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Remarks on certain sums involving floor function

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ABSTRACT: For each $a = 1, 2, 3, \ldots, 7$, there exists an integer b depending on a such that

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor = \left\lfloor \frac{(2n+b)^2}{8a} \right\rfloor \quad \text{for all} \quad n \in \mathbb{N}.$$

In this article, we give some remarks on this identity. In particular, we show that the range of a cannot be extended and the value of b is unique.

KEYWORDS: floor function, summation identity, residue class, fractional part

MSC2010: 11A25 11A07 05A19

INTRODUCTION

Recall that the floor function of a real number x, denoted by $\lfloor x \rfloor$, is defined to be the largest integer less than or equal to x; and the fractional part of x, denoted by $\{x\}$, is defined by $\{x\} = x - \lfloor x \rfloor$. Sums involving the floor function or the fractional part of real numbers have been a popular area of research. For example, in a proof of the quadratic reciprocity law, Gauss shows that for relatively prime positive integers a, b,

$$\sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

Dirichlet's divisor problem is to determine the smallest $\theta \geqslant 1/4$ such that

$$\sum_{n \leqslant x} \left\lfloor \frac{x}{n} \right\rfloor = x \log x + (2\gamma - 1)x + O\left(x^{\theta+\varepsilon}\right)$$

for any $\varepsilon > 0$. Hermite's identity states that for $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor nx \rfloor.$$

For some recent articles on sums involving the floor function, see for example in the articles by Aursukaree et al [1] for a generalization of Hermite's identity, by Kawsumarng et al [2, 3] for the floor function as additive bases, by Onphaeng and Pongsriiam [4] for upper and lower bounds of Jacobsthal-Tverberg sums, by Thanatipanonda and Wong [5] for predictions on sharp bounds for Jacobsthal-Tverberg sums, by Pongsriiam and Vaughan [6] for an improved formula in Dirichlet's divisor problem on arithmetic progressions, Ruankong and Kuhapatanakul [7] for sums involving the floor function and consecutive integral roots, by Phunphayap and Pongsriiam [8] for some applications of the floor function. For more references, see the books by Graham et al [9] and by Pongsriiam [10].

In particular, it is an exercise in Apostol's book [11] to show that for each $a = 1, 2, 3, \ldots, 7$, there exists an integer b depending on a such that

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor = \left\lfloor \frac{(2n+b)^2}{8a} \right\rfloor \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

In this article, we give some remarks on this identity. In particular, we show that a simple formula for $a \geqslant 8$ does not exist and the value of b for each $a \leqslant 7$ is unique.

PRELIMINARIES AND LEMMAS

In this section, we give some results which are useful in proving the main theorems. We also give a proof of (1) for completeness. Recall that for $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have $\lfloor n + x \rfloor = n + \lfloor x \rfloor$ and $0 \leqslant \{x\} < 1$. These are well-known and are often used without reference. Next, we prove a lemma that are applied throughout this article.

Lemma 1 *Let $n \geqslant 0$, $a \geqslant 1$, $0 \leqslant r < a$ be integers, and let $n \equiv r \pmod{a}$. Suppose $b = 2-a$. Then*

$$\sum_{k=1}^{n}\left\lfloor \frac{k}{a}\right\rfloor = \frac{n(n+b)-r(b+r)}{2a}. \tag{2}$$

Proof: If $a = 1$, then $r = 0$, $b = 1$, and

$$\sum_{k=1}^{n}\left\lfloor \frac{k}{a}\right\rfloor = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n(n+b)-r(b+r)}{2a}.$$

If $n < a$, then $r = n$ and both sides of (2) are zero. So we assume that $n \geqslant a \geqslant 2$. Since $n \equiv r \pmod{a}$ and $n \geqslant a > r$, there exists $q \in \mathbb{Z}^+$ such that $n = aq + r$. Therefore the left-hand side of (2) can be written as

$$\sum_{k=0}^{aq-1}\left\lfloor \frac{k}{a}\right\rfloor + \sum_{k=aq}^{aq+r}\left\lfloor \frac{k}{a}\right\rfloor.$$

