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Meromorphic solutions of some types of q-difference differential equation and delay differential equation

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ABSTRACT: In this paper, we investigate the existence of rational solutions and value distribution of non-rational meromorphic solutions with the finite order of delay differential equation

$$w(z+1)-w(z-1)+a\frac{w'(z)}{w(z)}=b,$$

where $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$ are constants. In addition, necessary conditions are obtained for q-difference differential equation

$$w(qz)-w(z/q)+a(z)\frac{w'(z)}{w(z)}=R(z,w(z))$$

to admit a non-rational meromorphic solution of zero-order, where $|q| \notin \{0,1\}$, $a(z)$ is rational and $R(z,w(z))$ is rational in $w(z)$ with rational coefficients in z.

KEYWORDS: value distribution, meromorphic solution, delay differential equation, q-difference differential equation, zero-order

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INTRODUCTION AND MAIN RESULTS

Quispel et al [1] have derived the delay differential equation

$$w(z+1)-w(z-1)+a\frac{w'(z)}{w(z)}=b, \tag{1}$$

where a,b are constants, as a reduction of the integrable Kac-van Moerbeke equation on the basis of its Lie symmetries. They showed that (1) admits a Lax representation, and reduces to the first Painlevé equation in a continuum limit.

In this paper, we continue to study (1) and obtain some properties of meromorphic solutions of (1), that is, we obtain the following result.

Theorem 1 *Consider the delay differential (1), where $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$ are constants. Then*

(i) *If $b \in \mathbb{C}\backslash\{0\}$, then (1) has no rational solution;*

(ii) *If $b = 0$, then (1) admits a non-constant rational solution of the form*

$$w(z)=-\frac{a}{2}+\frac{m(z)}{n(z)},$$

where $m(z)$ and $n(z)$ are polynomials with $\deg m(z)=m$, $\deg n(z)=n$ and $m<n$;

(iii) *If $b \in \mathbb{C}\backslash\{0\}$, suppose that $w(z)$ is a transcendental meromorphic solution of (1) with finite order, then $w(z)$ has at most one finite Borel exceptional value 0 unless*

$$w(z)=p\exp(k\pi i z) \quad \text{and} \quad \frac{b}{a}=k\pi i, \quad i=\sqrt{-1},$$

where p is a non-zero constant, $k \in \mathbb{Z}\backslash\{0\}$;

(iv) *If $b \in \mathbb{C}\backslash\{0\}$, suppose that $w(z)$ is a finite order transcendental meromorphic solution of (1), and has only finitely many zeros and poles, then*

$$w(z)=p\exp(k\pi i z) \quad \text{and} \quad \frac{b}{a}=k\pi i, \quad i=\sqrt{-1},$$

where p is a non-zero constant, $k \in \mathbb{Z}\backslash\{0\}$.

Example 1 ([1])

$$\begin{aligned} w(z) &= -\frac{a}{2}\frac{(z+c+1)(z+c-2)}{(z+c)(z+c-1)} \\ &= -\frac{a}{2}+\frac{a}{(z+c)(z+c-1)} \end{aligned}$$

is a rational solution of equation

$$w(z+1)-w(z-1)+a\frac{w'(z)}{w(z)}=0,$$

where $a \in \mathbb{C}\backslash\{0\}$ and $c \in \mathbb{C}$.

The difference analogue of the logarithmic derivative lemma, which was obtained independently by Halburd et al [2] and by Chiang et al [3], plays a key role in the value distribution of difference [4–9]. Also, q-difference analogue of the logarithmic derivative lemma, which was obtained by Barnett et al [10], plays an important role in the value distribution of q-difference [11–14].

Halburd et al [15] investigated the differential-difference equation

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=R(z,w(z)), \quad (2)$$

where $a(z)$ is rational and $R(z,w(z))$ is rational in $w(z)$ and meromorphic in z. In this paper, we study q-difference differential equation

$$\begin{aligned} w(qz)-w(z/q)+a(z)\frac{w'(z)}{w(z)} &= R(z,w(z)) \\ &= \frac{P(z,w(z))}{Q(z,w(z))}, \end{aligned} \quad (3)$$

where $|q|\notin\{0,1\}$, $a(z)$ is rational and $R(z,w(z))$ is rational in $w(z)$ with rational coefficients in z. Now, we state the main findings as follows.

Theorem 2 *Let $w(z)$ be a transcendental meromorphic solution of zero-order of (3), where $a(z)$ is rational in z, $P(z,w(z))$ is a polynomial in $w(z)$ having rational coefficients in z, and $Q(z,w(z))\not\equiv 0$ is a monic polynomial in $w(z)$ with roots that are rational in z and not roots of $P(z,w(z))$. Then*

$$\deg_w(P(z,w))=\deg_w(Q(z,w))+1\leqslant 3, \quad (4)$$

or the degree of $R(z,w(z))$ as a rational function in $w(z)$ is either 0 *or* 1.

Considering a meromorphic function $w(z)$ in the complex plane, we assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory [16, 17]. In addition, we use notations $\rho(w)$, $\lambda(w)$ and $\lambda\left(\frac{1}{w}\right)$ to denote the order of growth, the exponent of convergence of the zero-sequence and the pole-sequence of meromorphic function $w(z)$, respectively. And we denote by $S(r,w)$ any quantify satisfying $S(r,w)=o(T(r,w))$, as $r\to\infty$, outside of a possible exceptional set of finite logarithmic measure.

