Permanence and uniform asymptotical stability of a ratio-dependent Leslie system with feedback controls on time scales

Zhouhong Li, Jianwen Zhou, Tianwei Zhang

a Department of Mathematics, Yuxi Normal University, Yuxi 653100 China
b Department of Mathematics, Yunnan University, Kunming 650091 China
c Institute de Mathématiques, Kunming University of Science and Technology, Kunming 650500 China

Corresponding author, e-mail: zhang@kust.edu.cn

ABSTRACT: Using time-scale calculus and construction of a suitable Lyapunov functional, the permanence and uniform asymptotical stability of a ratio-dependent Leslie model with feedback control on time scales are studied. The results of this paper extend some recent research results. An illustrative example with numerical simulations is employed to visually manifest the theoretical findings.

KEYWORDS: permanence, uniform asymptotical stability, Leslie system, feedback control, time scale

MSC2010: 34D20 92D25

INTRODUCTION

Predator-prey is one of the important interactions among species commonly observed in social animals and human society. The dynamical relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Leslie [1] proposed a predator-prey model to describe the "carrying capacity" of the predator's environment that is proportional to the number of preys. Leslie emphasized that the rates of increase of both preys and predators are limited, which is not the same as those in the Lotka-Volterra model. In case the number of preys is large or the number of predators is small, the rates are able to reach their upper limits. In terms of continuous time, these discussions deduce Leslie predator-prey system as follows:

\[
\begin{align*}
\dot{x}_1 &= x_1[b - ax_1] - p(x_1, x_2)x_2, \\
\dot{x}_2 &= x_2[g - f(x_2)],
\end{align*}
\]

where \(x_1\) and \(x_2\) stand for the population (the density) of the preys and of the predators, respectively, \(p\) is the so-called predator functional response to predator and prey. In the last decades, the dynamical behaviors for the continuous-time Leslie predator-prey systems such as Hopf bifurcation [2, 3], permanence [4], periodic solution [5, 6], almost periodic solution [7, 8], and stability [4], etc., have been widely investigated.

Recently, more and more obvious evidences of biology and physiology showed that in many conditions, especially when the predators have to search for food (consequently, have to share or compete for food), a more realistic and general predator-prey system should rely on the theory of ratio-dependence, this theory is confirmed by lots of experimental results. In the last decades, much work has been done on the ecosystem with feedback controls [9, 10]. In particular, Wang et al [11] considered a ratio-dependent Leslie predator-prey model with feedback controls as follows:

\[
\begin{align*}
\dot{x}_1 &= x_1 b - ax_1 - p(x_1, x_2)x_2 - \frac{c(x_1)x_2}{h(t)x_2^2(t) + x_2^2(t)} - d(t)u_1(t), \\
\dot{x}_2 &= x_2 g - f(x_2) - p(t)u_2(t), \\
\dot{u}_1 &= \alpha(t) - \beta(t)u_1(t) + \gamma(t)x_2(t),
\end{align*}
\]

where \(u_1(t)\) and \(u_2(t)\) are control variables. Under the assumption that the coefficients of the above system are all \(T\)-periodic functions, they obtained the existence of a unique globally attractive positive \(T\)-periodic solution of the above system.

Many literatures have discussed that the discrete-time systems described by difference equations are more realistic and more appropriate than the continuous-time systems in case...
there exists non-overlapping generations in the populations. Furthermore, discrete-time systems can also provide efficient computational process of continuous-time systems for numerical simulations. As a result, various dynamical behaviors of discrete-time Leslie-Gower predator-prey systems, such as bifurcations [12, 13], chaos control [12], chaos [13], permanence [14, 15], almost periodic solutions [14, 15], and stability [14], have become research highlights for many scholars.

In [15], the following discrete ratio-dependent Leslie model was discussed:

\[
\begin{align*}
N^1_i(t) &= b(t) - a(t) \exp(N^2_i(t)) - d(t)u_1(t), \\
N^2_i(t) &= g(t) - f(t) \exp(N^2_i(t)) - p(t)u_2(t),
\end{align*}
\]

where \(a, b, c, f, g, h, d, p, r, s, i, j, k \) are bounded sequences defined on \( \mathbb{Z}^+ \), \( i = 1, 2 \). There exists a non-overlapping generations in the populations. Furthermore, discrete-time systems for numerical simulations.

