A local limit theorem for Poisson binomial random variables

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ABSTRACT: We investigate the local limit theorem for Poisson binomial random variables $S_n := \sum_{i=1}^{n} X_i$, where $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables. Using the idea from Neammanee [Int J Math Math Sci **5** (2005):717–728], we derive improved explicit bounds for the density of S_n.

KEYWORDS: Poisson binomial random variable, local limit theorem, normal density function

MSC2010: 60F05

INTRODUCTION

De Moivre and Laplace (see also [1]) established the local limit theorem for binomial case in 1754. For sums of independent random variables, the local limit theorem began with the work of Esseen [2] and well understood by Petrov [3]. Since then, many researchers gave the local limit theorem in various versions (see [4–9] for examples). In this work, we are interested in the density of Poisson binomial random variables, i.e. $P(S_n = k)$, where $S_n := \sum_{i=1}^{n} X_i$, and $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables with $p_i := P(X_i = 1), q_i := P(X_i = 0)$, and $p_i + q_i = 1$ for $i = 1, 2, \ldots, n$.

Let $\mu := \mathrm{E}\,S_n = \sum_{i=1}^{n} p_i$ and $\sigma^2 := \mathrm{Var}\,S_n = \sum_{i=1}^{n} p_i q_i$. To approximate this probability, we use a local limit theorem that describes how $P(S_n = k)$ approaches the normal density, $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}$. The error bounds can be defined by

$$\triangle_n = \sup_{k \in \{0,1,\ldots,n\}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right|.$$

In the case that $X_1, X_2, \ldots, X_n$ are independent identically distributed (i.i.d) integral-valued random variables with finite third moment. Petrov [3] showed that $\triangle_n \leq C_1/\sigma^2$, where $C_1 > 0$ is a constant.

In addition, if S_n is a symmetric binomial with $P(X_i = 0) = P(X_i = 1) = 1/2$, Petrov [3] (see also [10]) improved the rate of convergence from

$O\left(1/\sigma^2\right)$ to $O\left(1/\sigma^3\right)$, i.e.,

$$\triangle_n \leq \frac{C_2}{n\sqrt{n}},$$

where $C_2 > 0$ is a constant.

Explicit constants of the error bounds were not given until 1998 when Korolev and Zhukov [11] were able to give the rate of convergence with a constant of error bounds in the case of a continuous random variable. Shevtsova [12] later improved the result of Korolev and Zhukov and got a better constant in 2017 as in the following theorem.

Theorem 1 *Let $X_1, X_2, \ldots, X_n$ be i.i.d absolutely continuous random variables with the density $p(x)$, satisfying the conditions: $\mathrm{E}\,X_1 = 0$, $\mathrm{Var}\,X_1 = 1$ and $\mathrm{E}\,|X_1|^3 < \infty$. Assume that $A = \sup_x p(x) < \infty$. Let $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. Then, for all $n \geq 2$*

$$\triangle_n \leq \frac{1}{2\pi} \frac{\mathrm{E}\,|X_1|^3}{\sqrt{n}} + 1.2136 \frac{\mathrm{E}\,|X_1|^3}{\sqrt{n}} e^{-0.0344 \frac{n}{(\mathrm{E}\,|X_1|^3)^2}}$$

$$+ A\sqrt{n} \left[1 - \frac{0.0023}{A^2(\mathrm{E}\,|X_1|^3)^2} \left(1 - \frac{0.0047}{(\mathrm{E}\,|X_1|^3)^2} \right)^3 \right]^{n-2}.$$

It should be noted that the constants of the error bounds were given in the case of continuous random variables. In 2018, Zolotukhin et al [13] gave the rate of convergence with a constant of error bounds in the case when S_n is binomial with $P(X_i = 1) = p = 1 - P(X_i = 0)$. They showed that if $npq > 1$,

then

$$\triangle_n \leqslant \min\left\{\frac{1}{\sqrt{2enpq}}, \frac{0.516}{npq}\right\}. \qquad (1)$$

Observe that (1) gives a constant of the error bounds in the case when X_i's are identical. In this work, the condition of the random variables being identical will be relaxed.