LEMMAS

Lemma 1 ([10], Theorem 1.2) *Let $f(z)$ be a non-constant zero-order meromorphic function, and $q\in\mathbb{C}\backslash\{0\}$. Then*

$$m\left(r,\frac{f(qz)}{f(z)}\right)=o(T(r,f)) \quad (5)$$

on a set of logarithmic density 1.

Lemma 2 ([14], Theorem 1.1) *Let $f(z)$ be a non-constant zero-order meromorphic function, and $q\in\mathbb{C}\backslash\{0\}$. Then*

$$T(r,f(qz))=(1+o(1))T(r,f(z)) \quad (6)$$

on a set of logarithmic density 1.

Remark 1 Equation (6) implies that

$$T(r,f(qz))=T(r,f(z))+S(r,f).$$

Lemma 3 ([18], Lemma 4) *Let $c_1>1$, $c_2>1$ and $\rho\geqslant 0$. If $T:\mathbb{R}^+\to\mathbb{R}^+$ is an increasing function such that*

$$\limsup_{r\to\infty}\frac{\log T(r)}{\log r}=\rho,$$

then the logarithmic density of the set

$$E:=\{r:T(c_1r)\geqslant c_2T(r)\}$$

satisfies

$$\overline{\log\text{dens}}(E)\leqslant\frac{\rho\log c_1}{\log c_2}.$$

The following lemma plays a key role in the proof of Theorem 2. A q-difference differential polynomial in $w(z)$ is defined by

$$\begin{aligned} P(z,w(z))=\sum_{\lambda\in I}a_\lambda(z)w(z)^{\lambda_{0,0}}w(q_1z)^{\lambda_{1,0}}\cdots \\ \cdots w(q_\nu z)^{\lambda_{\nu,0}}w'(z)^{\lambda_{0,1}}\cdots w^{(\mu)}(q_\nu z)^{\lambda_{\nu,\mu}}, \end{aligned}$$

where $q_1,\dots,q_\nu$ are distinct complex constants and $|q_i|\notin\{0,1\}(1\leqslant i\leqslant\nu)$, I is a finite index set consisting of elements of the form $\lambda=(\lambda_{0,0},\cdots,\lambda_{\nu,\mu})$ and the coefficients $a_\lambda(z)$ are rational functions of z for all $\lambda\in I$.

Lemma 4 *Let $w(z)$ be a transcendental meromorphic solution of*

$$P(z,w(z))=0, \quad (7)$$

where $P(z,w(z))$ is q-difference differential polynomial in $w(z)$ with rational coefficients, and let $b_1,\dots,b_l$ be rational functions satisfying $P(z,b_j)\not\equiv 0$ for all $j\in\{1,\dots,l\}$. Denote $K=\max_{1\leqslant i\leqslant\nu}\{|q_i|,1/|q_i|\}$. If there exists $s>0$ and $\tau\in(0,1)$ such that

$$\sum_{j=1}^{l}n\left(r,\frac{1}{w-b_j}\right)\leqslant l\tau n(K^sr,w)+O(1), \quad (8)$$

then $\rho(w)>0$.

Proof: Suppose that $\rho(w)=0$. We first show that the assumption $P(z,b_j)\not\equiv 0$ implies that

$$m\left(r,\frac{1}{w-b_j}\right)=S(r,w), \quad (9)$$

on a set of logarithmic density 1. This result is an extension of Mohon'ko's Theorem [19] and its q-difference analogue [10] for differential-difference equations with meromorphic solutions of zero-order.

By substituting $w = g + b_j$ into (7) we obtain

$$Q(z,g) + R(z) = 0, \tag{10}$$

where

$$Q(z,g) = \sum_{\lambda\in I} a_\lambda(z) G_\lambda(z,g) \tag{11}$$

is a q-difference differential polynomial in $g(z)$ such for all λ in the finite index set I, $G_\lambda(z,g)$ is a non-constant product of derivatives and q-shift of $g(z)$. Also $R(z) \not\equiv 0$, since $P(z,b_j) \not\equiv 0$ for all $j \in \{1,\dots,l\}$, is a rational function. The coefficients $a_\lambda(z)$ in (11) are all rational. By defining $E_1 = \{\theta \in [0,2\pi) : |g(r\,\mathrm{e}^{i\theta})| \leqslant 1\}$ and $E_2 = [0,2\pi)\backslash E_1$, we have

$$\begin{aligned} m\left(r, \frac{1}{w-b_j}\right) &= m\left(r, \frac{1}{g}\right) \\ &= \int_{\theta\in E_1} \log^+ \left|\frac{1}{g(r\,\mathrm{e}^{i\theta})}\right| \frac{d\theta}{2\pi}. \end{aligned} \tag{12}$$

