RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2021.S010

ScienceAsia 47S (2021): **130–135**

On some $(C_\lambda, 1)(E_\lambda, q)$ ideal convergent sequence spaces

Ekrem Aljimi[a], **Penpark Sirimark**[b,*]

[a] Faculty of Computer Science, University of "Kadri Zeka", Gjilan 60000 Republic of Kosova
[b] Department of Science and Mathematics, Rajamangala University of Technology Isan Surin Campus, Surin 32000 Thailand

*Corresponding author, e-mail: penpark.si@rmuti.ac.th

Received 30 Nov 2020
Accepted 28 Apr 2021

ABSTRACT: In this paper, we introduce a new $(C_\lambda, 1)(E_\lambda, q)$ ideal convergence and new sequence spaces $C^I_{(C_\lambda,1)(E_\lambda,q)}$, $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$ and $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$. Moreover, we investigate some topological and geometrical properties of spaces $C^I_{(C_\lambda,1)(E_\lambda,q)}$, $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$ and $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$, showing these spaces as linear and paranormed spaces. Furthermore, we give some inclusion relations on $C^I_{(C_\lambda,1)(E_\lambda,q)} \subset (l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$.

KEYWORDS: statistical convergence, ideal convergence, linear spaces, paranormed spaces, $(C_\lambda, 1)(E_\lambda, q)$-summability method, $(C_\lambda, 1)(E_\lambda, q)$ ideal convergence

MSC2010: 40G15 40G05 40A35 40C05

INTRODUCTION

The notion of statistical convergence and some results related to this type of convergence were given by Fast, see [1]. The definition of statistical convergence is given as follows.

Let K be a subset of $\mathbb{N} = \{1, 2, 3, \dots\}$. We define the set $K_n = \{i \in K : i < n\}$. If the limit of the sequence $n^{-1}|\{i \in K : i < n\}| = n^{-1}|K_n|$ exists, we call $d(K) = \lim_n n^{-1}|K_n|$ the asymptotic density, and by $|M|$ we denote the cardinality of the set M. Using the definition of asymptotic density, we can define the statistical convergence of sequence $x = (x_n)$.

A sequence $x = (x_n)$ is statistically convergent to L if $\epsilon > 0$, $\lim_n n^{-1}|\{n \in \mathbb{N} : |x_n - L| \geqslant \epsilon\}| = 0$, and we denote it by st-$\lim x = L$.

Altay and Basar [2] defined Euler sequences spaces e_0^r and e_c^r; they studied properties of these spaces and proved that these spaces are linear spaces as well as proved some other theorems. Savas and Gurdal [3] presented brilliant results regarding statistical convergence in intuitionistic fuzzy norm and examined various problems that arise in this area. Also, applications of statistical convergence in other fields of mathematics can be seen in [4–6].

Connor [7] introduced the concept of p-Cesáro convergence of sequences. Later, weighted Norlund-Euler λ-statistical convergence was defined by Valdete Loku and Ekrem Alimi, see [8]. In 2016, Tuncer and Mohiuddine [9] defined statistical Cesáro-Euler summability method $(C, 1)(E, 1)$ as follows.

$$t_n^{(C,1)(E,1)} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} S_\nu.$$

If $t_n^{(C,1)(E,1)} \to S$ as $n \to \infty$, then we say that the series $\sum_{n=0}^{\infty} x_n$ or sequence $\{S_n\}$ is summable to S by $(C, 1)(E, 1)$ method and it is denoted by $S_n \to S$ by $(C, 1)(E, 1)$.

Remark 1 If we put $S_\nu = 1$ then $(C, 1)(E, 1)$ summability method is reduced to the $(C, 1)$ method.