Out results are as follows.

Theorem 2 *Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with $p_i := P(X_i = 1)$ and $q_i := P(X_i = 0)$, where $p_i + q_i = 1$ for $i = 1, 2, \ldots, n$. Let $S_n = \sum_{i=1}^n X_i$, $\mathrm{E}S_n = \mu$ and $\mathrm{Var}S_n = \sigma^2$. Then for $\sigma^2 > 1$,*

$$\triangle_n \leqslant \frac{0.1194}{\sigma^2\left(1-\frac{3}{4\sigma}\right)^3} + \frac{0.0749}{\sigma^3}$$
$$+ \frac{0.2107}{\sigma^3\left(1-\frac{3}{4\sigma}\right)^6} + \left(\frac{0.4579}{\sqrt{\sigma}} + \frac{0.4725}{\sigma\sqrt{\sigma}}\right)e^{-\frac{3}{2}\sigma}.$$

For a special case when X_i's are identical, i.e., $P(X_i = 1) = p = 1 - P(X_i = 0)$, S_n is a binomial random variable with parameter p. Moreover, if $p = 1/2$, the rate of convergence can be improved from $O(1/n)$ to $O(1/n\sqrt{n})$ as in the following theorem.

Theorem 3 *Let $S_n \sim Bi(p)$. If $npq > 1$, then*

$$\triangle_n \leqslant \frac{0.1194}{npq\left(1-\frac{3}{4\sqrt{npq}}\right)^3} + \frac{0.0749}{npq\sqrt{npq}}$$
$$+ \frac{0.2107}{npq\sqrt{npq}\left(1-\frac{3}{4\sqrt{npq}}\right)^6}$$
$$+ \left(\frac{0.4579}{(npq)^{1/4}} + \frac{0.4725}{(npq)^{3/4}}\right)e^{-\frac{3}{2}\sqrt{npq}}. \qquad (2)$$

Futhermore, if $p = 1/2$ and $n > 4$, then

$$\triangle_n \leqslant \frac{0.5992}{n\sqrt{n}} + \frac{3.3984}{n^2\left(1-\frac{3}{2\sqrt{n}}\right)^4} + \frac{337.8048}{n^3\sqrt{n}\left(1-\frac{3}{2\sqrt{n}}\right)^8}$$
$$+ \left(\frac{0.6476}{n^{1/4}} + \frac{1.3365}{n^{3/4}}\right)e^{-\frac{3}{4}\sqrt{n}}. \qquad (3)$$

One can see that the constants in Theorem 2 and Theorem 3 can be expressed in terms of npq only. Moreover, the constants of the error bounds in (2) are smaller than those in (1) when n is large enough. Theorem 3 can be applied to a random walk as in the following corollary.

Corollary 1 *Let S_n be a random walk, $P(X_i = 1) = p = 1 - P(X_i = -1)$ and*

$$\epsilon_n = \sup_{k \in \{0,1,\ldots,n\}} \left| P(S_n = k) - \frac{1}{\sqrt{npq}\sqrt{2\pi}} e^{-\frac{\left(\frac{k+n}{2}-np\right)^2}{2npq}} \right|.$$

If $npq > 1$, then

$$\epsilon_n \leqslant \frac{0.1194}{npq\left(1-\frac{3}{4\sqrt{npq}}\right)^3} + \frac{0.0749}{npq\sqrt{npq}}$$
$$+ \frac{0.2107}{npq\sqrt{npq}\left(1-\frac{3}{4\sqrt{npq}}\right)^6}$$
$$+ \left(\frac{0.4579}{(npq)^{1/4}} + \frac{0.4725}{(npq)^{3/4}}\right)e^{-\frac{3}{2}\sqrt{npq}}. \qquad (4)$$