Furthermore, for all $z = r\,\mathrm{e}^{i\theta}$ such that $\theta \in E_1$,

$$\begin{aligned} \left|\frac{Q(z,g)}{g}\right| &= \frac{1}{|g|}\Big|\sum_{\lambda\in I} a_\lambda(z) g(z)^{\lambda_{0,0}} g(q_1 z)^{\lambda_{1,0}} \cdots \\ &\qquad \cdots g(q_\nu z)^{\lambda_{\nu,0}} g'(z)^{\lambda_{0,1}} \cdots g^{(\mu)}(q_\nu z)^{\lambda_{\nu,\mu}}\Big| \\ &\leqslant \sum_{\lambda\in I} |a_\lambda(z)| \left|\frac{g(q_1 z)}{g(z)}\right|^{\lambda_{1,0}} \cdots \\ &\qquad \cdots \left|\frac{g(q_\nu z)}{g(z)}\right|^{\lambda_{\nu,0}} \left|\frac{g'(z)}{g(z)}\right|^{\lambda_{0,1}} \cdots \left|\frac{g^{(\mu)}(q_\nu z)}{g(z)}\right|^{\lambda_{\nu,\mu}}, \end{aligned}$$

since $\deg_g(G_\lambda) \geqslant 1$ for all $\lambda \in I$ with $\lambda = (\lambda_{0,0},\dots,\lambda_{\nu,\mu})$. Using (10) we have

$$\begin{aligned} \log^+ \left|\frac{1}{g(z)}\right| &\leqslant \log^+ \left|\frac{R(z)}{g(z)}\right| + \log^+ \left|\frac{1}{R(z)}\right| \\ &\leqslant \log^+ \left|\frac{Q(z,g)}{g(z)}\right| + \log^+ \left|\frac{1}{R(z)}\right|. \end{aligned}$$

By using (12) with $q_0 = 1$, and applying the lemma on the logarithmic derivative, Lemma 1 and Lemma 2, we have

$$\begin{aligned} m\left(r, \tfrac{1}{w-b_j}\right) &\leqslant \int_{\theta\in E_1} \log^+ \left|\frac{Q(z,g)}{g(re^{i\theta})}\right| \frac{d\theta}{2\pi} + O(\log r) \\ &\leqslant \sum_{n=0}^{\nu}\sum_{m=0}^{\mu} \lambda_{n,m} m\left(r, \frac{g^{(m)}(q_n z)}{g(z)}\right) + O(\log r) \\ &\leqslant \sum_{n=0}^{\nu}\sum_{m=0}^{\mu} \lambda_{n,m}\Big(m\left(r, \frac{g^{(m)}(q_n z)}{g(q_n z)}\right) \\ &\quad + m\left(r, \frac{g(q_n z)}{g(z)}\right)\Big) + O(\log r) = S(r,w), \end{aligned} \tag{13}$$

on a set of logarithmic density 1. It follows from (8) that

$$\sum_{j=1}^{l} N\left(r, \tfrac{1}{w-b_j}\right) \leqslant l(\tau+\varepsilon)N(K^s r, w) + O(\log r), \tag{14}$$

where $\varepsilon > 0$ is chosen such that $\tau + \varepsilon < 1$. By the first main theorem of Nevanlinna theory, we have

$$lT(r,w) = \sum_{j=1}^{l}\left(m\left(r, \tfrac{1}{w-b_j}\right) + N\left(r, \tfrac{1}{w-b_j}\right)\right) + O(\log r). \tag{15}$$

It follows from (9), (14) and (15) that

$$\begin{aligned} lT(r,w) &\leqslant l(\tau+\varepsilon)N(K^s r, w) + S(r,w) \\ &\leqslant l(\tau+\varepsilon)T(K^s r, w) + S(r,w). \end{aligned} \tag{16}$$

By Remark 1 and an observation due to Bergweiler et al [20], it follows that

$$T(K^s r, w(z)) + O(1) = T(r, w(K^s z)) = T(r,w) + S(r,w)$$

on a set of logarithmic density 1. By combining (16), we have

$$T(r,w) \leqslant (\tau+\varepsilon)T(r,w) + S(r,w),$$

since $\tau + \varepsilon < 1$, the above inequality is a contradiction. Then we conclude that $\rho(w) > 0$. □

PROOF OF Theorem 1

Proof: Suppose that $w(z)$ is a rational solution of (1). Denote

$$w(z) = h(z) + \frac{m(z)}{n(z)}, \tag{17}$$

where $h(z)$, $m(z)$ and $n(z)$ are polynomial with $\deg h(z) = l$, $\deg m(z) = m$, $\deg n(z) = n$ with $m < n$. Set

$$\begin{aligned} h(z) &= c_0 z^l + \cdots + c_l, \\ m(z) &= a_0 z^m + \cdots + a_m, \\ n(z) &= b_0 z^n + \cdots + b_n, \end{aligned} \tag{18}$$

where $c_0,\dots,c_l, a_0(\neq 0),\dots,a_m, b_0(\neq 0),\cdots,b_n$ are constants.

(i) If $b \in \mathbb{C}\backslash\{0\}$, in what follows, we consider three cases.

Case 1: $l > 0$. By (17) and (18), when z is large enough, $w(z)$ can be written as

$$w(z) = c_0 z^l (1 + o(1)). \tag{19}$$

Then

$$\begin{aligned} &w(z+1) = c_0(z+1)^l(1+o(1)), \\ &w(z-1) = c_0(z-1)^l(1+o(1)), \\ &w(z+1) - w(z-1) = 2c_0 l z^{l-1}(1+o(1)), \\ &w'(z) = c_0 l z^{l-1}(1+o(1)). \end{aligned} \tag{20}$$

