphic functions has become a subject of some interests, see [5–18].

In 2012, Chen and Chen [5] proved

Theorem C *Let $f(z)$ be a non-constant meromorphic function of finite order, let a,c be two nonzero finite values, and let $n\geqslant 7$ be a positive integer. If $[f(z)]^n$ and $[\Delta f(z)]^n$ share a CM, $f(z)$ and $\Delta f(z)$ share ∞ CM, then $f(z)\equiv t\Delta f(z)$, where $t^n=1, t\neq 1$.*

In 2018, Qi, Li and Yang [15] proved

Theorem D *Let $f(z)$ be a non-constant meromorphic function of finite order, let a,c be two nonzero finite values, and let $n\geqslant 9$ be a positive integer. If $[f'(z)]^n$ and $[f(z+c)]^n$ share a CM, $f'(z)$ and $f(z+c)$ share ∞ CM, then $f'(z)\equiv tf(z+c)$, where $t^n=1$.*

Theorem E *Let $f(z)$ be a non-constant entire function of finite order, let a,c be two nonzero finite values, and let $n\geqslant 5$ be a positive integer. If $[f'(z)]^n$ and $[f(z+c)]^n$ share a CM, $f'(z)$ and $f(z+c)$ share ∞ CM, then $f'(z)\equiv tf(z+c)$, where $t^n=1$.*

In 2020, Wang and Fang [16] removed the condition that the function $f(z)$ is of finite order in Theorems D and E, and proved

Theorem F *Let $f(z)$ be a non-constant meromorphic function, let a, c be two nonzero finite values, and let $n \geqslant 5$, k be two positive integers. If $[f^{(k)}(z)]^n$ and $[f(z+c)]^n$ share a CM, $f^{(k)}(z)$ and $f(z+c)$ share ∞ CM, then $f^{(k)}(z) \equiv t f(z+c)$, where $t^n = 1$.*

By above theorems, we naturally pose following problem:

Problem 1 *Are Theorem C, Theorem D, and Theorem F still valid if the constant a is replaced by a small function $a(z)$ of $f(z)$?*

In this paper, we study the problem and obtain the following results.

Theorem 1 *Let $f(z)$ be a non-constant meromorphic function, let c be a nonzero finite value and $n \geqslant 6$ a positive integer, and let $a(z)(\not\equiv 0)$ be a small function of $f(z)$. If $[f(z)]^n$ and $[\Delta_c f(z)]^n$ share $a(z)$ CM, $f(z)$ and $\Delta_c f(z)$ share ∞ CM, then $f(z) \equiv t\Delta_c f(z)$, where $t^n = 1$, $t \neq 1$.*

Hence, Theorem C is still valid if the constant a is replaced by a small function $a(z)$ of $f(z)$.

Theorem 2 *Let $f(z)$ be a non-constant meromorphic function, let c be a nonzero value and $n \geqslant 5$ a positive integer, and let $a(z)(\not\equiv 0)$ be a small function of $f(z)$. If $[f^{(k)}(z)]^n$ and $[f(z+c)]^n$ share $a(z)$ CM, $f^{(k)}(z)$ and $f(z+c)$ share ∞ CM, then either $f^{(k)}(z) \equiv t f(z+c)$, where $t^n = 1$ or $[f^{(k)}(z)]^n[f(z+c)]^n \equiv a^2(z)$.*

Remark 1 The following example shows that Theorem F is not valid if the constant a is replaced by a small function $a(z)$ of $f(z)$. In other words, $[f^{(k)}(z)]^n[f(z+c)]^n \equiv a^2(z)$ can not be removed in Theorem 2.

Example 1 Let $f(z) = e^{e^z}$, $c = k\pi i$ and let $a(z) = e^{zn/2}$. Then by calculation we have $[f'(z)f(z+c)]^n = e^{nz} = a^2(z)$.