In the case that $p = 1/2$ and $n > 4$, we have

$$\epsilon_n \leqslant \frac{0.5992}{n\sqrt{n}} + \frac{3.3984}{n^2\left(1-\frac{3}{2\sqrt{n}}\right)^4} + \frac{337.8048}{n^3\sqrt{n}\left(1-\frac{3}{2\sqrt{n}}\right)^8}$$
$$+ \left(\frac{0.6476}{n^{1/4}} + \frac{1.3365}{n^{3/4}}\right)e^{-\frac{3}{4}\sqrt{n}}. \qquad (5)$$

AUXILIARY RESULTS

Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with $p_j := P(X_j = 1)$ and $q_j := P(X_j = 0)$, where $p_j + q_j = 1$ for $j = 1, 2, \ldots, n$. Let $S_n = \sum_{i=1}^n X_i$, $\mu = \mathrm{E}S_n = \sum_{i=1}^n p_i$ and $\sigma^2 = \mathrm{Var}S_n = \sum_{i=1}^n p_i q_i$. Let $\psi_1, \psi_2, \ldots, \psi_n$ and ψ be the characteristic functions of $X_1, X_2, \ldots, X_n$ and S_n, respectively. Then, for $j = 1, 2, \ldots, n$,

$$\psi_j(t) = q_j + p_j e^{it} \quad \text{and} \quad \psi(t) = \prod_{j=1}^n (q_j + p_j e^{it}).$$

Note that

$$\psi_j(t) = (q_j + p_j \cos(t)) + ip_j \sin(t) = \rho_j(t)e^{i\theta_j(t)},$$

where

$$\theta_j(t) = \arctan\left(\frac{p_j \sin(t)}{q_j + p_j \cos(t)}\right),$$
$$\rho_j(t) = |\psi_j(t)| = \left(1 - 4p_j q_j \sin^2\left(\frac{t}{2}\right)\right)^{1/2}.$$

Hence, $\psi(t) = \rho(t)e^{i\theta(t)}$, where $\theta(t) = \sum_{j=1}^n \theta_j(t)$ (mod 2π), and $\rho(t) = \prod_{j=1}^n \rho_j(t)$.

Let $\alpha(t) = \theta(t) - \mu t$. In order to prove our main results, we use the ideas from Uspensky [14] and Neammanee [15] to obtain the following Lemma 1.

Lemma 1 *Let* $t \in \left[0, \sqrt{3/\sigma}\right]$. *For* $j = 1, 2, \ldots, n$, *we have*

(i) $\cos((k-\mu)t - \alpha(t)) = \cos((k-\mu)t) + \triangle_{11}$, *where* $|\triangle_{11}| \leqslant \frac{0.1875\sigma^2 t^3}{\left(1 - \frac{3}{4\sigma}\right)^3} + \frac{0.0352\sigma^4 t^6}{\left(1 - \frac{3}{4\sigma}\right)^6}$.

(ii) *If* $p_j = q_j$, *then*

$$\cos((k-\mu)t - \alpha(t)) = \cos((k-\mu)t) + \triangle_{12},$$

where $|\triangle_{12}| \leqslant \frac{0.0834\sigma^2 t^5}{\left(1 - \frac{3}{4\sigma}\right)^4} + \frac{0.007\sigma^4 t^{10}}{\left(1 - \frac{3}{4\sigma}\right)^8}$.

Proof: (i) Using the Taylor's expansion, we have for some t_0, t_1, t_2, t_3,

$$\cos(\alpha(t)) = 1 - \tfrac{1}{2}\cos(t_0)(\alpha(t))^2 \tag{6}$$

$$\sin(\alpha(t)) = \alpha(t) - \tfrac{1}{2}\sin(t_1)(\alpha(t))^2 \tag{7}$$

$$\theta_j(t) = \theta_j(0) + \theta_j^{(1)}(0)t + \tfrac{1}{2}\theta_j^{(2)}(0)t^2$$
$$+ \tfrac{1}{6}\theta_j^{(3)}(t_2)t^3 \tag{8}$$

$$\theta_j(t) = \theta_j(0) + \theta_j^{(1)}(0)t + \tfrac{1}{2}\theta_j^{(2)}(0)t^2$$
$$+ \tfrac{1}{6}\theta_j^{(3)}(0)t^3 + \tfrac{1}{24}\theta_j^{(4)}(t_3)t^4. \tag{9}$$

