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Normality of meromorphic functions and their differential polynomials

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ABSTRACT: In this paper, we study the normality of meromorphic families and prove the following theorem: Let k be a positive integer, $P(z)$ be a non-constant polynomial satisfying $P(0)=0$, $h(\not\equiv 0)$ be a holomorphic function in a domain D, $H(f,f',\dots,f^{(k)})$ be a differential polynomial with $\frac{\Gamma}{\gamma}|_H < k+1$, and $\mathscr{F}$ be a meromorphic family in D. If, for each $f\in\mathscr{F}$, $f\neq 0$ and $P(f^{(k)})+H(f,f',\dots,f^{(k)})\neq h$ for $z\in D$, then $\mathscr{F}$ is a normal family in D.

KEYWORDS: meromorphic functions, differential polynomials, normality

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INTRODUCTION AND MAIN RESULTS

In this paper, we suppose the reader is acquainted with standard symbols and primary results on Nevanlinna theory [1, 2].

At first, we give some definitions about differential monomial and differential polynomial.

Definition 1 Let f be a meromorphic function in domain D, and n_i be positive integers for all $i\in\{0,1,\dots,k\}$. We say that $M(f,f',\dots,f^{(k)})$ is a differential monomial of f, if

$$M(f,f',\dots,f^{(k)}) = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k},$$

where the degree of $M(f,f',\dots,f^{(k)})$ is $\gamma_M = n_0+n_1+\cdots+n_k$, and the weight of $M(f,f',\dots,f^{(k)})$ is $\Gamma_M = n_0+2n_1+\cdots+(k+1)n_k$.

Definition 2 Let $M_1(f,f',\dots,f^{(k)}), M_2(f,\dots,f^{(k)}), \dots, M_n(f,\dots,f^{(k)})$ be differential monomials of f, and $a_1(z),a_2(z),\dots,a_n(z)$ analytic in D. Then $H(f,f',\dots,f^{(k)})$ is called a differential polynomial of f, if

$$\begin{aligned}&H(f,f',\dots,f^{(k)})\\&= a_1(z)M_1(f,\dots,f^{(k)})+\cdots+a_n(z)M_n(f,\dots,f^{(k)}),\end{aligned}$$

where the degree of $H(f,f',\dots,f^{(k)})$ is $\gamma_H = \max\{\gamma_{M_1},\gamma_{M_2},\dots,\gamma_{M_n}\}$, and the weight of $H(f,f',\dots,f^{(k)})$ is $\Gamma_H = \max\{\Gamma_{M_1},\Gamma_{M_2},\dots,\Gamma_{M_n}\}$. If $\gamma_{M_1}=\gamma_{M_2}=\cdots=\gamma_{M_n}=m$, then $H(f,f',\dots,f^{(k)})$ is called the homogeneous differential polynomial of degree m. Set

$$\frac{\Gamma}{\gamma}\Big|_H = \max\left\{\frac{\Gamma_{M_1}}{\gamma_{M_1}},\frac{\Gamma_{M_2}}{\gamma_{M_2}},\dots,\frac{\Gamma_{M_n}}{\gamma_{M_n}}\right\}.$$

In 1959, Hayman [3] proved the following result.

Theorem 1 *Let k be a positive integer and f be a nonconstant meromorphic function in $\mathbb{C}$. Then f or $f^{(k)}-1$ has at least a zero. Moreover, if f is transcendental, then f or $f^{(k)}-1$ has infinitely many zeros.*

The normality corresponding to Theorem 1 was conjectured by Hayman [4] in 1967, and confirmed by Gu [5] in 1979.

Theorem 2 *Let k be a positive integer and let $\mathscr{F}$ be a meromorphic family in a domain D. If, for each $f\in\mathscr{F}$, $f\neq 0$ and $f^{(k)}\neq 1$ for $z\in D$, then $\mathscr{F}$ is a normal family in D.*

In 1986, Yang [6] extended Theorem 2 as follows.

Theorem 3 *Let k be a positive integer, $h(\not\equiv 0)$ be a holomorphic function in a domain D, and $\mathscr{F}$ be a* *meromorphic family in D. If for each $f \in \mathscr{F}$, $f \neq 0$ and $f^{(k)} \neq h$ for $z \in D$, then $\mathscr{F}$ is a normal family in D.*

In 1993, Yang [6] replaced $f^{(k)}$ in Theorem 2 by a linear differential polynomial, and proved the following result.