Theorem 3 *Let $f(z)$ be a transcendental meromorphic function of finite order, let c be a nonzero value and $n \geqslant 5$ a positive integer, and let $a(z)(\not\equiv 0)$ be a small function of $f(z)$. If $[f^{(k)}(z)]^n$ and $[f(z+c)]^n$ share $a(z)$ CM, $f^{(k)}(z)$ and $f(z+c)$ share ∞ CM, then $f^{(k)}(z) \equiv t f(z+c)$, where $t^n = 1$.*

Theorem 4 *Let $f(z)$ be a transcendental meromorphic function, let c be a nonzero value and $n \geqslant 5$ a positive integer, and let $a(z)(\not\equiv 0)$ be a rational function. If $[f^{(k)}(z)]^n$ and $[f(z+c)]^n$ share $a(z)$ CM, $f^{(k)}(z)$ and $f(z+c)$ share ∞ CM, then $f^{(k)}(z) \equiv t f(z+c)$, where $t^n = 1$.*

LEMMAS

Lemma 1 ([2, 3]) *Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Then*

$$m\Big(r, \tfrac{f^{(k)}}{f}\Big) = S(r, f).$$

Lemma 2 ([14]) *Let f be a non-constant meromorphic function, and let $n \geqslant 2$ be a positive integer. If f and $f^{(n)}$ have finite many zeros, then f is of finite order.*

Lemma 3 ([2]) *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are two non-constant meromorphic functions. If F and G share 1 CM and $H \not\equiv 0$, then

$$N_1(r, \tfrac{1}{F-1}) \leqslant N(r, H) + S(r, F) + S(r, G).$$

Remark 2 We know from the proof in [2] that Lemma 3 is valid when F and G share 1 CM almost.

Lemma 4 ([8, 10]) *Let f be a non-constant meromorphic function of finite order, let c be a nonzero complex number. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f),$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 5 ([7, 12]) *Let f be a non-constant meromorphic function of finite order, let c be a nonzero complex number. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f),$$
$$N(r, f(z+c)) = N(r, f) + S(r, f),$$
$$N(r, \tfrac{1}{f(z+c)}) = N(r, \tfrac{1}{f}) + S(r, f).$$

Lemma 6 ([6, 8]) *Let f be a non-constant meromorphic function of finite order, let c be a nonzero complex number. If $f(z+c) \equiv f(z)$, then f is of order at least 1.*

THE PROOF OF Theorem 1

Let

$$F = \frac{f^n}{a}, \quad G = \frac{[\Delta_c f]^n}{a}. \tag{1}$$

Since f^n and $[\Delta_c f]^n$ share a CM, we know that F and G share 1 CM almost. Set

$$\phi = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{2}$$

We discuss from following two cases.
Case 1: $\phi \equiv 0$. By (2) we have

$$\frac{F-1}{F} \equiv A\frac{G-1}{G}, \tag{3}$$

where A is a nonzero value.

If $A = 1$, then from (3) we get $f^n \equiv [\Delta_c f]^n$, that is $f \equiv t\Delta_c f$, where t is a complex number such that $t^n = 1$.

If $A\neq 1$, then from (3) we have

$$\frac{1}{F}-\frac{A}{G}\equiv 1-A. \tag{4}$$

By (4) we can obtain

$$\begin{aligned} &T(r,f)=T(r,\Delta_c f)+S(r,f),\\ &S(r,f)=S(r,\Delta_c f). \end{aligned} \tag{5}$$

According to (1), (4), (5) and Nevanlinna's Second Fundamental Theorem ([2], Page 19, Theorem 1.6) we get

$$\begin{aligned} nT(r,f)=T(r,F)+S(r,f)&\leqslant \overline{N}(r,F)\\ &+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{F-\frac{1}{1-A}})+S(r,f)\\ &\leqslant \overline{N}(r,f)+\overline{N}(r,\tfrac{1}{f})+\overline{N}(r,\Delta_c f)+S(r,f)\\ &\leqslant 3T(r,f)+S(r,f), \end{aligned} \tag{6}$$

it follows from (6) and $n\geqslant 6$ that $T(r,f)=S(r,f)$, a contradiction.