Note that

$$\theta_j^{(1)}(0) = p_j, \quad \theta_j^{(2)}(0) = 0,$$
$$\theta_j^{(3)}(0) = p_j q_j (p_j - q_j) \tag{10}$$

and for $t \in \left[0, \sqrt{3/\sigma}\right]$

$$\left|\theta_j^{(3)}(t)\right| \leqslant \frac{9p_j q_j}{8\left(1 - \frac{3}{4\sigma}\right)^3} \tag{11}$$

and

$$\left|\theta_j^{(4)}(t)\right| \leqslant \frac{2p_j q_j t}{\left(1 - \frac{3}{4\sigma}\right)^4}, \tag{12}$$

see page 722 [15] and also [14].

By (8), (10), and (11), we obtain

$$|\alpha(t)| \leqslant \frac{1}{6}\frac{9t^3}{8\left(1 - \frac{3}{4\sigma}\right)^3}\sum_{j=1}^{n} p_j q_j = \frac{0.1875\sigma^2 t^3}{\left(1 - \frac{3}{4\sigma}\right)^3} \tag{13}$$

and

$$|\alpha^2(t)| \leqslant \frac{0.0352\sigma^4 t^6}{\left(1 - \frac{3}{4\sigma}\right)^6}. \tag{14}$$

From (6) and (7), we get

$$\cos((k-\mu)t - \alpha(t))$$
$$= \cos((k-\mu)t)\cos(\alpha(t)) + \sin((k-\mu)t)\sin(\alpha(t))$$
$$= \cos((k-\mu)t)\left(1 - \tfrac{1}{2}\cos(t_0)\alpha^2(t)\right)$$
$$\quad + \sin((k-\mu)t)\left(\alpha(t) - \tfrac{1}{2}\sin(t_1)\alpha^2(t)\right)$$
$$= \cos((k-\mu)t) + \triangle_{11}, \tag{15}$$

where

$$|\triangle_{11}| \leqslant |\alpha(t)| + \alpha^2(t). \tag{16}$$

By (13), (14) and (16), we get

$$|\triangle_{11}| \leqslant \frac{0.1875\sigma^2 t^3}{\left(1 - \frac{3}{4\sigma}\right)^3} + \frac{0.0352\sigma^4 t^6}{\left(1 - \frac{3}{4\sigma}\right)^6}.$$

(ii) Since $p_j = q_j$, by (9), (10) and (12), we get

$$|\alpha(t)| \leqslant \frac{0.0834\sigma^2 t^5}{\left(1 - \frac{3}{4\sigma}\right)^4}, \quad |\alpha^2(t)| \leqslant \frac{0.007\sigma^4 t^{10}}{\left(1 - \frac{3}{4\sigma}\right)^8}. \tag{17}$$

The proof follows from (15), (16) and (17). □

Lemma 2 *For* $j = 1, 2, \ldots, n$,

(i) $\rho_j(t) \leqslant e^{-\frac{2}{\pi^2}p_j q_j t^2}$ *for* $t \in [0, \pi)$,

(ii) $\rho_j(t) \leqslant e^{-\frac{1}{2}p_j q_j t^2 + \frac{1}{24}p_j q_j t^4}$ *for* $t \in [0, \pi]$,

(iii) $\rho(t)$ *is decreasing on* $[0, \pi/2]$,

(iv) $\left|\rho(t) - e^{-\frac{1}{2}\sigma^2 t^2}\right| \leqslant \frac{1}{16}\sigma^2 t^4 e^{-\frac{1}{2}\sigma^2 t^2}$ *for* $t \in \left[0, \sqrt{\frac{3}{\sigma}}\right]$.

Proof: See page 720–723 in [15], and also [14]. □

In the next lemma, we give the probability of integral-valued random variable.