Time-scale calculus was proposed in 1990, which unifies the continuous-time analysis and the discrete-time analysis [16]. Later on, the theory of time scales was mainly developed by Bohner and Peterson [17, 18]. Recently, the theory of time scales has been applied in neural networks [19], ecological systems [20], as well as a variety of control systems [21]. With the mind of time scale scheme, the above continuous-time system (1) and discrete-time system (2) can be unified and integrated. It is therefore meaningful to study the dynamical systems on time scales that can unify differential and difference systems. Motivated by the results mentioned above, in this paper, we are concerned with the following ratio-dependent Leslie model with feedback controls on time scales:

\[
\begin{align*}
N^1_i(t) &= b(t) - a(t) \exp(N^2_i(t)) - d(t)u_1(t), \\
N^2_i(t) &= g(t) - f(t) \exp(N^2_i(t)) - p(t)u_2(t),
\end{align*}
\]

where \(a, b, c, f, g, h, d, p, r, s, i, j, k \) are bounded sequences defined on \( \mathbb{Z}^+ \), \( i = 1, 2 \). There exists a non-overlapping generations in the populations. Furthermore, discrete-time systems for numerical simulations.

PRELIMINARIES

Definition 1 [17, 18] Let \( \mathbb{T} \) be a nonempty closed subset (time scales) of \( \mathbb{R} \). For any subset \( \mathbb{I} \) of \( \mathbb{R} \), we denote \( \mathbb{I_T} = \mathbb{I} \cap \mathbb{T} \). The forward and backward jump operators \( \sigma, \rho : \mathbb{T} \rightarrow \mathbb{T} \) and the graininess \( \mu : \mathbb{T} \rightarrow \mathbb{R}^+ \) are defined, respectively, by

\[
\begin{align*}
\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\}, \\
\rho(t) &= \sup\{s \in \mathbb{T} : s < t\}, \\
\mu(t) &= \sigma(t) - t.
\end{align*}
\]

For \( y : \mathbb{T} \rightarrow \mathbb{R} \), \( y^\Delta(t) \) to be the number that exists in the property that, for any \( \epsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that

\[
\left| y^\Delta(t) - y(s) \right| < \epsilon |\sigma(t) - s| \quad \text{for all } s \in U.
\]

Let \( y \) be right-dense continuous. If \( y^\Delta(t) = y(t) \), then we define delta integral by

\[
\int_a^t y^\Delta(s) \Delta(s) = Y(t) - Y(a).
\]

A function \( p : \mathbb{T} \rightarrow \mathbb{R} \) is called regressive provided \( 1 + \mu(t)p(t) \neq 0 \) for all \( t \in \mathbb{T}^k \). The set of all regressive and rd-continuous functions \( p : \mathbb{T} \rightarrow \mathbb{R} \) will be denoted by \( \mathbb{R}_1 = \mathbb{R}(\mathbb{T}) = \mathbb{R}(\mathbb{T}, \mathbb{R}) \). If \( \mu(t)p(t) > 0 \) for all \( t \in \mathbb{T} \), then \( p \) is a positively regressive function from \( \mathbb{T} \). If \( r \) is a regressive function, then the generalized exponential function \( e_r \) is defined by

\[
e_r(t, s) = \exp \left( \int_s^t \xi_{\mu(s)}(r(\tau)) \Delta \tau \right), s, t \in \mathbb{T},
\]

with the cylinder transformation

\[
\xi_h(z) = \begin{cases} 
\frac{\log(1 + hz)}{h}, & h \neq 0, \\
\frac{z}{2}, & h = 0.
\end{cases}
\]

