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Rectangular M-tensors and strong rectangular M-tensors

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ABSTRACT: In this paper, two new classes of rectangular tensors called rectangular M-tensors and strong rectangular M-tensors are introduced. It is shown that an even-order partially symmetric rectangular M-tensor is positive semidefinite and an even-order partially symmetric strong rectangular M-tensor is positive definite. As a generalization of rectangular M-tensors, we introduce the rectangular H-tensors. In addition, some properties of (strong) rectangular M-tensors are established.

KEYWORDS: rectangular tensor, H-rectangular tensor, V-singular value, positive definiteness

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INTRODUCTION

Let $\mathbb{R}(\mathbb{C})$ be the real (complex) field, p, q, m, n be positive integers, $m,n \geqslant 2$, $[n]=\{1,2,\dots,n\}$. A (p,q)-th order $(m\times n)$-dimensional real rectangular tensor, denoted by $\mathscr{A}=(a_{i_1\dots i_p j_1\dots j_q})\in\mathbb{R}^{[p;q;m;n]}$, is defined as follows:

$$a_{i_1\dots i_p j_1\dots j_q}\in\mathbb{R},\quad i_1,\dots,i_p\in[m],\quad j_1,\dots,j_q\in[n].$$

$\mathscr{A}$ is called nonnegative if $a_{i_1\dots i_p j_1\dots j_q}\geqslant 0$, denoted by $\mathscr{A}\in\mathbb{R}_+^{[p;q;m;n]}$. A rectangular tensor $\mathscr{A}$ is called real partially symmetric, if $a_{i_1\dots i_p j_1\dots j_q}$ is invariant under any permutation of indices among $i_1\dots i_p$, and any permutation of indices among $j_1\dots j_q$, i.e.,

$$a_{\pi(i_1\dots i_p)\sigma(j_1\dots j_q)}=a_{i_1\dots i_p j_1\dots j_q},\quad \pi\in S_p,\ \ \sigma\in S_q,$$

where S_r is the permutation group of r indices. When p,q are even, $\mathscr{A}$ is called even-order partially symmetric.

For any vectors $x\in\mathbb{C}^m$, $y\in\mathbb{C}^n$, let $\mathscr{A}x^{p-1}y^q$ be a vector in $\mathbb{C}^m$ such that

$$\left(\mathscr{A}x^{p-1}y^q\right)_i=\sum_{i_2,\dots,i_p=1}^{m}\sum_{j_1,\dots,j_q=1}^{n}a_{ii_2\dots i_p j_1\dots j_q}x_{i_2}\dots x_{i_p}y_{j_1}\dots y_{j_q},$$

where $i\in[m]$. Let $\mathscr{A}x^p y^{q-1}$ be a vector in $\mathbb{C}^n$ such that

$$\left(\mathscr{A}x^{p}y^{q-1}\right)_j=\sum_{i_1,\dots,i_p=1}^{m}\sum_{j_2,\dots,j_q=1}^{n}a_{i_1i_2\dots i_p jj_2\dots j_q}x_{i_1}\dots x_{i_p}y_{j_2}\dots y_{j_q},$$

where $j\in[n]$.

Definition 1 [1] Let $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor, if there exist a number $\lambda\in\mathbb{C}$ and the vectors $x\in\mathbb{C}^m\backslash\{0\}$, $y\in\mathbb{C}^n\backslash\{0\}$ such that

$$\mathscr{A}x^{p-1}y^q=\lambda x^{[l-1]},\quad \mathscr{A}x^p y^{q-1}=\lambda y^{[l-1]},\quad (1)$$

where $x^{[\alpha]}=[x_1^\alpha,\dots,x_n^\alpha]^{\mathrm{T}}$ and $l=p+q$, then λ is called the *singular value* of $\mathscr{A}$, and (x,y) is the *left and right eigenvectors pair* of $\mathscr{A}$, associated with λ. If $\lambda\in\mathbb{R}$, $x\in\mathbb{R}^m$, and $y\in\mathbb{R}^n$, then λ is called the *H-singular value* of $\mathscr{A}$, and (x,y) is the *left and right H-eigenvectors pair* associated with λ.

In order to verify the positive definiteness of a (p,q)-th order $(m\times n)$-dimensional partially symmetric rectangular tensor, the definition of V-singular value is introduced as follows.