Case 2: $\phi\not\equiv 0$. Let z_0 be a common pole of f and $\Delta_c f$ with multiplicity l, then by (2) we know that z_0 is the zero of φ, and the multiplicity is at least $nl-1$. Since f and $\Delta_c f$ share ∞ CM, then

$$\begin{aligned} \overline{N}(r,F)=\overline{N}(r,G)&\leqslant \frac{1}{nl-1}N(r,\tfrac{1}{\varphi})+S(r,f)\\ &\leqslant \frac{1}{nl-1}T(r,\varphi)+S(r,f)\\ &\leqslant \frac{1}{n-1}[\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{G})]+S(r,f). \end{aligned} \tag{7}$$

Let H be defined as in Lemma 3. Suppose that $H\not\equiv 0$, by Lemma 3 and Remark 2 we have

$$\begin{aligned} N(r,H)\leqslant \overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{G})&\\ +N_0(r,\tfrac{1}{F'})+N_0(r,\tfrac{1}{G'})+S(r,f).& \end{aligned} \tag{8}$$

where $N_0(r,1/F')$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F-1$; $N_0(r,1/G')$ denotes the counting function corresponding to the zeros of G' which are not the zeros of G and $G-1$. By Nevanlinna's Second Fundamental Theorem, we get

$$\begin{aligned} T(r,F)+T(r,G)\leqslant \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})&\\ +\overline{N}(r,\tfrac{1}{F-1})+\overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}(r,\tfrac{1}{G-1})&\\ -N_0(r,\tfrac{1}{F'})-N_0(r,\tfrac{1}{G'})+S(r,f).& \end{aligned} \tag{9}$$

Since F and G share 1 almost CM, we have

$$\begin{aligned} &\overline{N}(r,\tfrac{1}{F-1})+\overline{N}(r,\tfrac{1}{G-1})\\ &\quad\leqslant N_1(r,\tfrac{1}{F-1})+\tfrac{1}{2}(N(r,\tfrac{1}{F-1})+N(r,\tfrac{1}{G-1})). \end{aligned} \tag{10}$$

By (8)–(10) we have

$$\begin{aligned} T(r,F)+T(r,G)\leqslant \overline{N}(r,F)+2\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,G)+2\overline{N}(r,\tfrac{1}{G})&\\ +\tfrac{1}{2}(N(r,\tfrac{1}{F-1})+N(r,\tfrac{1}{G-1}))& \end{aligned} \tag{11}$$

By Nevanlinna's First Fundamental Theorem ([2], Page 12, Theorem 1.2), we have

$$N(r,\tfrac{1}{F-1})+N(r,\tfrac{1}{G-1})\leqslant T(r,F)+T(r,G)+S(r,f). \tag{12}$$

By (7), (11) and (12), we can obtain

$$\begin{aligned} T(r,F)+T(r,G)&\leqslant 4\overline{N}(r,\tfrac{1}{f})+4\overline{N}(r,\tfrac{1}{\Delta_c f})+2\overline{N}(r,f)\\ &\quad+2\overline{N}(r,\Delta_c f)+S(r,f)\\ &\leqslant (4+\tfrac{4}{n-1})(T(r,F)+T(r,G))+S(r,f). \end{aligned} \tag{13}$$

Obviously, by (1) we have

$$\begin{aligned} \overline{N}(r,\tfrac{1}{F})&=\overline{N}(r,\tfrac{1}{f})+S(r,f),\\ \overline{N}(r,F)&=\overline{N}(r,f)+S(r,f),\\ \overline{N}(r,\tfrac{1}{G})&=\overline{N}(r,\tfrac{1}{\Delta_c f})+S(r,f),\\ \overline{N}(r,G)&=\overline{N}(r,\Delta_c f)+S(r,f),\\ T(r,F)&=nT(r,f)+S(r,f),\\ T(r,G)&=nT(r,\Delta_c f)+S(r,f). \end{aligned}$$