If $aq \leqslant k \leqslant aq+r$, then $\lfloor k/a\rfloor = q$, and so the second sum above is equal to $q(r+1)$. The first sum can be written as

$$\sum_{0\leqslant \ell<q}\Bigg(\sum_{a\ell\leqslant k<a(\ell+1)}\left\lfloor \frac{k}{a}\right\rfloor\Bigg) = \sum_{0\leqslant \ell<q} a\ell = \frac{aq(q-1)}{2}.$$

Combining the first and second sums and substituting $q = \frac{n-r}{a}$, we see that the left-hand side of (2) is equal to

$$\begin{aligned}
&\frac{1}{2}\left(aq^2 - aq + 2qr + 2q\right)\\
&\quad= \frac{1}{2a}\Big(a^2\big(\tfrac{n-r}{a}\big)^2 - a^2\big(\tfrac{n-r}{a}\big) + 2(n-r)r + 2(n-r)\Big)\\
&\quad= \frac{1}{2a}\left(n^2 - 2nr + r^2 - an + ar + 2nr - 2r^2 + 2n - 2r\right)\\
&\quad= \frac{1}{2a}\left(n(n+2-a) - r(r+2-a)\right),
\end{aligned}$$

which is equal to the right-hand side of (2). □

By applying Lemma 1, we can prove (1) conveniently as shown in the next theorem.

Theorem 1 *If $a \leqslant 7$ is a positive integer, then we can choose $b = 2-a$ so that*

$$\sum_{k=1}^{n}\left\lfloor \frac{k}{a}\right\rfloor = \left\lfloor \frac{(2n+b)^2}{8a}\right\rfloor \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Proof: We first consider the case $a = 1$. Then the left-hand side of (3) is $n(n+1)/2$ while the right-hand side of (3) is equal to

$$\left\lfloor \frac{(2n+1)^2}{8}\right\rfloor = \left\lfloor \frac{n(n+1)}{2} + \frac{1}{8}\right\rfloor = \frac{n(n+1)}{2},$$

where the last equality is obtained from the fact that $n(n+1)/2$ is an integer. The proofs for $a = 2$ to $a = 7$ are similar, so we show the details only in the cases $a = 6$ and $a = 7$. So suppose that $a = 6$. By Lemma 1, we obtain

$$\sum_{k=1}^{n}\left\lfloor \frac{k}{6}\right\rfloor = \begin{cases} \frac{n(n-4)}{12}, & \text{if } n \equiv 0,4 \pmod{6};\\ \frac{n(n-4)+3}{12}, & \text{if } n \equiv 1,3 \pmod{6};\\ \frac{n(n-4)+4}{12}, & \text{if } n \equiv 2 \pmod{6};\\ \frac{n(n-4)-5}{12}, & \text{if } n \equiv 5 \pmod{6}. \end{cases} \tag{4}$$

The right-hand side of (3) is equal to

$$\left\lfloor \frac{(2n-4)^2}{48}\right\rfloor = \left\lfloor \frac{4n^2-16n+16}{48}\right\rfloor = \left\lfloor \frac{n(n-4)}{12} + \frac{16}{48}\right\rfloor.$$

If $n \equiv 0,4,6,10 \pmod{12}$, then $n(n-4) \equiv 0 \pmod{12}$, and so

$$\left\lfloor \frac{n(n-4)}{12} + \frac{16}{48}\right\rfloor = \frac{n(n-4)}{12}.$$

If $n \equiv 1,3,7,9 \pmod{12}$, then $n(n-4) \equiv -3 \pmod{12}$, and therefore

$$\left\lfloor \frac{n(n-4)}{12} + \frac{16}{48}\right\rfloor = \left\lfloor \frac{n(n-4)+3}{12} + \frac{4}{48}\right\rfloor = \frac{n(n-4)+3}{12}.$$