Definition 2 [2] Let $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor, $p,q\geqslant 2$. If there exist a number $\lambda\in\mathbb{R}$, vectors $x\in\mathbb{R}^m\backslash\{0\}$, and

$y \in \mathbb{R}^n \backslash \{0\}$ such that

$$\mathscr{A} x^{p-1} y^q = \lambda x^{[p-1]}, \quad \mathscr{A} x^p y^{q-1} = \lambda y^{[q-1]},$$
$$\sum_{i=1}^{m} x_i^p = 1, \quad \sum_{j=1}^{n} y_j^q = 1, \tag{2}$$

then λ is called the V-singular value of $\mathscr{A}$, and (x, y) is the left and right eigenvectors pair of $\mathscr{A}$, associated with λ.

Suppose that $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular tensor, p and q are even. Then,

$$f(x,y) = \mathscr{A} x^p y^q = \sum_{i_1,\dots,i_p=1}^{m} \sum_{j_1,\dots,j_q=1}^{n} a_{i_1\dots i_p j_1 \dots j_q} x_{i_1} \dots x_{i_p} y_{j_1} \dots y_{j_q} > 0$$

for all nonzero vectors $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ if and only if $\mathscr{A}$ is positive definite. $\mathscr{A}$ is called an *elasticity tensor*, if $p = q = 2$, $m = n = 2$ or 3, and $\mathscr{A}$ is a real partially symmetric rectangular tensor. When $\mathscr{A}$ is an elasticity tensor, the strong ellipticity condition holds if and only if $\mathscr{A}$ is positive definite, the strong ellipticity condition plays an important role in the theory of elasticity [3–5]. The following necessary and sufficient conditions for the positive definiteness of a partially symmetric rectangular tensor are provided in as follows.

Theorem 1 ([1, 2]) *Suppose that $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular tensor, p and q are even. Then,*

(a) $\mathscr{A}$ is positive definite if and only if all of its H-singular values are positive.

(b) $\mathscr{A}$ is positive definite if and only if all of its V-singular values are positive.

Eigenvalue problems of square tensor have been drew widespread attention [6–8]. In order to verify the positive definiteness of an mth-order n dimensional real square symmetric tensor $\mathscr{A}$, the definition of H-eigenvalue is introduced by Qi in [9].

Definition 3 [9] Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be an mth-order n-dimensional real square tensor, if there exists a vector $x \in \mathbb{R}^n$ and a number $\lambda \in \mathbb{R}$ such that

$$\mathscr{A} x^{m-1} = \lambda x^{[m-1]},$$

where

$$\mathscr{A} x^{m-1} = \left(\sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m} \right)_{i \in [n]},$$
$$x^{[m-1]} = (x_i^{m-1})_{i \in [n]},$$

then λ is called an H-eigenvalue of $\mathscr{A}$ and x is called an H-eigenvector of $\mathscr{A}$ associated with λ.

Recently, M-tensors and strong M-tensors are introduced as the generalizations of the well-known M-matrices [10–12], the authors proved that an even-order symmetric M-tensor is positive semidefinite and an even-order symmetric strong M-tensor is positive definite [12–14].

Definition 4 [10, 12] A tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is called an M-tensor, if there exist a nonnegative tensor $\mathscr{B}$ and a positive real number $s \geqslant \rho(\mathscr{B})$ where

$$\rho_H(\mathscr{A}) = \max\{|\lambda| : \lambda \text{ is an H-eigenvalue of } \mathscr{A}\}$$

such that

$$\mathscr{A} = s\mathscr{I} - \mathscr{B},$$

in which $\mathscr{I} = (\delta_{i_1 \dots i_m})$ is the mth order n-dimensional identity tensor with

$$\delta_{i_1 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $s > \rho_H(\mathscr{B})$, then A is called a strong M-tensor.

Lately, Ding et al [5] introduced a structured partially symmetric tensor named elasticity M-tensors, and proved that a nonsingular elasticity M-tensor is positive definite.

Definition 5 [5] Let $\mathscr{A} \in \mathbb{R}^{[2;2;m;n]}$, $x = (x_i)_{i=1}^m \in \mathbb{R}^m \backslash \{0\}$, $y = (y_l)_{l=1}^n \in \mathbb{R}^n \backslash \{0\}$ and $\lambda \in \mathbb{R}$, such that

$$\begin{cases} \mathscr{A} x y y = \lambda x, \\ \mathscr{A} x x y = \lambda y, \\ x^{\mathrm{T}} x = 1, \ y^{\mathrm{T}} y = 1, \end{cases} \tag{3}$$

where

$$(\mathscr{A} x y y)_i = \sum a_{ijkl} x_j y_k y_l,$$
$$(\mathscr{A} x x y)_l = \sum a_{ijkl} x_i x_j y_k. \tag{4}$$

Then λ is called an M-eigenvalue of $\mathscr{A}$, the vectors x and y are called the corresponding M-eigenvectors.