Hence, by above formulas, (13) and Nevanlinna's First Fundamental Theorem, we get

$$\begin{aligned} &n(T(r,f)+T(r,\Delta_c f))\\ &\quad\leqslant (4+\tfrac{4}{n-1})(\overline{N}(r,\tfrac{1}{f})+\overline{N}(r,\tfrac{1}{\Delta_c f}))+S(r,f)\\ &\quad\leqslant (4+\tfrac{4}{n-1})(T(r,f)+T(r,\Delta_c f))+S(r,f), \end{aligned}$$

and it follows that

$$n(1-\tfrac{4}{n-1})(T(r,f)+T(r,\Delta_c f))\leqslant S(r,f). \tag{14}$$

Thus by (14) and $n\geqslant 6$, we get $T(r,f)=S(r,f)$, a contradiction.

Hence, $H\equiv 0$. Thus we have

$$\frac{F''}{F'}-2\frac{F'}{F-1}=\frac{G''}{G'}-2\frac{G'}{G-1}.$$

Solving above equation, we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B,\quad \frac{A}{G-1}=\frac{1+B-BF}{F-1} \tag{15}$$

where $A(\neq 0)$ and B are constants.

Now we consider three subcases.

Case 2.1: $B\neq 0,-1$. It follows from (15) that

$$\begin{aligned} &T(r,\Delta_c f)=T(r,f)+S(r,f),\\ &\overline{N}\Big(r,\frac{1}{F-\frac{B+1}{B}}\Big)=\overline{N}(r,G). \end{aligned} \tag{16}$$

So by (15), (16), Nevanlinna's Second Fundamental Theorem, and the fact that f and $\Delta_c f$ share ∞ CM, we get

$$\begin{aligned}nT(r,f) &\leqslant T(r,F)+S(r,f)\\ &\leqslant \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{F-\frac{B+1}{B}})+S(r,f)\\ &\leqslant \overline{N}(r,\tfrac{1}{F})+\overline{N}(r,F)+\overline{N}(r,G)+S(r,f)\\ &\leqslant \overline{N}(r,\tfrac{1}{f})+\overline{N}(r,f)+\overline{N}(r,\Delta_c f)+S(r,f)\\ &\leqslant 3T(r,f)+S(r,f). \qquad (17)\end{aligned}$$

Therefore, by (17) and $n \geqslant 6$, we can get $T(r,f) = S(r,f)$, a contradiction.

Case 2.2: $B=0$. By (15) we obtain

$$F=\frac{G+(A-1)}{A}, \quad G=AF-(A-1). \qquad (18)$$

If $A \neq 1$, by (18) we get

$$\overline{N}\Big(r,\frac{1}{F-\frac{A-1}{A}}\Big)=\overline{N}(r,\tfrac{1}{G})=\overline{N}(r,\tfrac{1}{\Delta_c f})+S(r,f). \quad (19)$$

By (16), (19), Nevanlinna's Second Fundamental Theorem, and the fact that f and $\Delta_c f$ share ∞ CM, we get

$$\begin{aligned}nT(r,f) &\leqslant T(r,F)+S(r,f)\\ &\leqslant \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{F-\frac{A-1}{A}})+S(r,f)\\ &\leqslant \overline{N}(r,\tfrac{1}{F})+\overline{N}(r,F)+\overline{N}(r,\tfrac{1}{G})+S(r,f)\\ &\leqslant \overline{N}(r,\tfrac{1}{f})+\overline{N}(r,f)+\overline{N}(r,\tfrac{1}{\Delta_c f})+S(r,f)\\ &\leqslant 3T(r,f)+S(r,f). \qquad (20)\end{aligned}$$

Therefore, by (20) and $n \geqslant 6$, we can get $T(r,f) = S(r,f)$, a contradiction.