Definition 1 ([9]) For sequence $x = (x_n)$, we write $t_n^{(C,1)(E,1)} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu$. If the sequence $\left(t_n^{(C,1)(E,1)}\right)$ converges to any number L, i.e. st-$\lim_{n\to\infty} t_n^{(C,1)(E,1)} = L$, then $x = (x_k)$ is statistically summable $(C, 1)(E, 1)$ and we write that as $L = (CE)_1(\text{st})\text{-}\lim x$.

Definition 2 ([10]) A family $I \subset 2^X$ of subsets of non-empty set X is called an ideal in X if I is hereditary (i.e. $A \in I$, $B \subset A$ implies $B \in I$) and additive (i.e. $A, B \in I$ implies $A \cup B \in I$).

The ideal I of X is proper if $I \neq 2^X$.

The proper ideal $I \subset 2^X$ is an admissible ideal in X if it contains $\{\{y\} : y \in X\}$.

A proper ideal I is maximal if there is no proper ideal $J \neq I$ containing I as a subset.

Definition 3 ([11, 12]) Let $I \subset 2^X$ denote an ideal on X. The family of sets $F(I) \subset 2^X$ that is not non-empty is called a filter on X corresponding to I if
(i) $\phi \in F(I)$,
(ii) $A, B \in F(I)$ implies $A \cap B \in F(I)$,
(iii) $A \in F(I)$ and $B \subset A$ implies $B \in F(I)$.
For each ideal I, there is a filter $F(I)$ corresponding to I. A non-empty family of sets $F(I)$ is called a filter according to the ideal I that can be written by

$$F(I) = \{K \subset 2^X : K^C \in I\}.$$

Here, $K^C = X \backslash K = X - K$.

Definition 4 ([10, 13–17]) A sequence $x = (x_n) \in \omega$ is said to be ideally convergent or I-convergent to a number L if for all $\epsilon > 0$

$$A(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geqslant \epsilon\} \in I$$

and it is represented by $I\text{-}\lim_n x_n = L$.

The space of all I-convergent sequences to L is indicated by C^I which can be written as

$$C^I = \{x = (x_n) \in \omega : \{n \in \mathbb{N} : |x_n - L| \geqslant \epsilon\} \in I\}.$$

Definition 5 ([10, 13, 17]) A sequence $x = (x_n) \in \omega$ is said to be I-null sequence if $I\text{-}\lim_n x_n = 0$. The space of all I-null sequences is defined by C_0^I, where $C_0^I = \{x = (x_n) \in \omega : \{n \in \mathbb{N} : |x_n| \geqslant \epsilon\} \in I\}$.

Definition 6 ([10]) The sequence $x = (x_n) \in \omega$ is said to be I-bounded if there exists a real constant $M > 0$, such that

$$l_\infty^I = \{n \in \mathbb{N} : |x_n| > M\} \in I.$$

Definition 7 ([11, 18, 19]) Let X be a linear space. A function $g : X \to \mathbb{R}$ is called a paranormed space if, for all $x, y \in X$, the following axioms are satisfied
(i) $g(x) = 0$ if $x = 0$,
(ii) $g(-x) = g(x)$,
(iii) $g(x + y) \leqslant g(x) + g(y)$,
(iv) If (μ_n) is sequence of scalars with $\mu_n \to \mu$ as $n \to \infty$ and $x_n, L \in X$ with $x_n \to L$ as $n \to \infty$ then $g(\mu_n x_n - \mu L) \to 0$ as $n \to \infty$.

Definition 8 ([17]) A sequence space X is said to be solid or normal if $x = (x_n) \in X$ implies that $\alpha x = (\alpha_n x_n) \in X$ for all sequence of scalars $\alpha = (\alpha_n)$ with $|\alpha_n| < 1$ for all $n \in \mathbb{N}$.

Definition 9 ([18]) Let $K = \{k_1 < k_2 < \ldots\} \subset \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_n) \in E\}$. A canonical preimage of a sequence $x_{k_n} \in \lambda_K^E$ is a sequence $y = (y_n) \in \omega$ defined as

$$y_n = \begin{cases} x_n, & \text{if} \quad n \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of step space λ_K^E is a set of canonical preimages of all elements in λ_K^E if and only if it is a canonical preimage of some $x \in \lambda_K^E$.