Definition 6 [5] A partially symmetric tensor $\mathscr{A} \in \mathbb{R}^{[2;2;n;n]}$ is called an elasticity M-tensor if there exist a nonnegative partially symmetric tensor $\mathscr{B} \in \mathbb{R}^{[2;2;n;n]}$ and a real number $s \geqslant \rho_M(\mathscr{B})$, where

$$\rho_M(\mathscr{A}) = \max\{|\lambda| : \lambda \text{ is an M-eigenvalue of } \mathscr{A}\}$$

such that

$$\mathscr{A} = s\mathscr{I}_E - \mathscr{B},$$

in which $\mathscr{I}_E = (e_{ijkl}) \in \mathbb{R}^{[2;2;n;n]}$ is the elasticity identity tensor with

$$e_{ijkl} = \begin{cases} 1, & \text{if } i = j \text{ and } k = l, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $s > \rho_M(\mathscr{B})$, then $\mathscr{A}$ is called a nonsingular elasticity M-tensor.

NOTATION AND PRELIMINARIES

In this section, we shall introduce some definitions and important properties related to eigenvalue of a tensor, which are needed in the subsequent analysis.

Let $\mathbb{R}^n_+$ denote the cone of nonnegative vectors. We use small letters $a, b, \ldots$ for scalars, small letters $x, y, \ldots$ for vectors, capital letters $A, B, \ldots$ for matrices, calligraphic letters $\mathscr{A}, \mathscr{B}, \ldots$ for tensors. The i-th entry of a vector x is denoted by x_i, the (i, j)-th entry of a matrix A is denoted by a_{ij} and the $(i_1 \ldots i_p j_1 \ldots j_q)$-th entry of a rectangular tensor $\mathscr{A}$ is denoted by $a_{i_1 \ldots i_p j_1 \ldots j_q}$. For any rectangular tensor $\mathscr{A} = (a_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}^{[p;q;m;n]}$, denote $|\mathscr{A}| = (|a_{i_1 \ldots i_p j_1 \ldots j_q}|)$.

The Perron-Frobenius theorem for nonnegative square tensors is introduced in [15], which states that the spectral radius of any nonnegative square tensor is an eigenvalue with a nonnegative eigenvector. Denote $\lambda_{\max}(\mathscr{A})$ as the maximal V-singular value of $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$. Then

$$\lambda_{\max}(\mathscr{A}) = \max \left\{ \mathscr{A} x^p y^q, \ \sum_{i=1}^m x_i^p = 1, \ \sum_{j=1}^n y_j^q = 1 \right\}.$$

Let $\sigma(\mathscr{A})$ be the set containing all V-singular values of $\mathscr{A} = (a_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}^{[p;q;m;n]}$. And we call

$$\rho_V(\mathscr{A}) = \max \left\{ |\lambda| : \lambda \in \sigma(\mathscr{A}) \right\}$$

is the largest V-singular value of $\mathscr{A}$. The following Perron-Frobenius theorem for V-singular values of nonnegative square rectangular tensors is introduced in [2].

Lemma 1 *Let $\mathscr{A} = (a_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}_+^{[p;q;m;n]}$. Then, $\rho_V(\mathscr{A}) = \lambda_{\max}(\mathscr{A})$, and there is a pair of nonnegative eigenvectors corresponding to the $\rho_V(\mathscr{A})$.*

RECTANGULAR M-TENSORS AND STRONG RECTANGULAR M-TENSORS

In this section, we firstly introduce (strong) rectangular M-tensors which are different from (strong) M-tensors. Moreover, the positive definiteness of an even-order partially symmetric strong M-tensor is proved. The identity tensor $\mathscr{I}$ plays an important role in the definition of M-tensor, because

$$\mathscr{I} x^{m-1} = x^{[m-1]}$$

always holds for any nonzero vector x. That is to say, 1 is an unique H-eigenvalue of the identity tensor $\mathscr{I}$. Similarly, the elasticity identity tensor $\mathscr{I}_E$ also plays an important role in the definition of elasticity M-tensor, because

$$\mathscr{I}_E x y^2 = x, \ \mathscr{I}_E x^2 y = y$$

always holds for any nonzero vectors x, y. That is to say, 1 is an unique M-eigenvalue of the identity tensor $\mathscr{I}$. We next introduce (strong) rectangular M-tensors based on the so-called rectangular identity tensor.