If $n \equiv 2,8 \pmod{12}$, then $n(n-4) \equiv -4 \pmod{12}$, and thus

$$\left\lfloor \frac{n(n-4)}{12} + \frac{16}{48}\right\rfloor = \left\lfloor \frac{n(n-4)+4}{12}\right\rfloor = \frac{n(n-4)+4}{12}.$$

If $n \equiv 5,11 \pmod{12}$, then $n(n-4) \equiv 5 \pmod{12}$ and hence

$$\left\lfloor \frac{n(n-4)}{12} + \frac{16}{48}\right\rfloor = \left\lfloor \frac{n(n-4)-5}{12} + \frac{36}{48}\right\rfloor = \frac{n(n-4)-5}{12}.$$

From these, we obtain

$$\begin{aligned}
\left\lfloor \frac{(2n+b)^2}{8a}\right\rfloor &= \left\lfloor \frac{n(n-4)}{12} + \frac{16}{48}\right\rfloor\\
&= \begin{cases} \frac{n(n-4)}{12}, & \text{if } n \equiv 0,4,6,10 \pmod{12};\\ \frac{n(n-4)+3}{12}, & \text{if } n \equiv 1,3,7,9 \pmod{12};\\ \frac{n(n-4)+4}{12}, & \text{if } n \equiv 2,8 \pmod{12};\\ \frac{n(n-4)-5}{12}, & \text{if } n \equiv 5,11 \pmod{12}. \end{cases}
\end{aligned} \tag{5}$$

Observe that $n \equiv 0,4 \pmod{6}$ if and only if $n \equiv 0,4,6,10 \pmod{12}$; $n \equiv 1,3 \pmod{6}$ if and only if $n \equiv 1,3,7,9 \pmod{12}$; $n \equiv 2 \pmod{6}$ if and only if $n \equiv 2,8 \pmod{12}$; $n \equiv 5 \pmod{6}$ if and only if $n \equiv 5,11 \pmod{12}$. Comparing (4) and (5), we see that

$$\sum_{k=1}^{n}\left\lfloor \frac{k}{6}\right\rfloor = \left\lfloor \frac{(2n+b)^2}{8a}\right\rfloor.$$

So this theorem is proved for $a=6$. Next, consider $a=7$. By Lemma 1, we have

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{7}\right\rfloor=\begin{cases}\frac{n(n-5)}{14}, & \text{if } n\equiv 0,5 \pmod 7;\\ \frac{n(n-5)+4}{14}, & \text{if } n\equiv 1,4 \pmod 7;\\ \frac{n(n-5)+6}{14}, & \text{if } n\equiv 2,3 \pmod 7;\\ \frac{n(n-5)-6}{14}, & \text{if } n\equiv 6 \pmod 7).\end{cases} \quad (6)$$

The right-hand side of (3) is equal to

$$\left\lfloor\frac{(2n-5)^2}{56}\right\rfloor=\left\lfloor\frac{4n^2-20n+25}{56}\right\rfloor=\left\lfloor\frac{n(n-5)}{14}+\frac{25}{56}\right\rfloor.$$

Similar to the case $a=6$, we calculate $n(n-5)$ mod 14 according to the residues of n modulo 14 and obtain that

$$\left\lfloor\frac{n(n-5)}{14}+\frac{25}{56}\right\rfloor =\begin{cases}\frac{n(n-5)}{14}, & \text{if } n\equiv 0,5,7,12 \pmod{14};\\ \frac{n(n-5)+4}{14}, & \text{if } n\equiv 1,4,8,11 \pmod{14};\\ \frac{n(n-5)+6}{14}, & \text{if } n\equiv 2,3,9,10 \pmod{14};\\ \frac{n(n-5)-6}{14}, & \text{if } n\equiv 6,13 \pmod{14}).\end{cases} \quad (7)$$

Comparing (6) and (7), we see that this theorem is verified for $a=7$. Hence the proof is complete. □

MAIN RESULTS

In this section, we show that b in Theorem 1, after a reduction, is necessarily equal to $2-a$ and the range of $a\leqslant 7$ cannot be extended to any positive integer larger than 7.