Lemma 3 *Let* X *be any integral-valued random variable with finite variance and the characteristic function* ψ_X. *Then*,

$$P(X = k) = \frac{1}{\pi}\int_0^{\pi}|\psi_X(t)|\cos\left((k - E(X))t - \alpha(t)\right)\mathrm{d}t,$$

where $\alpha(t) = \theta_X(t) - E(X)t$ *and* $\theta_X(t)$ *is the argument of* $\psi_X(t)$.

Proof: Note that, for each integer k,

$$\int_{-\pi}^{\pi} e^{-ikt}\psi_X(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} e^{-ikt}\left(\sum_{j=-\infty}^{\infty} P(X = j)e^{ijt}\right)\mathrm{d}t$$

$$= \sum_{j=-\infty}^{\infty} P(X = j)\int_{-\pi}^{\pi} e^{i(j-k)t}\,\mathrm{d}t$$

$$= 2\pi P(X = k),$$

and

$$\int_{-\pi}^{\pi} e^{-ikt}\psi_X(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} e^{-ikt}|\psi_X(t)|e^{i\theta_X(t)}\,\mathrm{d}t$$

$$= \int_{-\pi}^{\pi} |\psi_X(t)|e^{i(\theta_X(t) - kt)}\,\mathrm{d}t$$

$$= \int_{-\pi}^{\pi} |\psi_X(t)|\cos(\theta_X(t) - kt)\,\mathrm{d}t$$

$$+ i\int_{-\pi}^{\pi} |\psi_X(t)|\sin(\theta_X(t) - kt)\,\mathrm{d}t.$$

From the above results and the fact that $P(X = k)$ is real, we get

$$P(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\psi_X(t)| \cos(\theta_X(t) - kt) \, dt.$$

Observe that

$$\psi_X(t) = \sum_{j=-\infty}^{\infty} P(X = j) e^{ijt}$$

$$= \sum_{j=-\infty}^{\infty} P(X = j) \cos(jt) + i \sum_{j=-\infty}^{\infty} P(X = j) \sin(jt).$$

Therefore,

$$|\psi_X(t)| = \left[\left(\sum_{j=-\infty}^{\infty} P(X = j) \cos(jt) \right)^2 + \left(\sum_{j=-\infty}^{\infty} P(X = j) \sin(jt) \right)^2 \right]^{\frac{1}{2}},$$

$$\theta_X(t) = \arctan \left(\frac{\sum_{j=-\infty}^{\infty} P(X = j) \sin(jt)}{\sum_{j=-\infty}^{\infty} P(X = j) \cos(jt)} \right).$$

Since $|\psi_X(t)|$ is even and $\theta_X(t)$ is odd, $|\psi_X(t)| \cos(\theta_X(t) - kt)$ is even. Hence,

$$P(X = k) = \frac{2}{2\pi} \int_0^{\pi} |\psi_X(t)| \cos(\theta_X(t) - kt) \, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} |\psi_X(t)| \cos((k - E(X))t - \alpha(t)) \, dt.$$

$\square$

PROOF OF THE MAIN RESULTS

Proof of Theorem 2

Proof: By Lemma 3, we get

$$P(S_n = k) = \frac{1}{\pi} \int_0^{\pi} \rho(t) \cos((k - \mu)t - \alpha(t)) \, dt$$

$$= \frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} \rho(t) \cos((k - \mu)t - \alpha(t)) \, dt + \triangle_1,$$

where

$$\triangle_1 = \frac{1}{\pi} \int_{\sqrt{3/\sigma}}^{\pi} \rho(t) \cos((k - \mu)t - \alpha(t)) \, dt. \quad (18)$$