Definition 10 ([11, 18]) A sequence space X is said to be monotone if it contains the canonical preimages of all its step-spaces.

Lemma 1 ([20, 21]) *Solidness of sequence space X implies the monotonicity of X.*

MAIN RESULTS

Weighted $(C_\lambda, 1)(E_\lambda, q)$ ideal convergence

In the following, we will generalize $(C, 1)(E, 1)$-summability method, one can read [9] for more details. We call this new method generalized Cesáro-Euler summability method, which is denoted by $(C_\lambda, 1)(E_\lambda, q)$.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leqslant \lambda_n + 1, \qquad \lambda_1 = 1.$$

The collection such sequences can be designated by Δ. The new $(C_\lambda, 1)(E_\lambda, q)$ summability method is defined as

$$t_n^{(C_\lambda,1)(E_\lambda,q)} = \frac{1}{n+1} \sum_{k \in I_n} \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} S_\nu,$$

where $I_n = [n - \lambda_n + 1, n]$. If $t_n^{(C_\lambda,1)(E_\lambda,q)} \to S$ as $n \to \infty$, then we say that the series $\sum_{n=0}^{\infty} x_n$ or sequence $\{S_n\}$ is summable to S by $(C_\lambda, 1)(E_\lambda, q)$ method and it is denoted by $S_n \to S$ by $(C_\lambda, 1)(E_\lambda, q)$.

Remark 2 If we put $\lambda_n = n$ in $(C_\lambda, 1)(E_\lambda, q)$ we get $(C, 1)(E, q)$ summability method.

Remark 3 If we put $q = 1$ in $(C, 1)(E, q)$, then $(C, 1)(E, q)$ summability method is reduced to $(C, 1)(E, 1)$ summability method [9].

Definition 11 A sequence $x = (x_k)$ is said to be $(C_\lambda, 1)(E_\lambda, q)$ summable to L if

$$\lim_{n\to\infty} t_n^{(C_\lambda,1)(E_\lambda,q)} = L.$$

Definition 12 A sequence $x=(x_k)$ is said to be strongly $(C_\lambda,1)(E_\lambda,q)$ summable to L if

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|=0.$$

In this case, the sequence x_k can be written by $x_k\to L$ by $[(C_\lambda,1)(E_\lambda,q)]$ where $[(C_\lambda,1)(E_\lambda,q)]$ denotes the set of all strongly $(C_\lambda,1)(E_\lambda,q)$-summable sequences.

Definition 13 A sequence $x=(x_k)$ is said to be statistically summable $(C_\lambda,1)(E_\lambda,q)$ to L if st-lim $t_n^{(C_\lambda,1)(E_\lambda,q)}=L$. In this case, we can write in the form $(CE)_\lambda$(st)-lim $x=L$.

Definition 14 A sequence $x=(x_k)$ is called weighted $(C_\lambda,1)(E_\lambda,q)$ statistically convergent to L if for each $\epsilon>0$, such that

$$\lim_{n\to\infty}\left|\left\{k\leqslant(n+1):\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\times\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\geqslant\epsilon\right\}\right|=0.$$

Definition 15 The $(C_\lambda,1)(E_\lambda,q)$ mean is defined by the matrix $(CE)^t=(a^t_{nk})$ as

$$a^t_{nk}=\begin{cases}\frac{1}{n+1}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}, & 0\leqslant k\leqslant n\\ 0, & k>n.\end{cases}$$

for every $k,n\in\mathbb{N}$.