By substituting (19) and (20) into (1), it follows that

$$c_0z^l(1+o(1))2c_0lz^{l-1}(1+o(1))+ac_0lz^{l-1}(1+o(1)) = bc_0z^l(1+o(1)),$$

that is,

$$2c_0^2lz^{2l-1}(1+o(1)) = bc_0z^l(1+o(1)),$$

from which it follows that

$$2l-1=l \quad and \quad 2c_0^2=bc_0.$$

So $l=1$ and $b=2c_0$. Then $w(z)$ can be rewritten as

$$w(z)=c_0z+c_1+o(1), \quad c_0\neq 0, \tag{21}$$

and

$$\begin{aligned} w(z+1)-w(z-1) &= c_0(z+1)+c_1+o(1)-[c_0(z-1)+c_1+o(1)] \\ &= 2c_0+o(1), \\ w'(z) &= c_0+o(1). \end{aligned} \tag{22}$$

Substituting (21) and (22) into (1) yields

$$(c_0z+c_1+o(1))(2c_0+o(1))+a(c_0+o(1)) = b(c_0z+c_1+o(1)).$$

Since $b=2c_0$, from the above equality, we conclude that $a=0$, a contradiction.

Case 2: $l=0, c_0\neq 0$. By (17) and (18), when z is large enough, $w(z)$ can be written as

$$w(z)=c_0+\frac{m(z)}{n(z)}=c_0+o(1). \tag{23}$$

By calculation and $m<n$, we see that

$$\begin{aligned} & m(z+1)n(z-1)-m(z-1)n(z+1) \\ &\qquad = 2(m-n)a_0b_0z^{m+n-1}(1+o(1)), \\ & m'(z)n(z)-m(z)n'(z) \\ &\qquad = (m-n)a_0b_0z^{m+n-1}(1+o(1)), \\ & n(z+1)n(z-1)=b_0^2z^{2n}(1+o(1)), \\ & n^2(z)=b_0^2z^{2n}(1+o(1)). \end{aligned} \tag{24}$$

Then

$$\begin{aligned} w(z+1)-w(z-1) &= \frac{m(z+1)n(z-1)-m(z-1)n(z+1)}{n(z+1)n(z-1)} \\ &= 2(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1)), \\ w'(z) &= \frac{m'(z)n(z)-m(z)n'(z)}{n^2(z)} \\ &= (m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1)). \end{aligned} \tag{25}$$

Substituting (23)–(25) into (1) yields

$$(c_0+o(1))2(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1))+ a(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1))=b(c_0+o(1)),$$

that is,

$$(a+2c_0)(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1))=b(c_0+o(1)).$$

Since $m<n$ and $b,c_0\neq 0$, the above equality is a contradiction.

Case 3: $l=0$, $c_0=0$. Because $m<n$, we see that

$$w(z)=\frac{m(z)}{n(z)}=\frac{a_0}{b_0}z^{m-n}(1+o(1)). \tag{26}$$

By substituting (24)–(26) into (1), we have

$$\frac{a_0}{b_0}z^{m-n}(1+o(1))2(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1)) +a(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1))=b\frac{a_0}{b_0}z^{m-n}(1+o(1)),$$

that is,

$$a(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1))=b\frac{a_0}{b_0}z^{m-n}(1+o(1)).$$

Since $m<n$ and $ab\neq 0$, the above equality is a contradiction.

(ii) If $b=0$, suppose that $l>0$ or $l=0$ and $c_0=0$, we can conclude that (1) has no non-constant rational solution by using the same method of proof of Case 1 or Case 3, respectively. Suppose that $l=0$ and $c_0\neq 0$, substitute (23)–(25) into (1) yields

$$(c_0+o(1))2(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1))+ a(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1))=0,$$

that is,

$$(a+2c_0)(m-n)\frac{a_0}{b_0}z^{m-n-1}(1+o(1))=0.$$

Since $m<n$ and $a_0,b_0\neq 0$, then we have $c_0=-\frac{a}{2}$.

(iii) If $b\in\mathbb{C}\backslash\{0\}$, denote

$$P(z,w):=w(z)[w(z+1)-w(z-1)]+aw'(z)-bw(z)=0.$$

Then, we have

$$P(z,\alpha)=-b\alpha\neq 0, \quad \alpha\in\mathbb{C}\backslash\{0\}.$$

An application of the estimate (2.3) of Lemma 2.1 in [15], yields

$$m\left(r,\frac{1}{w-\alpha}\right)=S(r,w).$$

Then

$$N\left(r,\frac{1}{w-\alpha}\right)=T(r,w)+S(r,w)$$

for all $\alpha\in\mathbb{C}\backslash\{0\}$. Thus $w(z)$ has at most one finite Borel exceptional value 0. Suppose that $w(z)$ has two Borel exceptional values 0 and ∞, by Theorem 2.11 of [21], we see that $w(z)$ is of regular growth, then $w(z)$ can be written as

$$w(z)=p(z)\,\mathrm{e}^{h(z)}, \tag{27}$$

where $p(z)$ is a meromorphic function, and $h(z)$ is polynomial such that

$$\sigma(p(z))=\max\{\lambda(w),\lambda(1/w)\}<\deg h(z).$$

By (27), we obtain

$$\begin{aligned} w(z+1)&=p(z+1)\,\mathrm{e}^{h(z+1)}=p_+(z)\,\mathrm{e}^{h(z)},\\ w(z-1)&=p(z-1)\,\mathrm{e}^{h(z-1)}=p_-(z)\,\mathrm{e}^{h(z)},\\ w'(z)&=(p'(z)+p(z)h'(z))\,\mathrm{e}^{h(z)}, \end{aligned} \tag{28}$$

where $p_+(z)=p(z+1)\,\mathrm{e}^{h(z+1)-h(z)}$, $p_-(z)=p(z-1)\,\mathrm{e}^{h(z-1)-h(z)}$. Substituting (27) and (28) into (1) yields

$$\begin{aligned} p(z)(p_+(z)-p_-(z))\,\mathrm{e}^{h(z)}\\ =bp(z)-a(p'(z)+p(z)h'(z)). \end{aligned} \tag{29}$$