Definition 7 A tensor $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ is called a rectangular M-tensor if there exist a partially symmetric nonnegative tensor $\mathscr{B} \in \mathbb{R}^{[p;q;m;n]}$ and a real number $s \geqslant \rho_V(\mathscr{B})$, where

$$\rho_V(\mathscr{B}) = \max \{ |\lambda| : \lambda \text{ is a V-singular value of } \mathscr{B} \},$$

such that

$$\mathscr{A} = s\mathscr{I}_R - \mathscr{B},$$

in which $\mathscr{I}_R = (\epsilon_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}^{[p;q;n;n]}$ is the rectangular identity tensor with

$$\epsilon_{i_1 \ldots i_p j_1 \ldots j_q} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_p \text{ and } j_1 = \cdots = j_q, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $s > \rho_V(\mathscr{B})$, then $\mathscr{A}$ is called a strong rectangular M-tensor.

Remark 1 If $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular tensor, p and q are even. Then, there exist V-singular value of $\mathscr{A}$ and associated left and right eigenvectors [2].

Remark 2 From the definition of rectangular identity tensor, we can get, 1 is its unique V-singular value of $\mathscr{I}_R$.

Remark 3 If $\mathscr{A} \in \mathbb{R}^{[2;2;n;n]}$ and $\mathscr{B} \in \mathbb{R}^{[2;2;n;n]}$ are partially symmetric, then, the definition of the rectangular M-tensor is the same as the definition of the elasticity M-tensor. Therefore, the rectangular M-tensor can be regarded as a generalization of the elasticity M-tensor.

Theorem 2 *Let $\mathscr{B} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor, $\mathscr{A} = a(\mathscr{B} + b\mathscr{I}_R)$, where a and b are two real numbers. Then μ is a V-singular value of $\mathscr{A}$ if and only if $\mu = a(\lambda + b)$ and λ is a V-singular value of $\mathscr{B}$. In this case, they have the same eigenvectors pair.*

Proof: If λ is a V-singular value of $\mathscr{B}$ with eigenvectors pair (x, y), then

$$\begin{gathered} \mathscr{B}x^{p-1}y^q = \lambda x^{[p-1]}, \quad \mathscr{B}x^p y^{q-1} = \lambda y^{[q-1]}, \\ \sum_{i=1}^m x_i^p = 1, \quad \sum_{j=1}^n y_j^q = 1. \end{gathered} \tag{5}$$

Since $\mathscr{I}_R$ is a rectangular identity tensor, then

$$\mathscr{I}_R x^{p-1}y^q = x^{[p-1]}, \quad \mathscr{I}_R x^p y^{q-1} = y^{[q-1]} \tag{6}$$

From (5) and (6), we have

$$\begin{aligned} a(\mathscr{B} + b\mathscr{I}_R)x^{p-1}y^q &= a(\lambda + b)x^{[p-1]}, \\ a(\mathscr{B} + b\mathscr{I}_R)x^p y^{q-1} &= a(\lambda + b)y^{[q-1]}, \end{aligned}$$

which means

$$\mathscr{A}x^{p-1}y^q = \mu x^{[p-1]}, \quad \mathscr{A}x^p y^{q-1} = \mu y^{[q-1]},$$

i.e., μ is a V-singular value of $\mathscr{A}$ with eigenvectors pair (x, y).

On the other side, if $a = 0$, the result is trivial. If $a \neq 0$, suppose μ is a V-singular value of $\mathscr{A}$ with eigenvectors pair (x, y), then

$$\begin{aligned} \mathscr{B}x^{p-1}y^q &= \frac{1}{a}(\mu - ab)x^{[p-1]}, \\ \mathscr{B}x^p y^{q-1} &= \frac{1}{a}(\mu - ab)y^{[q-1]}, \end{aligned}$$

i.e., $\lambda = \frac{1}{a}(\mu - ab)$ is a V-singular value of $\mathscr{B}$ with eigenvectors pair (x, y). □

Corollary 1 *Suppose $\mathscr{B} \in \mathbb{R}^{[p;q;m;n]}$ is partially symmetric, s is a real numbers and $\mathscr{A} = s\mathscr{I}_R - \mathscr{B}$. Then for any V-singular value η of $\mathscr{A}$ with eigenvectors pair (x, y), there exists a V-singular value $\theta = s - \eta$ of $\mathscr{B}$ with same eigenvectors pair (x, y).*

It is showed that all H-eigenvalues of a M-tensor are nonnegative, and all H-eigenvalues of a strong M-tensor are positive [10, 12]. For rectangular M-tensors and strong rectangular M-tensors, the following spectral properties are presented.