Theorem 4 *Let k be a positive integer, $a_1(z), \ldots, a_k(z)$ be holomorphic functions in a domain D, and $\mathscr{F}$ be a meromorphic family in D. If for each $f \in \mathscr{F}$, $f \neq 0$ and $f^{(k)}(z) + a_1(z) f^{(k-1)} + \cdots + a_k(z) f(z) \neq 1$ for $z \in D$, then $\mathscr{F}$ is a normal family in D.*

In 1991, Gu [7] considered the situation of homogeneous differential polynomial with constant coefficient, and proved the following result.

Theorem 5 *Let $k, q \geqslant 3$ be two positive integers, $H(f, f', \ldots, f^{(k)}) = a_1 M_1(f, f', \ldots, f^{(k)}) + \cdots + a_n M_n(f, f', \ldots, f^{(k)})$ be a homogeneous differential polynomial of degree q with constant coefficient, and for each $i \in \{1, \ldots, n\}$, the degree of $f^{(k)}$ in $M_i(f, f', \ldots, f^{(k)})$ be $\leqslant q-2$, and let $\mathscr{F}$ be a meromorphic family in a domain D. If for each $f \in \mathscr{F}$, $f \neq 0$ and $(f^{(k)})^q + H(f, f', \ldots, f^{(k)}) \neq 1$ for $z \in D$, then $\mathscr{F}$ is a normal family in D.*

In this paper, we improve Theorem 5 as follows.

Theorem 6 *Let k be a positive integer, $P(z)$ be a non-constant polynomial satisfying $P(0) = 0$, $h(\not\equiv 0)$ be a holomorphic function in a domain D, $H(f, f', \ldots, f^{(k)})$ be a differential polynomial with $\frac{\Gamma}{\gamma}|_H < k+1$, and $\mathscr{F}$ be a meromorphic family in D. If for each $f \in \mathscr{F}$, $f \neq 0$ and $P(f^{(k)}) + H(f, f', \ldots, f^{(k)}) \neq h$ for $z \in D$, then $\mathscr{F}$ is a normal family in D.*

The following example shows that the condition $P(0) = 0$ is necessary.

Example 1 Suppose that $\mathscr{F} = \{f_n = 1/nz\}$, $P(z) = z^2 + 1$, and $D = \{z : |z| < 1\}$. Then, for each $f_n \in \mathscr{F}$, $f_n \neq 0$ and $P(f_n^{(k)}) = (k!)^2/n^2 z^{2k+2} + 1 \neq 1$ for $z \in D$, but $\mathscr{F}$ is not a normal family in D.

In 2000, Fang and Hong [8] proved the following result.

Theorem 7 *Let $k, q \geqslant 2$ be two positive integers, $H(f, f', \ldots, f^{(k)})$ be a differential polynomial with $\frac{\Gamma}{\gamma}|_H < k+1$, and $\mathscr{F}$ be a meromorphic family in a domain D. If for each $f \in \mathscr{F}$, the multiplicity of zeros of f are at least $k+1$ and $(f^{(k)})^q + H(f, f', \ldots, f^{(k)}) \neq 1$ for $z \in D$, then $\mathscr{F}$ is a normal family in D.*

In this paper, we improve Theorem 7 slightly as follows.

Theorem 8 *Let k, q be two positive integers, $P(z)$ be a non-constant polynomial satisfying $P(z) \not\equiv -(1-z)^q + 1$ and $P(0) = 0$, $H(f, f', \ldots, f^{(k)})$ be a differential polynomial with $\frac{\Gamma}{\gamma}|_H < k+1$, and $\mathscr{F}$ be a meromorphic family in a domain D. If for each $f \in \mathscr{F}$, the multiplicity of zeros of f are at least $k+1$ and $P(f^{(k)}) + H(f, f', \ldots, f^{(k)}) \neq 1$ for $z \in D$, then $\mathscr{F}$ is a normal family in D.*

The following example shows that the condition $P(z) \not\equiv -(1-z)^q + 1$ is necessary.

Example 2 Suppose that $\mathscr{F} = \{f_n = \frac{1}{k!}\frac{z^{k+1}}{z+1/n}\}$, $P(z) = -(1-z)^q + 1$ with any positive integer q, and $D = \{z : |z| < 1\}$. Then, for each $f_n \in \mathscr{F}$, the zeros of f_n are multiplicity $\geqslant k+1$, $f_n^{(k)} = 1 - \frac{1}{(nz+1)^{k+1}} \neq 1$, and $P(f_n^{(k)}) = -\frac{1}{(nz+1)^{qk+q}} + 1 \neq 1$ for $z \in D$, but $\mathscr{F}$ is not a normal family in D.