Note that

$$|\triangle_1| \leqslant \frac{1}{\pi} \int_{\sqrt{3/\sigma}}^{\pi} \rho(t) \, dt = \triangle_{11} + \triangle_{12},$$

where

$$\triangle_{11} = \frac{1}{\pi} \int_{\sqrt{\frac{3}{\sigma}}}^{\sqrt{\frac{3}{4\sigma}} \pi} \rho(t) \, dt,$$

$$\triangle_{12} = \frac{1}{\pi} \int_{\sqrt{\frac{3}{4\sigma}} \pi}^{\pi} \rho(t) \, dt. \quad (19)$$

By Lemma 2(ii) and Lemma 2(iii), we obtain

$$|\triangle_{11}| \leqslant \frac{1}{\pi} \rho\left(\sqrt{\frac{3}{\sigma}} \right) \int_{\sqrt{\frac{3}{\sigma}}}^{\sqrt{\frac{3}{4\sigma}} \pi} dt$$

$$\leqslant \frac{1}{\pi} e^{\frac{9}{24}} e^{-\frac{3}{2}\sigma} \int_{\sqrt{\frac{3}{\sigma}}}^{\sqrt{\frac{3}{4\sigma}} \pi} dt$$

$$= \frac{1}{\pi} e^{\frac{9}{24}} e^{-\frac{3}{2}\sigma} \left(\sqrt{\frac{3}{4\sigma}} \pi - \sqrt{\frac{3}{\sigma}} \right)$$

$$\leqslant \frac{0.4579}{\sqrt{\sigma}} e^{-\frac{3}{2}\sigma}, \quad (20)$$

and by Lemma 2(i), we get

$$|\triangle_{12}| \leqslant \frac{1}{\pi} \int_{\sqrt{\frac{3}{4\sigma}} \pi}^{\infty} e^{-\frac{2}{\pi^2} \sigma^2 t^2} dt$$

$$= \frac{1}{\pi} \int_{\sqrt{3\sigma/2}}^{\infty} \frac{\pi}{\sqrt{2}\sigma} e^{-t^2} dt$$

$$\leqslant \frac{1}{\sigma} \frac{\sqrt{2}}{\sqrt{6\sigma}} \int_{\sqrt{3\sigma/2}}^{\infty} t e^{-t^2} dt$$

$$= \frac{1}{\sigma\sqrt{3\sigma}} \left(\frac{e^{-\frac{3}{2}\sigma}}{2} \right)$$

$$\leqslant \frac{0.2887}{\sigma\sqrt{\sigma}} e^{-\frac{3}{2}\sigma}. \quad (21)$$

Hence, by (18)–(21),

$$P(S_n = k) = \frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} \rho(t) \cos((k - \mu)t - \alpha(t)) \, dt + \triangle_1, \quad (22)$$

where

$$|\triangle_1| \leqslant |\triangle_{11}| + |\triangle_{12}| \leqslant \frac{0.4579}{\sqrt{\sigma}} e^{-\frac{3}{2}\sigma} + \frac{0.2887}{\sigma\sqrt{\sigma}} e^{-\frac{3}{2}\sigma}.$$

By Lemma 2(iv), we obtain

$$\frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} \rho(t) \cos((k - \mu)t - \alpha(t)) \, dt$$

$$= \frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k - \mu)t - \alpha(t)) \, dt + \triangle_2,$$

where

$$|\triangle_2| \leqslant \frac{\sigma^2}{16\pi} \int_0^{\sqrt{3/\sigma}} t^4 \, e^{-\frac{1}{2}\sigma^2 t^2} \, dt$$

$$\leqslant \frac{\sigma^2}{16\pi} \int_0^\infty t^4 \, e^{-\frac{1}{2}\sigma^2 t^2} \, dt$$

$$= \frac{\sigma^2}{16\pi} \left(\frac{3\sqrt{\pi}}{\sqrt{2}\sigma^5} \right) \leqslant \frac{0.0749}{\sigma^3}. \qquad (23)$$