The motivation of this work comes from the results of Tug [12] and Tug and Başar [22]. In the current work, we give some new results about sequence spaces which are related to $(C_\lambda,1)(E_\lambda,q)$ summability method. We denote by: $C^I_{(C_\lambda,1)(E_\lambda,q)}$, weighted $(C_\lambda,1)(E_\lambda,q)$ I-convergent sequences spaces; $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$, weighted $(C_\lambda,1)(E_\lambda,q)$ I-null sequences spaces; and $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$, weighted $(C_\lambda,1)(E_\lambda,q)$ I-bounded sequences spaces. For spaces $C^I_{(C_\lambda,1)(E_\lambda,q)}$, $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$, and $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$, we give some topological and geometrical properties as well as prove some results and relations related to these spaces.

Definition 16 A sequence $x=(x_n)$ is said to be weighted $(C_\lambda,1)(E_\lambda,q)$ ideally convergent if for every $\epsilon>0$,

$$N(\epsilon)=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\times\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\geqslant\epsilon\right\}\in I.$$

Definition 17 The sets of all weighted $(C_\lambda,1)(E_\lambda,q)$ I-convergent sequence spaces are defined as

$$C^I_{(C_\lambda,1)(E_\lambda,q)}=\left\{x=(x_n)\in\omega:\left\{n\in\mathbb{N}:\frac{1}{n+1}\times\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\geqslant\epsilon\right\}\in I\right\}.$$

Definition 18 The sets of all weighted $(C_\lambda,1)(E_\lambda,q)$ I-null sequence spaces are defined as

$$(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}=\left\{x=(x_n)\in\omega:\frac{1}{n+1}\times\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v|\geqslant\epsilon\in I\right\}.$$

Definition 19 The sets of all weighted $(C_\lambda,1)(E_\lambda,q)$ I-bounded sequence spaces are defined as

$$(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}=\left\{x=(x_n)\in\omega:\left\{n\in\mathbb{N}:\exists M>0,\ \frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v|\geqslant M\right\}\in I\right\}.$$

Theorem 1 *The spaces* $C^I_{(C_\lambda,1)(E_\lambda,q)}$, $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$, *and* $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$ *are linear spaces.*

Proof: To prove that the space $C^I_{(C_\lambda,1)(E_\lambda,q)}$ is linear, we let $x=(x_n)\in C^I_{(C_\lambda,1)(E_\lambda,q)}$, $y=(y_n)\in C^I_{(C_\lambda,1)(E_\lambda,q)}$ and $\alpha,\beta\in\mathbb{C}$ be as given. For every $\epsilon>0$ we denote

$$A(\epsilon)=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\times\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L_1|\geqslant\frac{\epsilon}{2}\right\}\in I$$

$$B(\epsilon)=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\times\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L_2|\geqslant\frac{\epsilon}{2}\right\}\in I$$

for some $L_1,\ L_2 \in \mathbb{C}$. Now, we write the following inequalities.

$$\begin{aligned}
&\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|(\alpha x_\nu+\beta y_\nu)-(\alpha L_1+\beta L_2)|\\
&\leqslant \frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}(|\alpha||x_\nu-L_1|+|\beta||y_\nu-L_2|)\\
&\qquad\leqslant |\alpha|\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu-L_1|\\
&\qquad\qquad+|\beta|\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu-L_2|.
\end{aligned}$$

Then, using the above inequality, we obtain

$$\begin{aligned}
&\Big\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\times\\
&\qquad\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|(\alpha x_\nu+\beta y_\nu)-(\alpha L_1+\beta L_2)|\geqslant\epsilon\Big\}\\
&=\Big\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\times\\
&\qquad\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|\alpha(x_\nu-L_1)-\beta(y_\nu-L_2)|\geqslant\epsilon\Big\}\\
&\subseteq\Big\{n\in\mathbb{N}:\frac{|\alpha|}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu-L_1|\geqslant\frac{\epsilon}{2}\Big\}\\
&\cup\Big\{n\in\mathbb{N}:\frac{|\beta|}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu-L_2|\geqslant\frac{\epsilon}{2}\Big\}\\
&\hfill\subseteq A(\epsilon)\cup B(\epsilon)\in I.
\end{aligned}$$