LEMMAS

In order to prove our results, we need some lemmas as follows.

Lemma 1 ([9]) *Let $\alpha \in \mathbb{R}$ satisfy $-1 < \alpha < \infty$ and $\mathscr{F}$ be a meromorphic family in a domain D. If $\mathscr{F}$ is not a normal family at $z_0 \in D$, then there exist points $z_n (\in D) \to z_0$, functions $f_n \in \mathscr{F}$, and positive numbers $\rho_n \to 0^+$, such that $g_n(\xi) = \rho_n^{-\alpha} f(z_n + \rho_n \xi)$ converges locally spherically uniformly in $\mathbb{C}$ to a non-constant meromorphic function $g(\xi)$, and moreover, g is of order at most two.*

Lemma 2 *Let f be a meromorphic function, k be a positive integer, and $P(z)$ be a non-constant polynomial with $P(0) = 0$. If $f \neq 0$ and $P(f^{(k)}) \neq 1$, then f must be a constant.*

Proof: Since $P(z)$ is a non-constant polynomial and $P(0) = 0$, there exist a point $\omega (\neq 0) \in \mathbb{C}$ such that $P(\omega) - 1 = 0$, and then we have $f^{(k)}(z) \neq \omega$ by $P(f^{(k)}) \neq 1$. It follows from Hayman's inequality and $f(z) \neq 0$ that $T(r, f) \leqslant S(r, f)$. That is $f(z) \equiv C$, where C is a constant. □

Lemma 3 ([10]) *Let k be a positive integer, let $q(z)$ and $p(z)$ are two coprime polynomials with $\deg q(z) < \deg p(z)$, and let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + \frac{q(z)}{p(z)}$, where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$. If $f^{(k)}(z) \neq 1$, then we have*

(i) $n = k$, and $k! a_k = 1$;

(ii) $f(z)=\frac{1}{k!}z^k+\ldots+a_0+\frac{1}{(az+b)^m}$;

(iii) *If the zeros of* $f(z)$ *are of order at least* $k+1$, *then* $m=1$ *in (ii) and* $f(z)=\frac{(cz+d)^{k+1}}{az+b}$, *where* $c(\neq 0), d$ *are constants.*

Lemma 4 ([11]) *Let* f *be a transcendental meromorphic function in the complex plane,* $k\geqslant 1$ *be an integer, and* $\varepsilon>0$. *Then we have*

$$(1-\varepsilon)T(r,f)\leqslant \overline{N}(r,\tfrac{1}{f})+N(r,\tfrac{1}{f^{(k)}-1})+S(r,f). \quad (1)$$

Lemma 5 *Let* k,q *be two positive integers, let* $P(z)$ *be a non-constant polynomial with* $P(0)=0$, *and let* $f(z)$ *be a meromorphic function with finite order. If the zeros of* $f(z)$ *are of order* $\geqslant k+1$ *and* $P(f^{(k)})\neq 1$, *then*

(I) $f(z)\equiv C$, *where* C *is a constant, if* $P(z)\not\equiv -(1-z)^q+1$;

(II) $f(z)\equiv C$ *or* $f(z)\equiv \frac{(cz+d)^{k+1}}{az+b}$, *if* $P(z)\equiv -(1-z)^q+1$.

Proof: Since $P(z)$ is a non-constant polynomial and $P(0)=0$, there exists a point $\omega(\neq 0)\in\mathbb{C}$ such that $P(\omega)-1=0$. Without loss of generality we suppose that $\omega=1$. It is obvious that $f^{(k)}(z)\neq 1$ by $P(f^{(k)}(z))\neq 1$.

We claim that $f(z)$ is a rational function. Presume that $f(z)$ is a transcendental meromorphic function of finite order. Then by Lemma 4, setting $\varepsilon=1/2$ in (1) and taking into consideration of the zeros of $f(z)$ are multiplicity $\geqslant k+1\geqslant 3$, we obtain that

$$T(r,f)\leqslant 6N\left(r,\tfrac{1}{f^{(k)}-1}\right)+S(r,f). \quad (2)$$

Clearly, (2) is a contradiction of $f^{(k)}(z)\neq 1$. That is $f(z)$ is a rational function. Now we consider two cases.