Definition 2 [22] For each \( t \in \mathbb{T} \), let \( \mathbb{N} \) be a neighbourhood of \( t \). Then we define the generalized derivation (of Dini derivative), \( D + u\Delta(t) \), to mean
that, given \( \varepsilon > 0 \), there exists a right neighbourhood \( N(\varepsilon > 0) \subset N \) of \( t \) such that
\[
\frac{u(\sigma(t)) - u(t)}{\sigma(t) - s} < D^+ u^\Delta(t) + \varepsilon
\]
for each \( s \in N(\varepsilon > 0) \), \( s > t \). In case \( t \) is right-scattered and \( u(t) \) is continuous at \( t \), this reduces to
\[
D^+ u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\sigma(t) - s}.
\]

Lemma 1 [17] The following facts hold:

1. \((v_1 f + v_2 g)^\Delta = v_1 f^\Delta + v_2 g^\Delta\), for any constants \( v_1 \) and \( v_2 \);
2. \((f g)^\Delta = (t) = f^\Delta(t) g(t) + f(\sigma(t)) g^\Delta(t) = f(t) g^\Delta(t) + f^\Delta(t) g(\sigma(t))\);
3. if \( f^\Delta \geq 0 \), then \( f \) is nondecreasing.

Lemma 2 [23] Let \(-\alpha \in \mathfrak{N}\). The following facts are valid:

1. If \( N^\Delta(t) \leq b - a N^\Delta(t) \), then for \( t > t_0 \),
\[
N(t) \leq N(t_0) e^\alpha(t, t_0) + \left(\frac{b}{a}\right)^\alpha (1 - e^\alpha(t, t_0)).
\]
In particular, if \( a, b > 0 \), we have
\[
\limsup_{t \to \infty} N(t) \leq \left(\frac{b}{a}\right)^\alpha,
\]
where \( \alpha \) is a positive constant.

2. If \( N^\Delta(t) \geq b - a N^\Delta(t) \), then for \( t > t_0 \),
\[
N(t) \geq N(t_0) e^\alpha(t, t_0) + \left(\frac{b}{a}\right)^\alpha (1 - e^\alpha(t, t_0)).
\]
In particular, if \( a, b > 0 \), we have
\[
\liminf_{t \to \infty} N(t) \geq \left(\frac{b}{a}\right)^\alpha,
\]
where \( \alpha \) is a positive constant.

PERMANENCE

Definition 3 System (3) is said to be permanent if for any solution \((N_1(t), N_2(t), u_1(t), u_2(t))^T\) of system (3), there exist some constants \( m_i, q_i, M_i \) and \( Q_i \) \((i = 1, 2)\) such that
\[
m_i \leq \liminf_{t \to \infty} N_i(t) \leq \limsup_{t \to \infty} N_i(t) \leq M_i,
\]
\[
q_i \leq \liminf_{t \to \infty} u_i(t) \leq \limsup_{t \to \infty} u_i(t) \leq Q_i.
\]

Upper bounds of prey, predator and feedback control population

Proposition 1 Every solution \((N_1(t), N_2(t), u_1(t), u_2(t))^T\) of system (3) satisfies
\[
\limsup_{t \to \infty} N_1(t) \leq N_1^*, \quad \limsup_{t \to \infty} u_1(t) \leq u_1^*.
\]

Proof: Let \((N_1(t), N_2(t), u_1(t), u_2(t))^T\) be any solution of system (3), it follows from the first equation of system (3) and the Bernoulli inequality, \( \exp\{x\} \geq 1 + x \), for \( x \in \mathbb{R} \), we obtain
\[
N_1^*(t) = b(t) - \alpha(t) \exp\{N_1(t)\} \leq b(t) - \alpha(t)(N_1(t) + 1) \leq b^M - \alpha(t) N_1(t), \quad t \in \mathbb{T}.
\]

It follows from Lemma 2 that
\[
\limsup_{t \to \infty} N_1(t) \leq \frac{b^M - \alpha(t)}{\alpha(t)} := N_1^*.
\]

Now, for any \( \varepsilon > 0 \), there exists a \( t_0 \in \mathbb{T} \) such that \( N_1(t) \leq N_1^* + \varepsilon \) for all \( t \geq t_0 \). Then, from the second equation of system (3), we obtain
\[
N_2^*(t) = g(t) - f(t) \exp\{N_2(t) - N_1(t)\} \leq g(t) - \frac{f(t)}{\exp\{N_1^*\}}(N_2(t) + 1) \leq g^M - \frac{f_1^M}{\exp\{N_1^*\}} N_2(t), \quad t \in \mathbb{T}.
\]