Theorem 3 *If $\mathscr{A} = s\mathscr{I}_R - \mathscr{B}$ is a partially symmetric rectangular M-tensor and η is a V-singular value of $\mathscr{A}$, then η is nonnegative. If $\mathscr{A} = s\mathscr{I}_R - \mathscr{B}$ is a partially symmetric strong rectangular M-tensor and η is a V-singular value of $\mathscr{A}$, then η is positive.*

Proof: Let $\rho_V(\mathscr{B})$ be the largest V-singular value of $\mathscr{B}$, according to Corollary 1, there exists a V-singular value θ of $\mathscr{B}$ such that

$$\eta = s - \theta.$$

Since $\mathscr{A} = s\mathscr{I}_R - \mathscr{B}$ is a rectangular M-tensor, then

$$\eta = s - \theta \geqslant s - \rho_V(\mathscr{B}).$$

Similarly, the results about strong rectangular M-tensors can be obtained. □

Theorem 4 *Let a partially symmetric rectangular tensor $\mathscr{B}$ be nonnegative, irreducible and $\mathscr{A} = s\mathscr{I}_R - \mathscr{B}$ be a rectangular M-tensor. Then the smallest V-singular value of $\mathscr{A}$ is nonnegative and its corresponding eigenvectors are positive. If $\mathscr{A}$ is a strong rectangular M-tensor, then the smallest V-singular value of $\mathscr{A}$ is positive and its corresponding eigenvectors are positive.*

Proof: By Lemma 1, we know that $\rho_V(\mathscr{B})$ is a positive V-singular value with positive eigenvectors. By Corollary 1, we can see that $c = s - \rho_V(\mathscr{B}) \geqslant 0$ is a V-singular value of $\mathscr{A}$ and they have the same eigenvectors. If $\mathscr{A}$ is a strong rectangular M-tensor, note that $c = s - \rho_V(\mathscr{B}) > 0$. □

The entries $a_{i\ldots ij\ldots j}(i \in [m], j \in [n])$ are called diagonal, and other entries are called off-diagonal. A rectangular tensor in $\mathbb{R}^{[p;q;m;n]}$ is called a rectangular Z-tensor if all its off-diagonal entries are nonpositive.

Theorem 5 *Let $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular Z-tensor. Then $\mathscr{A}$ is a strong rectangular M-tensor if and only if $\alpha > \rho_V(\alpha\mathscr{I}_R - \mathscr{A})$, where $\alpha = \max_{i\in[m],j\in[n]}\{a_{i\ldots ij\ldots j}\}$.*

Proof: If $\alpha > \rho_V(\alpha\mathscr{I}_R - \mathscr{A})$, by $\mathscr{A} = \alpha\mathscr{I}_R - (\alpha\mathscr{I}_R - \mathscr{A})$ and the definition of strong rectangular M-tensors, then $\mathscr{A}$ is a strong rectangular M-tensor.

If $\mathscr{A}$ is a strong rectangular M-tensor, then $\mathscr{A}$ can be written as $\mathscr{A} = s\mathscr{I}_R - \mathscr{B}$, where $\mathscr{B}$ is a nonnegative rectangular tensor and $s > \rho_V(\mathscr{B})$. Then $\alpha\mathscr{I}_R - \mathscr{A} = (\alpha - s)\mathscr{I}_R + \mathscr{B}$, which yields $\alpha - \rho_v(\alpha\mathscr{I}_R - \mathscr{A}) = \rho_v(s\mathscr{I}_R - \mathscr{B}) > 0$, therefore $\alpha > \rho_v(\alpha\mathscr{I}_R - \mathscr{A})$. □

Theorem 6 *$\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ is a rectangular M-tensor if and only if $\mathscr{A} + t\mathscr{I}_R$ is a strong rectangular M-tensor for any $t > 0$.*

Proof: If $\mathscr{A} + t\mathscr{I}_R$ is a strong rectangular M-tensor for any $t > 0$, when t approaches 0,then $\mathscr{A}$ is a strong rectangular M-tensor.

If $\mathscr{A}$ is a strong rectangular M-tensor, then $\mathscr{A}$ can be written as $\mathscr{A}=s\mathscr{I}_R-\mathscr{B}$, where $\mathscr{B}$ is a nonnegative rectangular tensor and $s>\rho_V(\mathscr{B})$. Then $\mathscr{A}+t\mathscr{I}_R=(s+t)\mathscr{I}_R-\mathscr{B}$, which yields that, $\mathscr{A}+t\mathscr{I}_R$ is a strong rectangular M-tensor. □

Theorem 7 *When p,q are even, let $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular Z-tensor. Then $\mathscr{A}$ is a strong rectangular M-tensor if and only if $\mathscr{A}$ is positive definite, and $\mathscr{A}$ is a rectangular M-tensor if and only if $\mathscr{A}$ is positive semidefinite.*

Proof: When p,q are even, if $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ is a strong rectangular M-tensor, by Theorem 3 and Theorem 4, then $\mathscr{A}$ is positive definite.