Case 1 Suppose that $f(z)$ is a rational function $q(z)/p(z)$, where $q(z)$ and $p(z)$ are coprime polynomial with $\deg p(z)>0$. Then by Lemma 3 we have $f(z)=q(z)/p(z)=(cz+d)^{k+1}/(az+b)$. Hence $f^{(k)}(z)=1+(-1)^k k!a^k/(az+b)^{k+1}$, then $P(z)\equiv -(1-z)^q+1$ in this case.

In fact, if $P(z)\not\equiv -(1-z)^q+1$, then $P(z)\equiv -(1-z)^q+1+\varphi(z)$, where $\varphi(z)$ is a polynomial with $\varphi(0)=0$ and $\varphi(z)\not\equiv(1-z)^q-(1-z)^m$, and m is a nonnegative integer.

We claim that there exists a zero $\alpha(\neq 1)$ such that $P(\alpha)-1=-(1-\alpha)^q+\varphi(\alpha)=0$. Otherwise, $P(z)-1=0$ has only one zero $z=1$ that is $-(1-z)^q+\varphi(z)=A(1-z)^m$, where $A\neq 0$ is a constant and m is a positive integer. Then we have $\varphi(z)=A(1-z)^m+(1-z)^q$. By $\varphi(0)=A+1=0$ we have $A=-1$, so $\varphi(z)=-(1-z)^m+(1-z)^q$, a contradiction.

Hence there exists a point $\alpha(\neq 1)$ such that $P(\alpha)-1=0$. By $\alpha\neq 1$ we know that

$$f^{(k)}(z)-\alpha=1+\frac{(-1)^k k!a^k}{(az+b)^{k+1}}-\alpha=0$$

has solutions, which contradicts $P(f^{(k)})\neq 1$.

Case 2 Suppose that $f(z)$ is a non-constant polynomial, then $\deg f\geqslant k+1$, hence $f^{(k)}(z)$ is a polynomial with $\deg f^{(k)}\geqslant 1$. Therefore $f^{(k)}=1$ has solutions, which contradicts $f^{(k)}(z)\neq 1$. Hence $f(z)$ is a constant. □

THE PROOF OF Theorem 6

Proof: At first we explain that $\mathscr{F}$ is normal in the set $D'=\{z\in D: h(z)\neq 0\}$. Presume that $\mathscr{F}$ is not normal at $z_0\in D'$. Without loss of generality we may assume that $h(z_0)=1$. By Lemma 1, there exist $f_n\in\mathscr{F}$, $z_n\to z_0$ and $\rho_n\to 0^+$ such that $g_n(\xi)=\rho_n^{-k}f_n(z_n+\rho_n\xi)$ converges locally spherically uniformly in $\mathbb{C}$ to a nonconstant meromorphic function $g(\xi)$. And by Hurwitz's theorem we know that $g(\xi)\neq 0$. As

$$\begin{aligned}P(g_n^{(k)}(\xi))-1&=P(f_n^{(k)}(z_n+\rho_n\xi))-1\\&=P(f_n^{(k)}(z_n+\rho_n\xi))\\&\quad+H(f_n(z_n+\rho_n\xi),\ldots,f_n^{(k)}(z_n+\rho_n\xi))-1\\&\quad-H(f_n(z_n+\rho_n\xi),\ldots,f_n^{(k)}(z_n+\rho_n\xi)),\end{aligned}$$

and

$$\begin{aligned}&H(f_n(z_n+\rho_n\xi),\ldots,f_n^{(k)}(z_n+\rho_n\xi))\\&=\sum_{i=1}^m a_i(z_n+\rho_n\xi)M_i(f_n(z_n+\rho_n\xi),\ldots,f_n^{(k)}(z_n+\rho_n\xi))\\&=\sum_{i=1}^m a_i(z_n+\rho_n\xi)\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}M_i(g_n(\xi),\ldots,g_n^{(k)}(\xi)).\end{aligned}$$

Considering $a_i(z)$ are analytical on D for $i=1,2,\ldots,m$, we deduce that

$$|a_i(z_n+\rho_n\xi)|\leqslant M\left(\tfrac{1+r}{2},a_i(z)\right)<\infty$$

for sufficiently large n. Hence we come to the conclusion from $\frac{\Gamma}{\gamma}|_H<k+1$ that

$$\lim_{n\to\infty}\sum_{i=1}^m a_i(z_n+\rho_n\xi)\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}M_i(g_n(\xi),\ldots,g_n^{(k)}(\xi))=0.$$