By Lemma 1(i), we get

$$\frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t - \alpha(t)) \, dt$$

$$= \frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t) \, dt + \triangle_3, \quad (24)$$

where

$$|\triangle_3| \leqslant \frac{1}{\pi} \int_0^{\sqrt{\frac{3}{\sigma}}} \left(\frac{0.1875\sigma^2 t^3}{\left(1-\frac{3}{4\sigma}\right)^3} + \frac{0.0352\sigma^4 t^6}{\left(1-\frac{3}{4\sigma}\right)^6} \right) e^{-\frac{1}{2}\sigma^2 t^2} \, dt$$

$$\leqslant \frac{0.1875\sigma^2}{\pi\left(1-\frac{3}{4\sigma}\right)^3} \left(\frac{2}{\sigma^4} \right) + \frac{0.0352\sigma^4}{\pi\left(1-\frac{3}{4\sigma}\right)^6} \left(\frac{15\sqrt{\pi}}{\sqrt{2}\sigma^7} \right)$$

$$\leqslant \frac{0.1194}{\sigma^2\left(1-\frac{3}{4\sigma}\right)^3} + \frac{0.2107}{\sigma^3\left(1-\frac{3}{4\sigma}\right)^6}. \quad (25)$$

From (22)–(25),

$$P(S_n = k) = \frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t) \, dt + \triangle_4,$$

$$\qquad (26)$$

where

$$|\triangle_4| \leqslant |\triangle_1| + |\triangle_2| + |\triangle_3|$$

$$\leqslant \frac{0.4579}{\sqrt{\sigma}} e^{-\frac{3}{2}\sigma} + \frac{0.2887}{\sigma\sqrt{\sigma}} e^{-\frac{3}{2}\sigma} + \frac{0.0749}{\sigma^3}$$

$$+ \frac{0.1194}{\sigma^2\left(1-\frac{3}{4\sigma}\right)^3} + \frac{0.2107}{\sigma^3\left(1-\frac{3}{4\sigma}\right)^6}.$$

It is known that

$$\int_0^\infty e^{-at^2} \cos(bt) \, dt = \frac{1}{2}\sqrt{\frac{\pi}{a}} \, e^{-\frac{b^2}{4a}} \quad \text{for} \quad a > 0.$$

We thus obtain

$$\frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t) \, dt$$

$$= \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t) \, dt + \triangle_5$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(k-\mu)^2}{2\sigma^2}} + \triangle_5, \qquad (27)$$

where

$$|\triangle_5| \leqslant \frac{1}{\pi} \int_{\sqrt{3/\sigma}}^\infty e^{-\frac{1}{2}\sigma^2 t^2} \, dt$$

$$\leqslant \frac{1}{\pi} \sqrt{\frac{\sigma}{3}} \int_{\sqrt{3/\sigma}}^\infty t \, e^{-\frac{1}{2}\sigma^2 t^2} \, dt$$

$$= \frac{1}{\pi} \sqrt{\frac{\sigma}{3}} \left(\frac{e^{-\frac{3}{2}\sigma}}{\sigma^2} \right) \leqslant \frac{0.1838}{\sigma\sqrt{\sigma}} \, e^{-\frac{3}{2}\sigma}. \qquad (28)$$

From (26)–(28), we can conclude that

$$P(S_n = k) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(k-\mu)^2}{2\sigma^2}} + \triangle_6,$$

where

$$|\triangle_6| \leqslant |\triangle_4| + |\triangle_5|$$

$$\leqslant \frac{0.4579}{\sqrt{\sigma}} e^{-\frac{3}{2}\sigma} + \frac{0.2887}{\sigma\sqrt{\sigma}} e^{-\frac{3}{2}\sigma} + \frac{0.0749}{\sigma^3}$$

$$+ \frac{0.1194}{\sigma^2\left(1-\frac{3}{4\sigma}\right)^3} + \frac{0.2107}{\sigma^3\left(1-\frac{3}{4\sigma}\right)^6} + \frac{0.1838}{\sigma\sqrt{\sigma}} e^{-\frac{3}{2}\sigma}$$

$$= \frac{0.1194}{\sigma^2\left(1-\frac{3}{4\sigma}\right)^3} + \frac{0.0749}{\sigma^3} + \frac{0.2107}{\sigma^3\left(1-\frac{3}{4\sigma}\right)^6}$$