It follows from Lemma 2 that
\[
\limsup_{t \to \infty} N_2(t) \leq \frac{g^M - f_1^M}{\exp\{N_1^*\}} := N_2^*.
\]

For any \( \varepsilon > 0 \) and there exists a \( t_1 > t_0 \) in the above inequality, which leads to
\[
N_2(t) \leq N_2^* + \varepsilon, \quad t > t_1, t \in \mathbb{T}.
\]

Similarly, from the third equation of system (3), we have for \( t > t_1, t \in \mathbb{T}, i = 1, 2 \)
\[
u_1^\Delta(t) \leq \alpha_i(t) \nu_1(t) - \beta_i(t) \nu_1(t) + \gamma_i(t) \exp\{N_1^* + \varepsilon\} \leq \alpha_i^M - \beta_i^M \nu_1(t) + \gamma_i^M \exp\{N_1^* + \varepsilon\}.
\]

Let \( \varepsilon \to 0 \) in the above inequality, it follows from Lemma 2 that
\[
\limsup_{t \to \infty} u_i(t) \leq \frac{\alpha_i^M + \gamma_i^M \exp\{N_1^*\}}{\beta_i^M} := u_i^*, \quad i = 1, 2.
\]

The proof is completed. \( \square \)
Lower bounds of prey, predator and feedback control population

Proposition 2 Assume that \((H_1)\) \(b^i > d^M u^*_1\) and \((H_2)\) \(g^i > p^M u^*_2\), then every solution \((N_1(t), N_2(t), u_1(t), u_2(t))\) of system (3) satisfies

\[
\lim_{t \to \infty} N_j(t) > N_{j*} \quad \lim_{t \to \infty} u_i(t) > u_{i*}
\]

where

\[
N_{1*} = \ln \frac{b^i - d^M u^*_1}{a^M + \frac{c^M}{h^2 \exp[N^*_2]}}, \quad N_{2*} = \ln \frac{g^i - p^M u^*_2}{f^M},
\]

\[
u_i = \frac{a^i + \gamma_i^i \exp[N_{1*}]}{\beta_i^M}, \quad i = 1, 2.
\]

Proof: For any \(\epsilon > 0\), according to Proposition 1, there exists a \(T_2 \in \mathbb{T}\) such that \(N_j(t) < N_{j*} + \epsilon, u_i(t) < u_{i*} + \epsilon\) for all \(t > T_2\), \(i = 1, 2\). Then for \(t > T_2\), from the first equation of system (3), we obtain

\[
N_{1*}^\Delta(t) \geq b(t) - a(t) \exp(N_1(t)) - d(t)(u^*_1 + \epsilon) - c(t) \exp(N_{1*}) + \frac{e^M}{f^M} \exp[2N_{1*}]
\]

\[
= b(t) - a(t) \exp(N_1(t)) - d(t)(u^*_1 + \epsilon) - c(t) \exp(N_{1*}) + \frac{e^M}{f^M} \exp[2N_{1*}]
\]

\[
\geq b^i - d^M(u^*_1 + \epsilon) - \frac{a^M + \frac{c^M}{h^2 \exp[N^*_2]}}{N_2^*} \exp[N_1(t)].
\]

We claim that for \(t > T_2\)

\[
b^i - d^M(u^*_1 + \epsilon) - \frac{a^M + \frac{c^M}{h^2 \exp[N^*_2]}}{N_2^*} \exp[N_1(t)] < 0. (4)
\]

Otherwise, assume that there exists \(\bar{t} > T_2\) such that

\[
b^i - d^M(u^*_1 + \epsilon) - \left(\frac{a^M + \frac{c^M}{h^2 \exp[N^*_2]}}{N_2^*}\right) \exp(N_1(\bar{t})) > 0
\]

and for any \(t \in [T_2, \bar{t}]\),

\[
b^i - d^M(u^*_1 + \epsilon) - \left(\frac{a^M + \frac{c^M}{h^2 \exp[N^*_2]}}{N_2^*}\right) \exp(N_1(t)) \leq 0.
\]