If $\mathscr{A}$ is positive definite, then for any vectors $x,y\neq 0$, $\mathscr{A}x^py^q>0$. Denote $\mathscr{A}=s\mathscr{I}_R-\mathscr{B}$, where $\mathscr{B}$ is a nonnegative rectangular tensor, then $(s\mathscr{I}_R-\mathscr{B})x^py^q>0$, which yields $s>\rho_V(\mathscr{B})$ by $\sum_{i=1}^m x_i^p=1$, $\sum_{j=1}^n y_j^q=1$. The result for rectangular M-tensors can be obtained similarly. □

Let $\mathscr{A}x^p\in\mathbb{R}^{[q,n]}$ be a real qth-order n-dimensional square tensor, $\mathscr{A}y^q\in\mathbb{R}^{[p,m]}$ be a real pth-order m-dimensional square tensor, where

$$(\mathscr{A}x^p)_{j_1\dots j_q}=\sum_{j_1,\dots,j_q=1}^{n}a_{i_1i_2\dots i_pj_1\dots j_q}x_{i_1}\dots x_{i_p},$$
$$(\mathscr{A}y^p)_{i_1\dots i_p}=\sum_{i_1,\dots,i_p=1}^{m}a_{i_1i_2\dots i_pj_1\dots j_q}y_{j_1}\dots y_{j_q}.$$

The following propositions can be obtained from the definitions of $\mathscr{A}x^p$ and $\mathscr{A}y^q$.

Theorem 8 *When p,q are even, $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular Z-tensor. Then $\mathscr{A}$ is a strong rectangular M-tensor if and only if $\mathscr{A}x^p$ is a strong M-tensor for each $x\geqslant 0$, $\mathscr{A}$ is a rectangular M-tensor if and only if $\mathscr{A}x^p$ is a M-tensor for each $x\geqslant 0$.*

Proof: When p,q are even, if $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ is a strong rectangular M-tensor, by Theorem 3, then for any vectors $y\neq 0$, $\mathscr{A}x^py^q>0$, which yields $\mathscr{A}x^p$ is positive definite. And we find that, $\mathscr{A}x^p$ is a Z-tensor for each $x\geqslant 0$. Therefore, $\mathscr{A}x^p$ is a strong M-tensor.

If $\mathscr{A}x^p$ is a strong M-tensor, then for any vectors $y\neq 0$, $\mathscr{A}x^py^q>0$. Denote $\mathscr{A}=s\mathscr{I}_R-\mathscr{B}$, where $\mathscr{B}$ is a nonnegative rectangular tensor, then $s>\mathscr{B}x^py^q$ for each $x,y\geqslant 0$, which yields $s>\rho_V(\mathscr{B})$ by $\sum_{i=1}^m x_i^p=1$, $\sum_{j=1}^n y_j^q=1$ and Lemma 1. Therefore, $\mathscr{A}$ is a strong rectangular M-tensor. The result for rectangular M-tensors can be obtained similarly. □

Theorem 9 *When p,q are even, $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular Z-tensor. Then $\mathscr{A}$ is a strong rectangular M-tensor if and only if $\mathscr{A}y^q$ is a strong M-tensor for each $y\geqslant 0$, $\mathscr{A}$ is a rectangular M-tensor if and only if $\mathscr{A}y^q$ is a M-tensor for each $y\geqslant 0$.*

The following theorem can be obtained by Theorem 8 and Theorem 9.

Theorem 10 *When p,q are even, $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular Z-tensor. Then $\mathscr{A}$ is a strong rectangular M-tensor if and only if one of the following conditions satisfies:*

(1) For each $x\geqslant 0$, there exists $y\geqslant 0$ such that $\mathscr{A}x^py^{q-1}>0$;

(2) For each $x\geqslant 0$, there exists $y>0$ such that $\mathscr{A}x^py^{q-1}>0$;

(3) For each $y\geqslant 0$, there exists $x\geqslant 0$ such that $\mathscr{A}x^{p-1}y^q>0$;

(4) For each $y\geqslant 0$, there exists $x>0$ such that $\mathscr{A}x^{p-1}y^q>0$.