$$+ \left(\frac{0.4579}{\sqrt{\sigma}} + \frac{0.4725}{\sigma\sqrt{\sigma}} \right) e^{-\frac{3}{2}\sigma}.$$

$$\square$$

Proof of Theorem 3

Proof: Since $S_n \sim Bi(p)$, $\sigma^2 = npq$. By Theorem 2, we get (2). By Lemma 1(ii), we get

$$\frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t - \alpha(t)) \, dt$$

$$= \frac{1}{\pi} \int_0^{\sqrt{3/\sigma}} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t) \, dt + \triangle_7, \quad (29)$$

where

$$|\triangle_7| \leqslant \frac{1}{\pi} \int_0^{\sqrt{\frac{3}{\sigma}}} \left(\frac{0.0834\sigma^2 t^5}{\left(1-\frac{3}{4\sigma}\right)^4} + \frac{0.007\sigma^4 t^{10}}{\left(1-\frac{3}{4\sigma}\right)^8} \right) e^{-\frac{1}{2}\sigma^2 t^2} \, dt$$

$$\leqslant \frac{0.0834\sigma^2}{\pi\left(1-\frac{3}{4\sigma}\right)^4} \left(\frac{8}{\sigma^6} \right) + \frac{0.007\sigma^4}{\pi\left(1-\frac{3}{4\sigma}\right)^8} \left(\frac{945\sqrt{\pi}}{\sqrt{2}\sigma^{11}} \right)$$

$$\leqslant \frac{0.2124}{\sigma^4\left(1-\frac{3}{4\sigma}\right)^4} + \frac{2.6391}{\sigma^7\left(1-\frac{3}{4\sigma}\right)^8}. \qquad (30)$$

By (22)–(23), (27)–(28) and (29)–(30), we can conclude that

$$P(S_n = k) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(k-\mu)^2}{2\sigma^2}} + \triangle_8,$$

where

$$|\triangle_8| \leqslant |\triangle_1| + |\triangle_2| + |\triangle_5| + |\triangle_7|$$

$$\leqslant \frac{0.4579}{\sqrt{\sigma}} \, e^{-\frac{3}{2}\sigma} + \frac{0.2887}{\sigma\sqrt{\sigma}} \, e^{-\frac{3}{2}\sigma} + \frac{0.0749}{\sigma^3}$$

$$+ \frac{0.1838}{\sigma\sqrt{\sigma}} \, e^{-\frac{3}{2}\sigma} + \frac{0.2124}{\sigma^4 \left(1 - \frac{3}{4\sigma}\right)^4} + \frac{2.6391}{\sigma^7 \left(1 - \frac{3}{4\sigma}\right)^8}$$

$$= \frac{0.0749}{\sigma^3} + \frac{0.2124}{\sigma^4 \left(1 - \frac{3}{4\sigma}\right)^4} + \frac{2.6391}{\sigma^7 \left(1 - \frac{3}{4\sigma}\right)^8}$$

$$+ \left(\frac{0.4579}{\sqrt{\sigma}} + \frac{0.4725}{\sigma\sqrt{\sigma}}\right) e^{-\frac{3}{2}\sigma}.$$

Hence, by $\sigma^2 = n/4$, we have (3). □

Proof of Corollary 1

Proof: Let $Y_i = (X_i + 1)/2$. Then, $\mathrm{E}\left(\sum_{i=1}^{n} Y_i\right) = np$, $\mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) = npq$ and

$$P(S_n = k) = P\left(\sum_{i=1}^{n} X_i = k\right)$$

$$= P\left(\sum_{i=1}^{n} \left(\frac{X_i + 1}{2}\right) = \frac{k+n}{2}\right)$$

$$= P\left(\sum_{i=1}^{n} Y_i = \frac{k+n}{2}\right).$$

By Theorem 2, we obtain (4). Moreover, if $p = 1/2$ and $n > 4$, we get (5). □

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