Hence,

\[
N_1(\bar{t}) < \ln \frac{b^i - d^M(u^*_1 + \epsilon)}{a^M + \frac{c^M}{h^2 \exp[N^*_2] + \epsilon}}
\]

and for any \(t \in [T_2, \bar{t}]\)

\[
N_1(t) \geq \ln \frac{b^i - d^M(u^*_1 + \epsilon)}{a^M + \frac{c^M}{h^2 \exp[N^*_2]}}.
\]

which implies \(N_{1*}^\Delta(\bar{t}) < 0\). It is a contradiction. Therefore, (4) holds for \(t > T_2\). Consequently, for \(t > T_2\),

\[
N_1(t) \geq \ln \frac{b^i - d^M(u^*_1 + \epsilon)}{a^M + \frac{c^M}{h^2 \exp[N^*_2]}}.
\]

Let \(\epsilon \to 0\), then

\[
\lim_{t \to \infty} N_1(t) \geq \ln \frac{b^i - d^M(u^*_1)}{a^M + \frac{c^M}{h^2 \exp[N^*_2]}} := N_{1*}.
\]

Now, for any small enough \(\epsilon > 0\), there exists a \(T_3 > T_2\) such that \(N_1(t) > N_{1*} - \epsilon\) and \(u_i(t) < u_{i*} + \epsilon\) for all \(t > T_3, i = 1, 2\). Similarly, from the second equation of system (3), we obtain for \(t > T_3\)

\[
N_{2*}^\Delta(t) \geq g(t) - f(t) \exp(N_2(t)) + \frac{e^M}{f^M} \exp[N_1(t)] - p(t)(u^*_2 + \epsilon)
\]

\[
\geq g^i - p^M(u^*_2 + \epsilon) - \frac{f^M}{\exp[N_{1*}]} \exp[N_2(t)].
\]

We claim that for \(t > T_3\)

\[
g^i - p^M(u^*_2 + \epsilon) - \frac{f^M}{\exp[N_{1*}]} \exp[N_2(t)] \leq 0. (5)
\]

Otherwise, assume that there exists \(\hat{t} > T_3\) such that

\[
g^i - p^M(u^*_2 + \epsilon) - \frac{f^M}{\exp[N_{1*}]} \exp[N_2(\hat{t})] > 0
\]

and for any \(t \in [T_3, \hat{t}]\),

\[
g^i - p^M(u^*_2 + \epsilon) - \frac{f^M}{\exp[N_{1*}]} \exp[N_2(t)] \leq 0.
\]

Hence,

\[
N_2(\hat{t}) < \ln \frac{g^i - p^M(u^*_2 + \epsilon)}{\frac{f^M}{\exp[N_{1*}]}},
\]

and for any \(t \in [T_3, \hat{t}]\),

\[
N_2(t) \geq \ln \frac{g^i - p^M(u^*_2 + \epsilon)}{\frac{f^M}{\exp[N_{1*}]}},
\]

which implies \(N_{2*}^\Delta(\hat{t}) < 0\). It is a contradiction. Therefore, (5) holds for \(t > T_3\). Consequently, for \(t > T_3\),

\[
N_2(t) \geq \ln \frac{g^i - p^M(u^*_2 + \epsilon)}{\frac{f^M}{\exp[N_{1*}]}},
\]

Let \(\epsilon \to 0\), then

\[
\lim_{t \to \infty} N_2(t) \geq \ln \frac{g^i - p^M(u^*_2)}{\frac{f^M}{\exp[N_{1*}]}},
\]

www.scienceasia.org
Now, for any small enough \( \epsilon > 0 \), there exists a \( t_4 \in \mathbb{T} \) such that \( N_i(t) \geq N_i - \epsilon \) for all \( t \geq t_4, i = 1, 2 \). From the third equation of system (3), we have

\[
\begin{align*}
\dot{u}_i(t) &\geq a_i(t) - \beta_i(t)u_i(t) + \gamma_i(t)\exp\{N_i + \epsilon\} \\
&\geq \alpha_i - \beta_i^M u_i(t) + \gamma_i^M\exp\{N_i - \epsilon\}, \quad i = 1, 2.
\end{align*}
\]