If $p_+(z)-p_-(z)\not\equiv 0$, rewrite (29) as

$$\mathrm{e}^{h(z)}=\frac{bp(z)-a(p'(z)+p(z)h'(z))}{p(z)(p_+(z)-p_-(z))}.$$

Since $\sigma(e^{h(z)})=\deg h(z)$ and

$$\sigma\left(\frac{bp(z)-a(p'(z)+p(z)h'(z))}{p(z)(p_+(z)-p_-(z))}\right)\leqslant\max\{\deg h(z)-1,\sigma(p(z))\},$$

which contradicts with $\sigma(p(z))<\deg h(z)$. Thus $p_+(z)-p_-(z)\equiv 0$, then $a(p'(z)+p(z)h'(z))-bp(z)\equiv 0$. By $p(z)\not\equiv 0$ and $a(p'(z)+p(z)h'(z))-bp(z)\equiv 0$, it follows that $p'(z)/p(z)\equiv b/a-h'(z)$, that is, $p(z)=c\,\mathrm{e}^{bz/a-h(z)}$, where c is a non-zero constant. Since $1\leqslant\deg h(z)$ and $\sigma(p(z))<\deg h(z)$, we have $h(z)=bz/a+c_1$ and $p(z)\equiv p$, where p is a non-zero constant, c_1 is a constant. It follows from $p_+(z)-p_-(z)\equiv 0$ and $p(z)\equiv p$ that $p(z+1)\,\mathrm{e}^{h(z+1)-h(z)}\equiv p(z-1)e^{h(z-1)-h(z)}$, then $\mathrm{e}^{2b/a}=1$, that is, $b/a=k\pi i$, $k\in\mathbb{Z}\backslash\{0\}$. So $w(z)=p\exp(k\pi iz)$.

(iv) Suppose that $w(z)$ has only finitely many zeros and poles, and is a finite order transcendental meromorphic solution of (1), then 0 and ∞ are two Borel exceptional values of $w(z)$, the conclusion follows immediately by Theorem 1 (iii). □

PROOF OF Theorem 2

Proof: Suppose that (3) has a transcendental meromorphic solution of zero-order. By (3) and an identity due to Valiron [22] and Mohon'ko [23] (see also [17], Theorem 2.2.5), we have

$$\begin{aligned} &T\left(r,w(qz)-w(z/q)+a(z)\frac{w'(z)}{w(z)}\right)\\ &\quad=T(r,R(z,w(z)))\\ &\quad=\deg_w(R(z,w(z)))T(r,w(z))+O(\log r). \end{aligned}$$

By applying the lemma on the logarithmic derivative and an observation due to Bergweiler et al [20], we have for all sufficiently large $r>0$,

$$\begin{aligned} &\deg_w(R(z,w(z)))T(r,w(z))\\ &\leqslant T(r,w(qz))+T(r,w(z/q))+T\left(r,\tfrac{w'(z)}{w(z)}\right)+O(\log r)\\ &\leqslant T(r,w(qz))+T(r,w(\tfrac{z}{q}))+\overline{N}(r,w)+\overline{N}(r,\tfrac{1}{w})+S(r,w)\\ &\leqslant T(|q|r,w(z))+T(\tfrac{r}{|q|},w(z))+2T(r,w(z))+S(r,w)\\ &\leqslant(2+\varepsilon)T(Kr,w(z))+2T(r,w(z)), \end{aligned}$$

that is,

$$\begin{aligned} &(\deg_w(R(z,w(z)))-2)T(r,w(z))\\ &\qquad\leqslant(2+\varepsilon)T(Kr,w(z)), \end{aligned} \tag{30}$$

with $K=\max\{|q|,1/|q|\}$ and $\varepsilon>0$. Let

$$G=\{r>0:\text{inequality (30) is true}\}.$$

Then the logarithmic density of G satisfies

$$\overline{\log\mathrm{dens}}(G)=\limsup_{r\to\infty}\frac{\int_{[0,r]\cap G}\frac{dt}{t}}{\log r}=1.$$

By Lemma 3 and inequality (30), it follows that

$$\overline{\log\mathrm{dens}}(G)\leqslant\frac{\rho(w)\log K}{\log\left(\frac{\deg_w(R(z,w(z)))-2}{2+\varepsilon}\right)},$$

and so

$$\log\left(\frac{\deg_w(R(z,w(z)))-2}{2}\right)\leqslant\rho(w)\log K, \tag{31}$$

by letting $\varepsilon\to 0$. Hence, if (3) has a non-rational meromorphic solution of zero-order, then it follows from (31) that

$$\deg_w(R(z,w(z)))\leqslant 4.$$

In what follows, similar to the proof of Theorem 1.1 in [15], we consider five cases.