RECTANGULAR H-TENSOR AND STRONG RECTANGULAR H-TENSOR

$\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ is a copositive rectangular tensor, if for any $x\in\mathbb{R}_+^m$, $y\in\mathbb{R}_+^n$, $\mathscr{A}x^py^q\geqslant 0$, $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ is a strictly copositive rectangular tensor, if for any $0\neq x\in\mathbb{R}_+^m$, $0\neq y\in\mathbb{R}_+^n$, $\mathscr{A}x^py^q>0$ [16]. The definition of H-tensor was introduced in [10]. In this section, we extend rectangular M-tensors to rectangular H-tensors as follows.

Definition 8 Let $\mathscr{A}\in\mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor. Then $\mathscr{M}(\mathscr{A})=(m_{i_1\dots i_pj_1\dots j_q})\in\mathbb{R}^{[p;q;m;n]}$ is called comparison rectangular tensor of $\mathscr{A}$, whose entries are defined as:

$$m_{i_1\dots i_pj_1\dots j_q}=\begin{cases}+|a_{i_1\dots i_pj_1\dots j_q}|, & \text{if } i_1=\dots=i_p,\ j_1=\dots=j_q,\\ -|a_{i_1\dots i_pj_1\dots j_q}|, & \text{otherwise.}\end{cases}$$

A rectangular tensor is called a rectangular H-tensor, if its comparison tensor is a rectangular M-tensor, and a rectangular tensor is called a strong rectangular H-tensor, if its comparison tensor is a strong rectangular M-tensor.

Theorem 11 ([16]) *Let $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor. Then $\mathscr{A}$ is copositive if and only if*

$$N^1_{\min}(\mathscr{A}) \equiv \min\left\{\mathscr{A}x^p y^q : x \in \mathbb{R}^m_+,\ y \in \mathbb{R}^n_+,\right. \left.\sum_{i=1}^m x_i^l = 1,\ \sum_{j=1}^n y_j^l = 1\right\} \geqslant 0. \quad (7)$$

$\mathscr{A}$ is strictly copositive if and only if

$$N^1_{\min}(\mathscr{A}) \equiv \min\left\{\mathscr{A}x^p y^q : x \in \mathbb{R}^m_+,\ y \in \mathbb{R}^n_+,\right. \sum_{i=1}^m x_i^l = 1,\ \sum_{j=1}^n y_j^l = 1\} > 0. \quad (8)$$

A general case of above theorem is given as follows.

Theorem 12 *Let $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor. Then $\mathscr{A}$ is copositive if and only if*

$$N^2_{\min}(\mathscr{A}) \equiv \min\{\mathscr{A}x^p y^q : x \in \mathbb{R}^m_+,\ y \in \mathbb{R}^n_+, \sum_{i=1}^m x_i^p = 1,\ \sum_{j=1}^n y_j^q = 1\} \geqslant 0. \quad (9)$$

$\mathscr{A}$ is strictly copositive if and only if

$$N^2_{\min}(\mathscr{A}) \equiv \min\{\mathscr{A}x^p y^q : x \in \mathbb{R}^m_+,\ y \in \mathbb{R}^n_+, \sum_{i=1}^m x_i^p = 1,\ \sum_{j=1}^n y_j^q = 1\} > 0. \quad (10)$$

Proof: For any $0 \neq x \in \mathbb{R}^m_+, 0 \neq y \in \mathbb{R}^n_+$, let

$$\bar{x} = \frac{x}{(\sum_{i=1}^m x_i^p)^{\frac{1}{p}}}, \quad \bar{y} = \frac{y}{(\sum_{j=1}^n y_j^q)^{\frac{1}{q}}},$$

then $\sum_{i=1}^m \bar{x}_i^p = 1,\ \sum_{j=1}^n \bar{y}_j^q = 1$, and

$$\mathscr{A}\bar{x}^p\bar{y}^q = \frac{\mathscr{A}x^p y^q}{\sum_{i=1}^m x_i^p \sum_{j=1}^n y_j^q}.$$

Therefore, $N^1_{min}(\mathscr{A}) \geqslant 0$ if and only if $N^2_{min}(\mathscr{A}) \geqslant 0$. The second conclusion is obtained similarly. □

Theorem 13 *When p, q are even, then a partially symmetric rectangular M-tensor is copositive, and a partially symmetric strong rectangular M-tensor is strictly copositive.*