Hence $A=1$. It follows from (18) that $F \equiv G$. Thus by (1) we deduce that $f \equiv t\Delta_c f$, where $t^n=1$, $t \neq -1$.

Case 2.3: $B=-1$, by (15) we have

$$F=\frac{A}{-G+A+1}, \quad G=\frac{(A+1)F-A}{F}. \qquad (21)$$

If $A \neq -1$, we get from (19) that $\overline{N}(r,\frac{1}{F-\frac{A}{A+1}})=\overline{N}(r,\frac{1}{G})$. Using the same argument as in the Case 2.1, we get a contradiction. Thus, $A=-1$.

By (21), we get $FG \equiv 1$. It follows from $FG \equiv 1$ and (1) that

$$f^n[\Delta_c f]^n \equiv a^2. \qquad (22)$$

Set $f\Delta_c f = b$, then we get $b^n = a^2$. It follows that $T(r,b)=\frac{2}{n}T(r,a)$. Thus $b \not\equiv 0$ is a small function of f. Since f and $\Delta_c f$ share ∞ CM, we deduce from $f\Delta_c f = b$ that

$$N(r,\tfrac{1}{f}) \leqslant N(r,\tfrac{1}{b}) \leqslant T(r,b)+O(1)=S(r,f), \quad (23)$$

$$N(r,f) \leqslant N(r,b) \leqslant T(r,b)=S(r,f). \qquad (24)$$

Thus by Nevanlinna's Second Fundamental Theorem, (23), (24), and Lemma 5, we get

$$\begin{aligned}2T(r,f)=T(r,f^2) &\leqslant T(r,\tfrac{f^2}{b})+T(r,b)+O(1)\\ &\leqslant N(r,\tfrac{f^2}{b})+N(r,\tfrac{b}{f^2})+N(r,\tfrac{1}{\frac{f^2}{b}-1})+S(r,f)\\ &\leqslant N(r,\tfrac{b}{ff_c})+S(r,f) \leqslant S(r,f), \qquad (25)\end{aligned}$$

that is $T(r,f)=S(r,f)$, a contradiction.

Hence, we prove that $f \equiv t\Delta_c f$, where $t^n=1$.

THE PROOF OF Theorem 2

Let

$$F=\frac{(f^{(k)})^n}{a}, \quad G=\frac{(f_c)^n}{a}. \qquad (26)$$

Since f_c and $f^{(k)}$ share ∞ CM, $f^{(k)}$ has no pole with multiplicity 1. Then we use the same argument as in the proof of Theorem 1 and note that (26) is replaced by the following formula:

$$\begin{aligned}\overline{N}(r,\tfrac{1}{F})&=\overline{N}(r,\tfrac{1}{G})\\ &\leqslant \tfrac{1}{2n-1}N(r,\tfrac{1}{\phi})+S(r,f)\\ &\leqslant \tfrac{1}{2n-1}T(r,\phi)+S(r,f)\\ &\leqslant \tfrac{1}{2n-1}[\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{G})]+S(r,f), \qquad (27)\end{aligned}$$

and we prove either $f^{(k)} \equiv tf_c$, with $t^n=1$, or $(f^{(k)}f_c)^n \equiv a^2$.