Case 1: Suppose that the denominator of $R(z,w(z))$ has at least two distinct non-zero rational roots for $w(z)$ as a function of z, say $b_1(z) \not\equiv 0$ and $b_2(z) \not\equiv 0$. Then we written (3) as

$$w(qz) - w(z/q) + a(z)\frac{w'(z)}{w(z)} = \frac{P(z,w(z))}{(w(z)-b_1(z))(w(z)-b_2(z))\hat{Q}(z,w(z))}, \quad (32)$$

where $P(z,w(z)) \not\equiv 0$ and $\hat{Q}(z,w(z)) \not\equiv 0$ are polynomials in $w(z)$ of at most degree 4 and 2, respectively. We do not rule out the possibility that $\hat{Q}(z,b_1(z)) \equiv 0$ or $\hat{Q}(z,b_2(z)) \equiv 0$. We also assume that $P(z,w(z))$ and $\hat{Q}(z,w(z))$ are two mutually prime polynomials in $w(z)$. Then neither $b_1(z)$, nor $b_2(z)$ is a solution of (32), and so they satisfy the first condition of Lemma 4. Assume that $\hat{z} \in \mathbb{C}$ is any point satisfying

$$w(\hat{z}) = b_1(\hat{z}), \quad (33)$$

and such that none of the coefficients of (32) have a zero or pole at $\hat{z}$ and $P(\hat{z},w(\hat{z})) \neq 0$. Let p denote the order of the zero of $w(z)-b_1(z)$ at $\hat{z}$. We call such a $\hat{z}$ a generic root of $w(z)-b_1(z)$ of order p. Without further comment, we only consider generic roots. Since the coefficients are rational and thus have finitely many zeros or poles, the contribution can be included in a bounded error term, leading to an error term of the type $O(\log r)$. It follows from (32) that either $w(qz)$ or $w(z/q)$ has a pole at $z = \hat{z}$ of order at least p. Without loss of generality, we assume that $w(qz)$ has such a pole at $z = \hat{z}$. By q-shifting the (32), we have

$$w(q^2z) - w(z) + a(qz)\frac{qw'(qz)}{w(qz)} = \frac{P(qz,w(qz))}{(w(qz)-b_1(qz))(w(qz)-b_2(qz))\hat{Q}(qz,w(qz))}, \quad (34)$$

which implies that $w(q^2z)$ has such a pole of order one at $z = \hat{z}$ provided that

$$\deg_w(P) \leqslant \deg_w(\hat{Q}) + 2. \quad (35)$$

Suppose that (35) holds, by iterating (32) one more step, we have

$$w(q^3z) - w(qz) + a(q^2z)\frac{q^2w'(q^2z)}{w(q^2z)} = \frac{P(q^2z,w(q^2z))}{(w(q^2z)-b_1(q^2z))(w(q^2z)-b_2(q^2z))\hat{Q}(q^2z,w(q^2z))}. \quad (36)$$

If $p > 1$, then $w(q^3z)$ must have a pole of order at least p at $z = \hat{z}$. Hence, we can pair up the zero of $w(z) - b_1(z)$ at $z = \hat{z}$ together with the pole of $w(z)$ at $z = q\hat{z}$ without possibility of a similar iteration process starting from another point, say $z = q^3\hat{z}$, and resulting in pairing the pole at $z = q\hat{z}$ with another root of $w(z) - b_1(z)$, or of $w(z) - b_2(z)$. Therefore, we have found a pole of order at least p which can be uniquely associated with the zero of $w(z) - b_1(z)$ at $z = \hat{z}$. If $p = 1$, it may in principle be possible that there is another root of $w(z)-b_1(z)$ or of $w(z)-b_2(z)$ at $z = q^3\hat{z}$, which needs to be paired with the pole of $w(z)$ at $z = q^2\hat{z}$. But since now all of the poles in the iteration are simple, we still pair up the root of $w(z)$ at $z = \hat{z}$ and the pole of $w(z)$ at $z = q\hat{z}$. If there is another root of $w(z)-b_1(z)$ at $z = q^3\hat{z}$ such that $w(q^4\hat{z})$ is finite, we can pair it up with pole of $w(z)$ at $z = q^2\hat{z}$. Thus for any $p \geqslant 1$, there is a pole of order at least p which can be paired up with the root of $w(z) - b_1(z)$ at $z = \hat{z}$. By adding up all points $\hat{z}$ such that (33) holds, similarly, for $b_2(\not\equiv b_1)$, it follows that

$$n\left(r, \tfrac{1}{w-b_1}\right) + n\left(r, \tfrac{1}{w-b_2}\right) \leqslant n(Kr, w) + O(1), \quad (37)$$

with $K = \max\{|q|, 1/|q|\}$. Hence, it satisfies the second condition of Lemma 4, so $\rho(w) > 0$, a contradiction.

Suppose that

$$\deg_w(P) > \deg_w(\hat{Q}) + 2.$$

If $\deg_w(P) = 3$, it follows that $\deg_w(Q) = 2$, and the assertion (4) holds. If

$$4 = \deg_w(P) > \deg_w(\hat{Q}) + 2 = 2, \quad (38)$$

and $\hat{z}$ is a generic root of $w(z) - b_1(z)$ of order p. It follows from (32) that either $w(qz)$ or $w(z/q)$ must have a pole at $z = \hat{z}$ of order at least p, and we suppose as above that $w(qz)$ has the pole at $z = \hat{z}$. Then, it follows that $w(q^2\hat{z})$ has a pole of order $2p$, and $w(q^3\hat{z})$ has a pole of order of $4p$. Hence we can pair the root of $w(z) - b_1(z)$ at $z = \hat{z}$ and the pole of $w(z)$ at $z = q\hat{z}$ the same way as in the case (35). Similarly, for the roots of $w(z) - b_2(z)$, and so (37) holds. By Lemma 4, $\rho(w) > 0$, a contradiction.