It follows from Lemma 2 that

\[
\lim_{t \to \infty} \inf_{t \geq t_0} u_i(t) \geq \frac{\alpha_i + \gamma_i^M\exp\{N_i\}}{\beta_i^M}, \quad i = 1, 2.
\]

Let \( \epsilon \to 0 \), we get

\[
\lim_{t \to \infty} \inf_{t \geq t_0} u_i(t) \geq \frac{\alpha_i + \gamma_i^M\exp\{N_i\}}{\beta_i^M} := u_{i*,} \quad i = 1, 2.
\]

The proof is completed. \( \Box \)

**Permanence result**

Now the main result of this section is obtained as follows.

**Theorem 1** Assume that (H_1) and (H_2) hold, then system (3) is permanent.

**Remark 1** If \( \mathbb{T} = \mathbb{R} \) or \( \mathbb{T} = \mathbb{Z} \), Theorem 1 gives a decision theorem for permanence of continuous or discrete Leslie predator-prey model with feedback controls, respectively. These results were obtained in [11, 15]. Therefore, our work in this paper extends the corresponding results in [11, 15].

**UNIFORM ASYMPTOTICAL STABILITY**

Set \( \mathbb{T}_1 = \{t \in \mathbb{T} : \sigma(t) > t\} \), \( \mu^T = \inf_{t \in \mathbb{T}_1} \mu(t) = \inf_{t \in \mathbb{T}_1} [\sigma(t) - t] \) and \( \mu^r = \sup_{t \in \mathbb{T}_1} \mu(t) = \sup_{t \in \mathbb{T}_1} [\sigma(t) - t] \).

**Theorem 2** Assume that (H_1) and (H_2) hold and suppose further that one of the following cases holds:

\( \mathbb{T}_1 = \emptyset \) and \( \exists i > 0 \), where

\[
\begin{align*}
\Xi_1 &= \min \left\{ a_i^e N_i - c_i M H - f_i^e N_i - \gamma_i^M e_i, \\
&\quad \frac{f_i}{e_i} N_i - c_i M H - f_i^e N_i - \gamma_i^M e_i, \beta_i^M - \mu_i^r - d_i M, \beta_i^r - d_i M \right\}, \\
H &= \left[ \left(h_i^r + e_i^2 N_i\right) e_i^2 N_i \right]^{1/2}, \\
\max \left\{ \left| e_i^2 N_i - (h_i^r)^2 e_i^2 N_i \right|, \left| h_i^r e_i^2 N_i - e_i^2 N_i \right| \right\}.
\end{align*}
\]

\( \mathbb{T}_1 = \emptyset \) and \( \exists i > 0 \), where

\[
\begin{align*}
\Xi_2 &= \min \left\{ \frac{1}{\mu_i^r} - a_i^e M H - f_i^e N_i - \gamma_i^M e_i, \\
&\quad \frac{1}{\mu_i^r} - f_i^e M H - f_i^e N_i - \gamma_i^M e_i, \\
&\quad \frac{1}{\mu_i^r} - \beta_i^M - d_i^r - M, \frac{1}{\mu_i^r} - \beta_i^r - d_i - M \right\},
\end{align*}
\]

\[
a^* = \max \left\{ \left| \frac{1}{\mu_i^r} - a_i^e N_i \right|, \left| a_i^e N_i - \frac{1}{\mu_i^r} \right| \right\},
\]

\[
f^* = \max \left\{ \left| \frac{1}{\mu_i^r} - f_i^e N_i \right|, \left| f_i^e N_i - \frac{1}{\mu_i^r} \right| \right\},
\]

\[
\beta_i^* = \max \left\{ \left| \frac{1}{\mu_i^r} - \beta_i^M \right|, \left| \beta_i^M - \frac{1}{\mu_i^r} \right| \right\}, \quad i = 1, 2.
\]

\( \mathbb{T}_1 \cap \mathbb{T}_1 \neq \emptyset, \mathbb{T}_1 \not\supset \mathbb{T}, \Xi_1 > 0 \) and \( \Xi_2 > 0 \),

then system (3) is uniformly asymptotically stable.