This proves that space $C^I_{(C_\lambda,1)(E_\lambda,q)}$ is linear. In a completely analogous way, we can prove that the spaces $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$ and $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$ are also linear. □

Theorem 2 *The spaces* $C^I_{(C_\lambda,1)(E_\lambda,q)}$, $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$ *and* $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$ *are paranormed spaces with the paranorm*

$$g(x)=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|.$$

Proof: We will prove only for the space $C^I_{(C_\lambda,1)(E_\lambda,q)}$. It is trivial that if $x=(x_k)=0$, then $g(x)=0$. For $x=(x_k)\neq 0$, then $g(x)\neq 0$, we have that
(i) For all $x\in C^I_{(C_\lambda,1)(E_\lambda,q)}$

$$g(x)=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|\geqslant 0.$$

(ii) For all $x\in C^I_{(C_\lambda,1)(E_\lambda,q)}$

$$\begin{aligned}
g(-x)&=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|-x_\nu|\\
&=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|=g(x).
\end{aligned}$$

(iii) For every $x,y\in C^I_{(C_\lambda,1)(E_\lambda,q)}$

$$\begin{aligned}
g(x+y)&=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu+y_\nu|\\
&\leqslant\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|\\
&\quad+\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|y_\nu|\\
&=g(x)+g(y).
\end{aligned}$$

Let (μ_n) be a sequence of scalars with $\mu_n\to\mu$ as $n\to\infty$ and $x_n\in C^I_{(C_\lambda,1)(E_\lambda,q)}$ such that

$$\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|\to L$$

when $n\to\infty$ in the sense that

$$g\Big(\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|\to L\Big)\to 0$$

when $n\to\infty$. Therefore,

$$\begin{aligned}
&\Big(\mu_n\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|-\mu L\Big)\\
&\leqslant g\Big(\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|(\mu_n-\mu)\Big)\\
&\quad+g\Big(\mu\Big(\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|-L\Big)\Big).
\end{aligned}$$

It is then clear that

$$\mu_n\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu}|x_\nu|\to\mu L$$

as $n\to\infty$. The proof is complete. The proof for spaces $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$ and $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$ is similar to the proof for the space $C^I_{(C_\lambda,1)(E_\lambda,q)}$. □

Theorem 3 *The space* $C^I_{(C_\lambda,1)(E_\lambda,q)}$ *is solid and monotone.*

Proof: Suppose that $x=(x_k)\in C^I_{(C_\lambda,1)(E_\lambda,q)}$ and (α_k) is a sequence of scalars with $|\alpha_k|\leqslant 1$ for all $k\in\mathbb{N}$. We then notice that

$$\begin{aligned}\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|\alpha_v x_v| &\leqslant \frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|\alpha_v||x_v|\\ &\leqslant \frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v|.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|\alpha_v x_v|\geqslant\epsilon\right\}\\ &\subseteq\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v|\geqslant\epsilon\right\}\end{aligned} \tag{1}$$

By using (1), we obtain $(\alpha_k x_k)\in C^I_{(C_\lambda,1)(E_\lambda,q)}$. The proof is complete. □

Theorem 4 *The space $C^I_{(C_\lambda,1)(E_\lambda,q)}$ is a closed subset of $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$.*

Proof: We begin by taking a Cauchy sequence $x^{(n)}_k\in C^I_{(C_\lambda,1)(E_\lambda,q)}$ such that $x^{(n)}_k\to x$ as $n\to\infty$. We need to show that $x\in C^I_{(C_\lambda,1)(E_\lambda,q)}$. Since $x^{(n)}_k\in C^I_{(C_\lambda,1)(E_\lambda,q)}$ then there exists a sequence of complex number α_k such that

$$A=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\times\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-\alpha_k|\geqslant\epsilon\right\}\in I. \tag{2}$$

To prove we mention that $\alpha_n\to x$ as $n\to\infty$ and $(A')^C\in I$ whenever

$$A'=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-\alpha|\geqslant\epsilon\right\}.$$

Since $x^{(n)}_k$ is a Cauchy sequence in $C^I_{(C_\lambda,1)(E_\lambda,q)}$, then for a given $\epsilon>0$, there exists $k_0\in\mathbb{N}$ such that

$$\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-x^{(l)}_v|<\frac{\epsilon}{3}$$

for all $m,n\geqslant k_0$.