Theorem 2 *Let $a\in\mathbb{N}$, $b,c,d\in\mathbb{R}$, and $d\neq 0$. Suppose that $A\subseteq\mathbb{N}$ is an infinite set and*

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{a}\right\rfloor=\left\lfloor\frac{(cn+b)^2}{da}\right\rfloor \quad \text{for all } n\in A. \quad (8)$$

Then

$$\frac{(cn+b)^2}{da}=\frac{(2n+2-a)^2}{8a} \quad \text{for all } n\in A.$$

Proof: Let $n\in A$. By Lemma 1, the left-hand side of (8) is equal to

$$\frac{n(n+2-a)-r(2-a+r)}{2a}=\frac{n^2}{2a}+\frac{n(2-a)}{2a}-\frac{r(2-a+r)}{2a},$$

where $0\leqslant r<a$ and $n\equiv r \pmod a$. Recall that $\lfloor x\rfloor=x-\{x\}$ and $0\leqslant\{x\}<1$. So the right-hand side of (8) can be written as

$$\frac{n^2c^2}{da}+\frac{2nbc}{da}+\frac{b^2}{da}-f_1(n,a,b,c,d),$$

where $0\leqslant f_1(n,a,b,c,d)<1$. Dividing both sides of (8) by n^2, we obtain

$$\frac{1}{2a}+\frac{2-a}{2an}-\frac{r(2-a+r)}{2an^2} =\frac{c^2}{da}+\frac{2bc}{dan}+\frac{b^2}{dan^2}-\frac{f_1(n,a,b,c,d)}{n^2}. \quad (9)$$

Since (9) holds for all $n\in A$, we can take limit as $n\in A$ and $n\to\infty$ on both sides of (9) which leads to $\frac{1}{2a}=\frac{c^2}{da}$. Therefore $c^2=\frac{d}{2}$ and (9) reduces to

$$\frac{2-a}{2an}-\frac{r(2-a+r)}{2an^2} =\frac{2bc}{dan}+\frac{b^2}{dan^2}-\frac{f_1(n,a,b,c,d)}{n^2}. \quad (10)$$

Multiplying both sides of (10) by n and taking limit as $n\in A$ and $n\to\infty$, we obtain $\frac{2-a}{2a}=\frac{2bc}{da}$. Then

$$\frac{(2-a)c}{2}=\frac{2bc^2}{d}=b.$$

From these, we obtain

$$\begin{aligned}\frac{(cn+b)^2}{da}&=\frac{c^2n^2}{da}+\frac{2bcn}{da}+\frac{b^2}{da}\\ &=\frac{n^2}{2a}+\frac{(2-a)n}{2a}+\frac{(2-a)^2c^2}{4da}\\ &=\frac{4n^2}{8a}+\frac{4(2-a)n}{8a}+\frac{(2-a)^2}{8a}\\ &=\frac{(2n+2-a)^2}{8a}.\end{aligned}$$

This completes the proof. □

Theorem 2 immediately implies that it is necessary to choose $b=2-a$ in Theorem 1.

Corollary 1 *The value $b=2-a$ in Theorem 1 is unique. That is, if $b\in\mathbb{R}$, $a\in\mathbb{N}$, $a\leqslant 7$, and*

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{a}\right\rfloor=\left\lfloor\frac{(2n+b)^2}{8a}\right\rfloor \quad \text{for infinitely many } n\in\mathbb{N},$$

then $b=2-a$.

Proof: By Theorem 2, we have

$$\frac{(2n+b)^2}{8a}=\frac{(2n+2-a)^2}{8a}. \quad (11)$$

Since (11) holds for infinitely many $n\in\mathbb{N}$, we can choose distinct positive integers n_0 and n_1 and substitute $n=n_0$ and $n=n_1$ in (11) to obtain

$$4n_0b+b^2=4n_0(2-a)+(2-a)^2, \quad (12)$$

$$4n_1b+b^2=4n_1(2-a)+(2-a)^2. \quad (13)$$

Subtracting (13)−(12), we obtain $b = 2 - a$, as desired. □

Next, we show that the range of $a \leqslant 7$ in Theorem 1 cannot be extended.