Thus we know that

$$P\left[g^{(k)}(\xi)\right]-1=\lim_{n\to\infty}\left\{P\left[g_n^{(k)}(\xi)\right]+\sum_{i=1}^{m}a_i(z_n+\rho_n\xi)\right.$$
$$\left.\times\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}M_i(g_n(\xi),\dots,g_n^{(k)}(\xi))\right\}-h(z_0)$$
$$=\lim_{n\to\infty}\left\{P\left[f_n^{(k)}(z_n+\rho_n\xi)\right]\right.$$
$$\left.+H(f_n(z_n+\rho_n\xi),\dots,f_n^{(k)}(z_n+\rho_n\xi))-h(z_n+\rho_n\xi)\right\}.$$

Since $P(f^{(k)}(z))+H(f(z),\dots,f^{(k)}(z))-h(z)\neq 0$, by Hurwitz's theorem we deduce that $P(g^{(k)}(\xi))\equiv 1$ or $P(g^{(k)}(\xi))\neq 1$. If $P(g^{(k)}(\xi))\equiv 1$, then there exists a value $\omega\in\mathbb{C}$ such that $g^{(k)}(\xi)\equiv\omega$ and $P(\omega)=1$, which contradicts $g(\xi)\neq 0$. Therefore, $P(g^{(k)}(\xi))\neq 1$. Then we conclude that $g(\xi)$ is a constant by Lemma 2, a contradiction.

Now we prove that $\mathscr{F}$ is normal at $\{z: h(z)=0\}$. Without loss of generality we may assume that $h(0)=0$, and we distinguish two cases.

Case 1 $P(z)$ has at least two distinct zeros such as a and b. Suppose that $\mathscr{F}$ is not normal at 0. Then by Lemma 1, there exist $z_n\to 0$, $f_n\in\mathscr{F}$, and $\rho_n\to 0^+$, such that $g_n(\xi)=\rho_n^{-k}f_n(z_n+\rho_n\xi)$ converges locally spherically uniformly to a nonconstant meromorphic function $g(\xi)$ in $\mathbb{C}$. Obviously $g(\xi)\neq 0$.

Similarly to the previous argument, we have

$$P\left[g^{(k)}(\xi)\right]=\lim_{n\to\infty}P\left[g_n^{(k)}(\xi)\right]$$
$$=\lim_{n\to\infty}\left\{P\left[f_n^{(k)}(z_n+\rho_n\xi)\right]\right.$$
$$\left.+H(f_n(z_n+\rho_n\xi),\dots,f_n^{(k)}(z_n+\rho_n\xi))-h(z_n+\rho_n\xi)\right\}.$$

Since $P(f^{(k)}(z))+H(f(z),\dots,f^{(k)}(z))-h(z)\neq 0$ and $g(\xi)\neq 0$, by Hurwitz's theorem we deduce that $P(g^{(k)}(\xi))\neq 0$. Hence $g^{(k)}(\xi)\neq a,b$. By $a\neq b$, then one of a and b is not 0. We can assume that $a\neq 0$, that is $g(\xi)\neq 0$ and $g^{(k)}(\xi)\neq a(\neq 0)$. Then from Hayman's inequality, we have $g(\xi)$ is a constant, that is a contradiction.

Case 2 $P(z)$ only has one zero. Hence $P(z)=az^q$ $(a\neq 0, q\geqslant 1)$. By $h(0)=0$, then there exists a real constant $\delta>0$ such that $\overline{\Delta}(0,\delta)=\{z:|z|\leqslant\delta\}\subset D$, and $h(z)\neq 0$ in $\Delta'(0,\delta)=\{z:0<|z|<\delta\}$. From the previous discussion we have $\mathscr{F}$ is normal in $\Delta'(0,\delta)$. For each $\{f_n\}\subset\mathscr{F}$, there exists a subsequence that we may also note as $\{f_n\}$, such that $f_n(z)$ converges locally spherically uniformly to $f(z)$ (meromorphic function or ∞) in $\Delta'(0,\delta)$. Now we separate two subcases.