Let us define the following sets for $\epsilon>0$.

$$A_1=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-x^{(l)}_v|<\frac{\epsilon}{3}\right\}$$

$$A_2=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-\alpha_k|<\frac{\epsilon}{3}\right\}$$

$$A_3=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-\alpha_l|<\frac{\epsilon}{3}\right\}.$$

For all $m,n\geqslant k_0$, whenever $A^C_1, A^C_2, A^C_3\in I$, we have

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|\alpha_k-\alpha_l|<\epsilon\right\}\\ &\supseteq\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-x^{(l)}_v|<\frac{\epsilon}{3}\right\}\\ &\cap\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-\alpha_k|<\frac{\epsilon}{3}\right\}\\ &\cap\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-\alpha_l|<\frac{\epsilon}{3}\right\}.\end{aligned}$$

We can see that (a_k) is a Cauchy sequence in $\mathbb{C}$ and converge to the scalar a as $n\to\infty$. Now, let take $0<\delta<1$, we need to show that if

$$A'=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-\alpha|<\delta\right\}$$

then $(A')^C\in I$. Since

$$\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-x_v|\to 0$$

as $n\to\infty$, then there exists $n_0\in\mathbb{N}$ such that

$$E_1=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-x_v|<\frac{\delta}{3}\right\}.$$

Which implies that $(E_1)^C\in I$ for all $n\geqslant n_0$. And we already have from the first part that

$$E_2=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|\alpha_v-\alpha|<\frac{\delta}{3}\right\}.$$

Which gives us $(E_2)^C\in I$ for all $n\geqslant n_0$. Since the set $A\in I$ defined as in (2) number δ instead of ϵ, then we have a subset $E_3\subset\mathbb{N}$ such that $(E_3)^C\in I$ whenever

$$E_3=\left\{n\in\mathbb{N}:\frac{1}{n+1}\sum_{k\in I_n}\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x^{(k)}_v-\alpha_l|<\frac{\delta}{3}\right\}.$$

Consequently, we may easily say that $(A')^C\supseteq E_1\cap E_2\cap E_3$. And by the definition of filter on the ideal, we can say that $C^I_{(C_\lambda,1)(E_\lambda,q)}\subset(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$. The proof is complete. □

CONCLUSION

We have generalized $(C,1)(E,1)$ summability method and we have termed this new generalized method as Cesáro-Euler λ-summability method, which is symbolically denoted by $(C_\lambda,1)(E_\lambda,q)$. We have given the definition of weighted $(C_\lambda,1)(E_\lambda,q)$ ideal convergent and definition of sequence spaces $C^I_{(C_\lambda,1)(E_\lambda,q)}$, $(C_0)^I_{(C_\lambda,1)(E_\lambda,q)}$ and $(l_\infty)^I_{(C_\lambda,1)(E_\lambda,q)}$, for which we have proved some of their topological and geometrical properties to make a modest contribution in the field of I-convergence. Furthermore, the results in this study could open a new direction of proving some new results on the space of $(C_\lambda,1)(E_\lambda,q)$ null and $(C_\lambda,1)(E_\lambda,q)$ convergent sequences, which will be the subject of future study.

Acknowledgements: The authors would like to acknowledge the University of "Kadri Zeka" and Rajamangala University of Technology Isan Surin Campus for providing the library and computer facilities necessary for conducting this research.

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