Proof: If $\mathscr{A}$ is a partially symmetric rectangular M-tensor, when p, q are even, from Theorem 7 in [2], we have

$$N^2_{\min}(\mathscr{A}) \geqslant \lambda_{\min}(\mathscr{A}) = \min\{\mathscr{A}x^p y^q : \sum_{i=1}^m x_i^p = 1, \sum_{j=1}^n y_j^q = 1\} \geqslant 0,$$

which yields that, $\mathscr{A}$ is copositive. The second conclusion can be obtained similarly. □

Theorem 14 *When p, q are even, $\mathscr{A}$ is a partially symmetric rectangular H-tensor with nonnegative diagonal entries, then $\mathscr{A}$ is positive semidefinite. If $\mathscr{A}$ is a strong partially symmetric rectangular H-tensor with positive diagonal entries, then $\mathscr{A}$ is positive definite.*

Proof: Let $\mathscr{A} = D - \mathscr{B}$, where D is the diagonal part of $\mathscr{A}$. Then its comparison tensor $\mathscr{M}(\mathscr{A}) = D - |\mathscr{B}|$ is a partially symmetric rectangular M-tensor. By Theorem 12, $\mathscr{M}(\mathscr{A})$ is copositive, which yields

$$\mathscr{M}(\mathscr{A})\bar{x}^p\bar{y}^q = D\bar{x}^p\bar{y}^q - |\mathscr{B}|\bar{x}^p\bar{y}^q \geqslant 0,$$

where $\bar{x} \in \mathbb{R}^m_+, \bar{y} \in \mathbb{R}^n_+, \sum_{i=1}^m \bar{x}_i^p = 1,\ \sum_{j=1}^n \bar{y}_j^q = 1$. Then

$$\begin{aligned}\mathscr{A}x^p y^q &= Dx^p y^q - \mathscr{B}x^p y^q \\ &\geqslant Dx^p y^q - |\mathscr{B}||x|^p|y|^q \geqslant 0,\end{aligned}$$

where $x \in \mathbb{R}^m, y \in \mathbb{R}^n, \sum_{i=1}^m x_i^p = 1,\ \sum_{j=1}^n y_j^q = 1$. Therefore, $\mathscr{A}$ is positive semidefinite. The second conclusion can be obtained similarly. □

RECTANGULAR TENSOR COMPLEMENTARITY PROBLEMS

Let $\mathscr{A} = (a_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q}) \in \mathbb{R}^{[p;q;m;n]}$, $q_m \in \mathbb{R}^m$ and $q_n \in \mathbb{R}^n$. The rectangular tensor complementarity problem [17], denoted by RTCP($\mathscr{A}, q_m, q_n$), is to find vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ such that

$$\begin{aligned} q_m + \mathscr{A}x^{p-1}y^q \geqslant 0,\ x \geqslant 0, x^{\mathrm{T}}(q_m + \mathscr{A}x^{p-1}y^q) = 0, \\ q_n + \mathscr{A}x^p y^{q-1} \geqslant 0,\ y \geqslant 0, y^{\mathrm{T}}(q_n + \mathscr{A}x^p y^{q-1}) = 0. \end{aligned} \quad (11)$$

Vectors x and y are said to be feasible if and only if x and y satisfy the following inequalities:

$$\begin{aligned} q_m + \mathscr{A}x^{p-1}y^q \geqslant 0, \quad x \geqslant 0, \\ q_n + \mathscr{A}x^p y^{q-1} \geqslant 0, \quad y \geqslant 0. \end{aligned} \quad (12)$$

A rectangular tensor $\mathscr{A} = (a_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q}) \in \mathbb{R}^{[p;q;m;n]}$ is called a rectangular S-tensor if and only if there exists $0 < x \in \mathbb{R}^m$, $0 < y \in \mathbb{R}^n$ such that

$$\mathscr{A} x^{p-1} y^q > 0, \quad \mathscr{A} x^p y^{q-1} > 0. \tag{13}$$

Then, a strong rectangular M-tensor is a rectangular S-tensor [17]. From Theorem 11 in [17], the following conclusion can be obtained easily.

Corollary 2 *Let $\mathscr{A} \in \mathbb{R}^{[p;q;m;n]}$ be a strong rectangular M-tensor. Then, the RTCP($\mathscr{A}, q_m, q_n$) is feasible for all $q_m \in \mathbb{R}^m$, $q_n \in \mathbb{R}^n$.*

CONCLUSION

In this paper, based on the definition of V-singular value for rectangular tensors, we extend elasticity M-tensors to rectangular M-tensors. Some properties of rectangular M-tensors are also presented. Finally, we prove that, an even-order partially symmetric rectangular H-tensor with nonnegative diagonal entries is positive semidefinite and an even order partially symmetric rectangular H-tensor with positive diagonal entries is positive definite.

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