Suppose that

$$4 = \deg_w(P) > \deg_w(\hat{Q}) + 2 = 3, \quad (39)$$

and that $\hat{z}$ is a point satisfying (33), and of order p. Since $\deg_w(\hat{Q}) = 1$, without loss of generality, we assume that

$$\hat{Q} = w(z) - b_3(z),$$

where $b_3(z)$ is rational in z. Assume that $b_3(z) \not\equiv b_j(z)$, for $j \in \{1,2\}$. It follows by the assumption $Q(z,w) \not\equiv 0$ that $b_3(z) \not\equiv 0$. As above, it follows by (32) that either $w(qz)$ or $w(z/q)$ has a pole of order at least p at $z = \hat{z}$, and we may again suppose that $w(qz)$ has that pole. If $p > 1$, then (34) implies that $w(q^2z)$ has a pole at $z = \hat{z}$ of order at least p. Even though, $w(z)-b_j(z)$ has a root at $z = q^3\hat{z}$ for some $j \in \{1,2,3\}$, we have found at least one pole for each root of $w(z) - b_1(z)$ in this iteration sequence, counting multiplicities. Therefore we can pair the root of $w(z)-b_1(z)$ at $z=\hat{z}$ and the pole of $w(z)$ at $z=q\hat{z}$ the same way as in (35) and (38). If $p=1$, it may in principle be possible that the pole of right hand side of (34) at $z=\hat{z}$ cancels with the pole of the term $a(qz)qw'(qz)/w(qz)$ at $z=\hat{z}$ in such a way that $w(q^2\hat{z})$ remains finite. If $w(q^2\hat{z})\neq b_j(q^2\hat{z})$ for $j\in\{1,2,3\}$, then it follows from (36) that $w(q^3z)$ has a pole at $z=\hat{z}$, and we can pair up the root of $w(z)-b_1(z)$ at $z=\hat{z}$ and the pole of $w(z)$ at $z=q\hat{z}$. If $w(q^2\hat{z})=b_j(q^2\hat{z})$ for $j\in\{1,2,3\}$, it may happen that $w(q^3\hat{z})$ remains finite. If all points $\hat{z}$ such that $w(\hat{z})=b_j(\hat{z})$ are a part of an iteration sequence of this form, i.e.,

$$w(\hat{z})=b_{j_1}(\hat{z}),\quad w(q\hat{z})=\infty,\quad w(\hat{z})=b_{j_2}(\hat{z}),\qquad j_1,j_2\in\{1,2,3\},$$

by adding up all roots of $w(z)-b_j(z)$, $j\in\{1,2,3\}$, we have

$$n\left(r,\tfrac{1}{w-b_1}\right)+n\left(r,\tfrac{1}{w-b_2}\right)+n\left(r,\tfrac{1}{w-b_3}\right)\leqslant 2n(Kr,w)+O(1),$$

with $K=\max\{|q|,1/|q|\}$. Note that, all of b_1, b_2 and b_3 are not solutions of (32), it follows from Lemma 4 that $\rho(w)>0$, a contradiction.

Case 2: Suppose that the denominator of $R(z,w(z))$ in (3) has at least one non-zero rational root, say $b_1(z)\not\equiv 0$. Then (3) can be written as

$$w(qz)-w(z/q)+a(z)\frac{w'(z)}{w(z)}=\frac{P(z,w(z))}{(w(z)-b_1(z))^n\check{Q}(z,w(z))},\tag{40}$$

where $P(z,w(z))\not\equiv 0$ and $(w(z)-b_1(z))^n\check{Q}(z,w(z))$ are two mutually prime polynomials in $w(z)$ with $\deg_w(P)\leqslant 4$ and $\deg_w((w(z)-b_1(z))^n\check{Q}(z,w(z)))=n+\deg_w(\check{Q})=n+m\leqslant 4$. Then $b_1(z)$ is not a solution of (40), and satisfies the first condition of Lemma 4. Assume that $n\in\{2,3,4\}$, and suppose that $\hat{z}$ is a generic root of $w(z)-b_1(z)$ of order p. Then either $w(qz)$ or $w(z/q)$ has a pole at $z=\hat{z}$ of order at least np, and without loss of generality, we suppose that $w(qz)$ has such that a pole at $z=\hat{z}$. Next, suppose that

$$\deg_w(P)\leqslant n+m.\tag{41}$$

Then $w(q^2\hat{z})$ is a pole of order one, and $w(q^3\hat{z})$ is a pole of order at least np. By continuing the iteration, it follows that $w(q^4\hat{z})$ is again a simple pole or finite value. Thus it may be $w(q^4\hat{z})=b_1(q^4\hat{z})$, and so it is at least in principle possible that $w(q^5\hat{z})$ is a finite value. By adding up all roots of $w(z)-b_1(z)$ and poles of $w(z)$ in the set $\{\hat{z},q\hat{z},\cdots,q^4\hat{z}\}$, and taking into account multiplicities of these points, we find that there at least $2np+1$ poles for $2p$ roots of $w(z)-b_1(z)$. If $w(q^4\hat{z})\neq b_1(q^4\hat{z})$, or $w(q^4\hat{z})$ is a root of $\check{Q}(z,w(z))$ with multiplicity p, then $w(q^5\hat{z})$ is a pole of order np, and we have even more poles for every root of $w(z)-b_1(z)$. By adding up the contribution from all points $z=\hat{z}$ to corresponding counting functions, it follows that

$$n\left(r,\frac{1}{w-b_1}\right)\leqslant\frac{1}{n}n(K^4r,w)+O(1),$$

with $K=\max\{|q|,1/|q|\}$. Then the second condition of Lemma 4 is satisfied, so $\rho(w)>0$, a contradiction.