Case 2.1 $f(z)\not\equiv 0$. If $f(z)\not\equiv\infty$, then from Hurwitz's theorem we have $f(z)\neq 0$ in $\Delta'(0,\delta)$. Hence

$$\min_{|z|=\delta/2}|f(z)|=A>0,$$

where A is a constant. Then

$$\min_{|z|=\delta/2}|f_n(z)|>\frac{A}{2}>0$$

for sufficiently large n. Since $f_n(z)$ are zero-free meromorphic functions in $\Delta'(0,\delta)$, then $1/f_n(z)$ are holomorphic in $\Delta'(0,\delta)$. Therefore, $1/f_n(z)$ are holomorphic in $\overline{\Delta}(0,\delta/2)$, and

$$\max_{|z|=\delta/2}\left|\frac{1}{f_n(z)}\right|<\frac{2}{A}.$$

By the Maximum Principle, we have

$$\max_{|z|\leqslant\delta/2}\left|\frac{1}{f_n(z)}\right|<\frac{2}{A},$$

That is

$$\min_{|z|\leqslant\delta/2}|f_n(z)|>\frac{A}{2}>0.$$

Hence there exists a subsequence of $\{f_n\}$ converges locally spherically uniformly in $\Delta(0,\delta/2)$. That is $\mathscr{F}$ is normal in $\Delta(0,\delta/2)$.

If $f\equiv\infty$, then $\{f_n\}$ converges locally spherically uniformly to ∞ in $\Delta'(0,\delta)$. That is $f_n(z)$ converges locally spherically uniformly to ∞ in $\{z:|z|=\delta/2\}$. For any $M>0$ and sufficiently large n, we have

$$\min_{|z|=\delta/2}|f_n(z)|>M>0.$$

Similarly to the previous argument, there exists a subsequence of $\{f_n\}$ converges locally spherically uniformly in $\Delta(0,\delta/2)$. Hence $\mathscr{F}$ is normal in $\Delta(0,\delta/2)$.

Case 2.2 $f(z)\equiv 0$. Then $\{f_n\}$ converges locally spherically uniformly to 0 in $\Delta'(0,\delta)$. Hence both

$$\frac{a\left(f_n^{(k)}(z)\right)^q+H\left(f_n(z),\dots,f_n^{(k)}(z)\right)}{h(z)}$$

and

$$\left(\frac{a\left(f_n^{(k)}(z)\right)^q+H\left(f_n(z),\dots,f_n^{(k)}(z)\right)}{h(z)}\right)'$$

also converge locally spherically uniformly to 0. By

Argument Principle, we have

$$\left|N\left(\frac{\delta}{2},0,\frac{a\left(f_n^{(k)}\right)^q+H\left(f_n,\dots,f_n^{(k)}\right)}{h}-1\right)\right.$$
$$\left.-N\left(\frac{\delta}{2},0,\frac{1}{\frac{a\left(f_n^{(k)}\right)^q+H\left(f_n,\dots,f_n^{(k)}\right)}{h}-1}\right)\right|$$
$$=\left|\frac{1}{2\pi i}\int_{|z|=\frac{\delta}{2}}\frac{\left(\frac{a\left(f_n^{(k)}\right)^q+H\left(f_n,\dots,f_n^{(k)}\right)}{h}\right)'}{\frac{a\left(f_n^{(k)}\right)^q+H\left(f_n,\dots,f_n^{(k)}\right)}{h}-1}\mathrm{d}z\right|<1$$

for sufficiently large n. Thus it can be seen that

$$N\left(\frac{\delta}{2},0,f_n\right)\leqslant N\left(\frac{\delta}{2},0,\frac{a\left(f_n^{(k)}\right)^q+H\left(f_n,\dots,f_n^{(k)}\right)}{h}-1\right)$$
$$=N\left(\frac{\delta}{2},0,\frac{1}{\frac{a\left(f_n^{(k)}\right)^q+H\left(f_n,\dots,f_n^{(k)}\right)}{h}-1}\right)$$

for sufficiently large n. By $a\left(f_n^{(k)}\right)^q+H\left(f_n,\dots,f_n^{(k)}\right)\neq h$, we have that f_n are holomorphic in $\Delta(0,\delta/2)$. That is $\{f_n\}$ converges locally spherically uniformly to 0 in $\Delta(0,\delta/2)$. Hence $\mathscr{F}$ is normal at 0. In conclusion, $\mathscr{F}$ is a normal family in D. □

THE PROOF OF Theorem 8

Proof: For any $z_0\in D$, presume that $\mathscr{F}$ is not normal at z_0. Then by Lemma 1, there exist $f_n\in\mathscr{F}$, $z_n\to z_0$ and $\rho_n\to 0^+$, such that $g_n(\xi)=\rho_n^{-k}f_n(z_n+\rho_n\xi)$ converges locally spherically uniformly in $\mathbb{C}$ to a nonconstant meromorphic function $g(\xi)$, and moreover, g is of order at most two. By Hurwitz's theorem we know that the zeros of g are multiplicity $\geqslant k+1$.