THE PROOF OF Theorem 3

By Theorem 2, we obtain either $f^{(k)} \equiv tf_c$, with $t^n=1$, or $(f^{(k)}f_c)^n \equiv a^2$. We claim that $f^{(k)} \equiv tf_c$, with $t^n=1$. Otherwise, we suppose

$$[f^{(k)}]^n f_c^n \equiv a^2. \qquad (28)$$

Since f is a meromorphic function of finite order, It follows from (28) that $N(r,1/f_c)=S(r,f)$. Thus by Lemma 1, Lemma 4, Lemma 5, and Nevanlinna's First Fundamental Theorem, we have

$$\begin{aligned}2nT(r,f)&=T(r,tfrac1f^{2n})+S(r,f)\\ &=m(r,\tfrac{1}{f^{2n}})+N(r,\tfrac{1}{f^{2n}})+S(r,f)=m(r,\tfrac{1}{f^{2n}})+S(r,f)\\ &\leqslant m\Big(r,\tfrac{[f^{(k)}]^n f_c^n}{f^2}\Big)+2T(r,a)+O(1)\\ &\leqslant nm(r,\tfrac{f^{(k)}}{f})+nm(r,\tfrac{f_c}{f})+2T(r,a)+S(r,f)\\ &=S(r,f). \qquad (29)\end{aligned}$$

which is $T(r,f)=S(r,f)$, a contradiction.

THE PROOF OF Theorem 4

By Theorem 2, we know that either $f^{(k)}(z) \equiv t f_c(z)$, where $t^n = 1$ or $[f^{(k)}(z)]^n f_c^n(z) = a^2(z)$. We claim that

$$f^{(k)}(z) \equiv t f_c(z), \tag{30}$$

where $t^n = 1$. Otherwise, we have

$$[f^{(k)}(z)]^n f_c^n(z) \equiv a^2(z). \tag{31}$$

Since $a(z)$ is a rational function, it follows from (31) that both $f(z)$ and $f^{(k)}(z)$ have finite many zeros and poles. If $k \geqslant 2$, by Lemma 2 we know that $f(z)$ is a transcendental meromorphic function of finite order. Thus by Theorem 3 we get a contradiction.

Next, we consider the case of $k = 1$. Since $f(z)$ has finite many zeros and poles, we assume that

$$f(z) = b(z)\,\mathrm{e}^{\alpha(z)}, \tag{32}$$

where $b(z)$ is a rational function and $\alpha(z)$ is an entire function. By (32) we get

$$f_c(z) = b_c(z)\,\mathrm{e}^{\alpha_c(z)}, \tag{33}$$

$$f'(z) = (b'(z) + \alpha'(z)b(z))\,\mathrm{e}^{\alpha(z)}, \tag{34}$$

It follows from (31)–(34) that

$$[(b'(z) + \alpha'(z)b(z))b_c(z)]^n\,\mathrm{e}^{n(\alpha(z)+\alpha_c(z))} \equiv a^2(z). \tag{35}$$

Thus, we have

$$(b'(z) + \alpha'(z)b(z))b_c(z) = d(z)\,\mathrm{e}^{\beta(z)}, \tag{36}$$

where $d(z)$ is a rational function, and $\beta(z)$ is an entire function.

By (35) and (36), we get

$$d^n(z)\,\mathrm{e}^{n\beta(z)}\,\mathrm{e}^{n(\alpha(z)+\alpha_c(z))} \equiv a^2(z). \tag{37}$$

It follows from (37) that

$$\beta(z) + \alpha(z) + \alpha_c(z) \equiv A, \tag{38}$$

where A is a finite complex number. Differential both sides of (38) we obtain

$$\beta'(z) + \alpha'(z) + \alpha_c'(z) \equiv 0. \tag{39}$$

By (36) we have

$$\alpha'(z) \equiv \frac{c}{b(z)b_c(z)}\,\mathrm{e}^{\beta(z)} - \frac{b'(z)}{b(z)}. \tag{40}$$

Therefore,

$$\frac{d(z)}{b(z)b_c(z)}\,\mathrm{e}^{\beta(z)} + \frac{d(z)}{b_c(z)b_{2c}(z)}\,\mathrm{e}^{\beta_c(z)} \equiv \frac{b'(z)}{b(z)} + \frac{b_c'(z)}{b_c(z)} - \beta'(z). \tag{41}$$

Next, we consider two cases.