Suppose that

$$\deg_w(P)>n+m+1.\tag{42}$$

Suppose that $z=\hat{z}$ is a generic root of $w(z)-b_1(z)$ of order p. As in the case (41) either $w(q\hat{z})$ or $w(\hat{z}/q)$, say $w(q\hat{z})$, is a pole of order np at least. This implies that $w(q^2\hat{z})$ is a pole of order np at least, and so, the only way that $w(q^4\hat{z})$ can be finite is that $w(q^3\hat{z})=b_1(q^3\hat{z})$, or $w(q^3\hat{z})$ is a root of $\check{Q}(z,w(z))$, with multiplicity p. This is the “best case”, we have found $2np$ poles, counting multiplicities, that correspond uniquely to $2p$ roots of $w(z)-b_1(z)$. Thus, we have

$$n\left(r,\frac{1}{w-b_1}\right)\leqslant\frac{1}{n}n(K^3r,w)+O(1),$$

with $K=\max\{|q|,1/|q|\}$. Lemma 4 implies that $\rho(w)>0$, a contradiction.

Case 3: Suppose that $Q(z,w)$ in (3) has only one simple root, and assume that

$$\deg_w(P)\geqslant 3.\tag{43}$$

Then the denominator of the right hand side of (3) can be written as $Q(z,w(z))=w(z)-b_1(z)$. Let $z=\hat{z}$ be a generic root of $w(z)-b_1(z)$ of order p. Then, either $w(q\hat{z})$ or $w(\hat{z}/q)$ is a pole of order p at least. Without loss of generality, we assume that $w(q\hat{z})$ is a pole of order p. Then $w(q^2\hat{z})$ is a pole of order $2p$ at least, and $w(q^3\hat{z})$ is a pole of order $4p$ at least, and so on. Then we have

$$n\left(r,\frac{1}{w-b_1}\right)\leqslant\frac{1}{3}n(K^2r,w)+O(1),$$

with $K=\max\{|q|,1/|q|\}$. Lemma 4 implies that $\rho(w)>0$, a contradiction.

Case 4: Suppose that $Q(z,w)$ in (3) has only one simple root, and

$$\deg_w(P)\leqslant 2.\tag{44}$$

If $\deg_w(P)=2$, then $\deg_w(P)=\deg_w(Q)+1$, and thus (4) holds. If $\deg_w(P)\leqslant 1$, then $\deg_w(R)=1$.

Case 5: $R(z,w(z))$ is a polynomial in $w(z)$. Then (3) can be written as

$$w(qz)-w(z/q)+a(z)\frac{w'(z)}{w(z)}=P(z,w(z)),\tag{45}$$

where $\deg_w(P) \leqslant 4$. If $\deg_w(P) = 1$, then (4) holds, and if $\deg_w(P) = 0$, it follows that $R(z, w(z))$ in (3) is a polynomial of degree 0 as asserted. Now, we assume that $\deg_w(P) \geqslant 2$ and suppose that $w(z)$ has either infinitely many zeros or poles(or both). Suppose that there is a pole or a zero of $w(z)$ at $z = \hat{z}$. Note that the coefficients of (45) are rational, we can always choose a zero or pole of $w(z)$ in such way that there is no cancellation with the coefficients. Without loss of generality, suppose that there is a pole of $w(z)$ at $z = q\hat{z}$. By iterating (45), it follows that $w(z)$ has a pole of order $\deg_w(P)$, at least, at $z = q^2\hat{z}$, and a pole of order $(\deg_w(P))^2$ at $z = q^3\hat{z}$, and so on. The string of poles with exponential growth in the multiplicity can not terminate. And note that the coefficients are rational and thus have finitely many zeros, and $w(z)$ has infinitely many zeros or poles, we can choose the starting point $\hat{z}$ of the iteration from outside a sufficiently large disc in such a way that no cancellation occurs. Thus

$$n(K^s|\hat{z}|, w) \geqslant (\deg_w(P))^{s-1},$$

for all $s \in \mathbb{N}$, with $K = \max\{|q|, 1/|q|\}$, and so

$$\begin{aligned}\lambda\left(\frac{1}{w}\right) &= \limsup_{r\to\infty} \frac{\log n(r,w)}{\log r}\\ &\geqslant \limsup_{s\to\infty} \frac{\log n(K^s|\hat{z}|, w)}{\log(K^s|\hat{z}|)}\\ &\geqslant \limsup_{s\to\infty} \frac{\log(\deg_w(P))^{s-1}}{\log(K^s|\hat{z}|)}\\ &= \frac{\log(\deg_w(P))}{\log K} > 0.\end{aligned}$$

Hence, $\rho(w) \geqslant \lambda\left(\frac{1}{w}\right) > 0$, a contradiction.

Suppose that $w(z)$ has finitely many poles and zeros, then $w(z)$ is a rational function, which contradicts with our assumption. □

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