Theorem 3 *For each positive integer $a \geqslant 8$ and for any choice of $b, c, d \in \mathbb{R}$ with $d \neq 0$, there are infinitely many $n \in \mathbb{N}$ such that*

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor \neq \left\lfloor \frac{(cn+b)^2}{da} \right\rfloor.$$

Proof: Suppose for a contradiction that there exist $a \in \mathbb{N}$ and $b, c, d \in \mathbb{R}$ such that $a \geqslant 8$, $d \neq 0$, and

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor \neq \left\lfloor \frac{(cn+b)^2}{da} \right\rfloor$$

for only a finite number of $n \in \mathbb{N}$. Then there exists $M \in \mathbb{N}$ such that

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor = \left\lfloor \frac{(cn+b)^2}{da} \right\rfloor \quad \text{for all } n \geqslant M.$$

By Theorem 2, we have

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor = \left\lfloor \frac{(2n+2-a)^2}{8a} \right\rfloor \quad \text{for all } n \geqslant M. \quad (14)$$

Let $n \geqslant M$ and $n \equiv a-1 \pmod{2a}$. Then $n(n+2-a) \equiv a-1 \pmod{2a}$ and $n \equiv a-1 \pmod{a}$. By Lemma 1, we obtain

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor = \frac{n(n+2-a)-(a-1)}{2a} \in \mathbb{Z}. \quad (15)$$

Next, we calculate the right-hand side of (14). We have

$$\begin{aligned}\left\lfloor \frac{(2n+2-a)^2}{8a} \right\rfloor &= \left\lfloor \frac{n(n+2-a)-(a-1)}{2a} + \frac{a-1}{2a} + \frac{(2-a)^2}{8a} \right\rfloor \\ &= \frac{n(n+2-a)-(a-1)}{2a} + \left\lfloor \frac{a-1}{2a} + \frac{(2-a)^2}{8a} \right\rfloor. \quad (16)\end{aligned}$$

But $\frac{a-1}{2a} + \frac{(2-a)^2}{8a} = \frac{a}{8} \geqslant 1$, and so

$$\left\lfloor \frac{a-1}{2a} + \frac{(2-a)^2}{8a} \right\rfloor \geqslant 1. \quad (17)$$

By (15), (16), and (17), we obtain

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor < \left\lfloor \frac{(2n+2-a)^2}{8a} \right\rfloor,$$

which contradicts (14). Hence the proof is complete. □

Remark 1 Obviously, the sum $\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor$ depends on a and n. If $a = 1, 2, 3, \dots, 7$, then Theorem 1 simply says that a simple formula for this sum exists; but if a is a positive integer larger than 7, then Theorem 3 states that such a simple formula does not exist. Nevertheless, we can always use Lemma 1 to evaluate this sum though it may lead to many cases of residues modulo a as shown in the following example.

Example 1 If $a = 8$, we can apply Lemma 1 to obtain

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{8} \right\rfloor = \begin{cases} \frac{n(n-6)}{16}, & \text{if } n \equiv 0, 6 \pmod{8}; \\ \frac{n(n-6)+5}{16}, & \text{if } n \equiv 1, 5 \pmod{8}; \\ \frac{n(n-6)+8}{16}, & \text{if } n \equiv 2, 4 \pmod{8}; \\ \frac{n(n-6)+9}{16}, & \text{if } n \equiv 3 \pmod{8}; \\ \frac{n(n-6)-7}{16}, & \text{if } n \equiv 7 \pmod{8}. \end{cases}$$

Questions: We have obtained the results for all positive integers a. Can we extend them to negative integers? What about nonzero rational numbers? Can we say something nontrivial about the sum $\sum_{k=1}^{n} \left\lfloor \frac{k}{a} \right\rfloor$ when a is positive irrational? What happen if we replace the floor by the ceiling function? We leave these questions to the interested readers.

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