Case 1:

$$\frac{b'(z)}{b(z)}\frac{b_c'(z)}{b_c(z)} - \beta'(z) \equiv 0. \tag{42}$$

Then, we claim that $\beta'(z) \equiv 0$. Otherwise, by (42) we get

$$b(z)b_c(z) \equiv B\,\mathrm{e}^{\beta(z)}, \tag{43}$$

where B is a nonzero constant. Since $b(z)$ is a rational function, so it is impossible. Thus, $\beta'(z) \equiv 0$, that is $\beta(z)$ is a constant. And then we deduce that

$$\frac{d(z)}{b(z)b_c(z)} + \frac{d_c(z)}{b_c(z)b_{2c}(z)} \equiv 0.$$

Let $A(z) = \frac{d(z)}{b(z)b_c(z)}$. Then $A(z) + A_c(z) \equiv 0$, $A_c(z) + A_{2c}(z) \equiv 0$, $A(z) \equiv A_{2c}(z)$. Then by Lemma 6, $A(z)$ is a meromorphic function of order $\geqslant 1$, but $A(z)$ is a rational function, it is impossible.

Case 2:

$$\frac{b'(z)}{b(z)} + \frac{b_c'(z)}{b_c(z)} - \beta'(z) \not\equiv 0. \tag{44}$$

We claim that $\beta'(z) \not\equiv 0$. Otherwise, $\beta(z) = D$ is a constant. It follows (41) that

$$\left[\frac{d(z)}{b(z)b_c(z)} + \frac{d_c(z)}{b_c(z)b_{2c}(z)}\right]\mathrm{e}^D \equiv \frac{b'(z)}{b(z)} + \frac{b_c'(z)}{b_c(z)}. \tag{45}$$

We can rewrite above as

$$\frac{d(z)}{b(z)b_c(z)}\,\mathrm{e}^D - \frac{b'(z)}{b(z)} \equiv \frac{d_c(z)}{b_c(z)b_{2c}(z)}\,\mathrm{e}^D - \frac{b_c'(z)}{b_c(z)}. \tag{46}$$

Let $H(z) \equiv \frac{d(z)}{b(z)b_c(z)}\,\mathrm{e}^D - \frac{b'(z)}{b(z)}$. Then (46) implies that $H_c(z) + H_{2c}(z) \equiv 0$, $H(z) + H_c(z) \equiv 0$ and $H(z) + H_{2c}(z) \equiv 0$. It follows from Lemma 6 that $H(z)$ is a meromorphic function of order $\geqslant 1$, a contradiction. Hence $\beta'(z) \not\equiv 0$, and $\mathrm{e}^{\beta(z)}$ is a transcendental entire function. By (41) we have

$$A_1(z)\,\mathrm{e}^{\beta(z)} + A_2(z)\,\mathrm{e}^{\beta_c(z)} \equiv 1, \tag{47}$$

where

$$A_1(z) = \frac{d(z)}{b(z)b_c(z)\left(\frac{b'(z)}{b(z)} + \frac{b_c'(z)}{b_c(z)} - \beta'(z)\right)}, \tag{48}$$

and

$$A_2(z) = \frac{d_c(z)}{b_c(z)b_{2c}(z)\left(\frac{b'(z)}{b(z)} + \frac{b_c'(z)}{b_c(z)} - \beta'(z)\right)}. \tag{49}$$

Since $T(r, \mathrm{e}^{\beta'}) = m(r, (\mathrm{e}^{\beta})'/\mathrm{e}^{\beta}) = S(r, \mathrm{e}^{\beta})$, it follows from (48) and (49) that $A_1(z)$ and $A_2(z)$ are small functions of $\mathrm{e}^{\beta(z)}$. It follows from above and the Nevanlinna's Second Fundamental Theorem, we can get $T(r, \mathrm{e}^{\beta}) = S(r, \mathrm{e}^{\beta})$, a contradiction.

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