If $P(g^{(k)}(\xi))\neq 1$, then we conclude that $g(\xi)$ is a constant by Lemma 5, a contradiction. Thus there exists ξ_0 such that $P(g^{(k)}(\xi_0))=1$. Clearly, $g(\xi_0)\neq\infty$. Therefore there exists $\delta>0$ such that $g(\xi)$ is holomorphic function in $D_{2\delta}=\{\xi:|\xi-\xi_0|<2\delta\}$. Thus $g_n^{(i)}(\xi)(i=0,1,\dots,k)$ are holomorphic in $D_\delta=\{\xi:|\xi-\xi_0|<\delta\}$ for large n and $g_n^{(i)}(\xi)(i=0,1,\dots,k)$ converges uniformly to $g^{(i)}(\xi)(i=0,1,\dots,k)$ on $\overline{D}_\delta=\{\xi:|\xi-\xi_0|\leqslant\delta\}$. As

$$\begin{aligned}P(g_n^{(k)}(\xi))-1&=P(f_n^{(k)}(z_n+\rho_n\xi))-1\\&=P(f_n^{(k)}(z_n+\rho_n\xi))\\&\quad+H(f_n(z_n+\rho_n\xi),\dots,f_n^{(k)}(z_n+\rho_n\xi))-1\\&\quad-H(f_n(z_n+\rho_n\xi),\dots,f_n^{(k)}(z_n+\rho_n\xi)),\end{aligned}$$

and

$$\begin{aligned}&H(f_n(z_n+\rho_n\xi),\dots,f_n^{(k)}(z_n+\rho_n\xi))\\&=\sum_{i=1}^m a_i(z_n+\rho_n\xi)M_i(f_n(z_n+\rho_n\xi),\dots,f_n^{(k)}(z_n+\rho_n\xi))\\&=\sum_{i=1}^m a_i(z_n+\rho_n\xi)\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}M_i(g_n(\xi),\dots,g_n^{(k)}(\xi)),\end{aligned}$$

considering $a_i(z)$ $(i=1,2,\dots,m)$ are analytic on D, we have

$$|a_i(z_n+\rho_n\xi)|\leqslant M\left(\tfrac{1+r}{2},a_i(z)\right)<\infty,$$

with sufficiently large n. For $\frac{\Gamma}{\gamma}|_H<k+1$ we conclude that

$$\sum_{i=1}^m a_i(z_n+\rho_n\xi)\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}M_i(g_n(\xi),\dots,g_n^{(k)}(\xi))$$

converges uniformly to 0 on $D_{\delta/2}=\{\xi:|\xi-\xi_0|<\delta/2\}$. Thus we know that

$$\begin{aligned}&P(g_n^{(k)}(\xi))+\sum_{i=1}^m\Big[a_i(z_n+\rho_n\xi)\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}\\&\qquad\times M_i(g_n(\xi),\dots,g_n^{(k)}(\xi))\Big]-1\\&=P(f_n^{(k)}(z_n+\rho_n\xi))\\&\quad+H(f_n(z_n+\rho_n\xi),\dots,f_n^{(k)}(z_n+\rho_n\xi))-1\end{aligned}$$

converges uniformly to $P(g^{(k)}(\xi))-1$ on $D_{\delta/2}=\{\xi:|\xi-\xi_0|<\delta/2\}$.

Since $P(f^{(k)})+H(f,f',\dots,f^{(k)})\neq 1$, by Hurwitz's theorem we deduce that $P(g^{(k)}(\xi))\equiv 1$ on $D_{\delta/2}=\{\xi:|\xi-\xi_0|<\delta/2\}$. Thus there exists a point $\omega\in\mathbb{C}$ such that $P(\omega)=1$ and $g^{(k)}(\xi)\equiv\omega$, which contradicts the zeros of g are multiplicity $\geqslant k+1$. □

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