**Proof:** Suppose that \( X(t) = (N_1(t), N_2(t), u_1(t), u_2(t))^T \) and \( \bar{X}(t) = (\bar{N}_1(t), \bar{N}_2(t), \bar{u}_1(t), \bar{u}_2(t))^T \) are any two solutions of system (3), it has

\[
N_1^\Delta(t) = b(t) - a(t)\exp\{N_i(t)\} - d(t)u_i(t)
\]

\[
- \frac{c(t)\exp\{N_i(t) + N_2(t)\}}{h(t)\exp\{2N_i(t)\} + \exp\{2N_1(t)\}},
\]

\[
N_2^\Delta(t) = g(t) - f(t)\exp\{N_i(t) - N_2(t) - p(t)u_2(t),
\]

\[
\bar{u}_1^\Delta(t) = a_i(t) - \beta_i(t)u_i(t) + \gamma_i(t)\exp\{N_i(t)\},
\]

\[
\bar{N}_1^\Delta(t) = b(t) - a(t)\exp\{\bar{N}_i(t)\}
\]

\[
- \frac{c(t)\exp\{\bar{N}_i(t) + \bar{N}_2(t)\}}{h(t)\exp\{2\bar{N}_2(t)\} + \exp\{2\bar{N}_1(t)\}},
\]

\[
\bar{N}_2^\Delta(t) = g(t) - f(t)\exp\{\bar{N}_i(t) - \bar{N}_2(t) - p(t)\bar{u}_2(t),
\]

\[
\bar{u}_2^\Delta(t) = a_i(t) - \beta_i(t)\bar{u}_i(t) + \gamma_i(t)\exp\{\bar{N}_i(t)\}, \quad i = 1, 2.
\]

Considering the Lyapunov function \( V(t, X, \bar{X}) \) on \( \mathbb{T} \) defined by

\[
V(t) = V(t, X, \bar{X}) = \sum_{i=1}^{\bar{X}} \left| N_i(t) - \bar{N}_i(t) \right| + \sum_{i=1}^{\bar{X}} \left| u_i(t) - \bar{u}_i(t) \right|.
\]

**Case 1:** \( \mathbb{T}_1 = \emptyset, \) i.e., \( \sigma(t) = t, \forall t \in \mathbb{T} \). In view of system (6), we get

\[
(N_i(t) - \bar{N}_i(t))^\prime = -a_i(t)\exp\{N_i(t)\} - \exp\{\bar{N}_i(t)\})
\]

\[
- d(t)u_i(t) - \bar{u}_i(t)
\]

\[
- \frac{c(t)\exp\{N_i(t) + N_2(t)\}}{h(t)\exp\{2N_1(t)\} + \exp\{2N_i(t)\}}
\]

\[
- \frac{c(t)\exp\{\bar{N}_i(t) + \bar{N}_2(t)\}}{h(t)\exp\{2\bar{N}_2(t)\} + \exp\{2\bar{N}_1(t)\}}
\]

\[
(N_i(t) - \bar{N}_i(t))^\prime = -f(t)\exp\{N_i(t) - N_2(t) - \bar{N}_i(t))
\]

\[
- \exp(\bar{N}_2(t) - \bar{N}_i(t))) + p(t)u_2(t) - \bar{u}_2(t),
\]

\[
(u_i(t) - \bar{u}_i(t))^\prime = -\beta_i(t)u_i(t) - \bar{u}_i(t)
\]

\[
+ \gamma_i(t)\exp\{\bar{N}_i(t) - \exp(\bar{N}_i(t))\}, \quad i = 1, 2.
\]

Calculating the right derivative \( D^+ V^\Delta(t) \) of \( V \) along the solution of system (7) leads to

\[
D^+ V^\Delta(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t),
\]

where

\[
V_1(t) = D^+|N_i(t) - \bar{N}_i(t)|, \quad V_2(t) = D^+|N_2(t) - \bar{N}_2(t)|,
\]

\[
V_3(t) = D^+|u_1(t) - \bar{u}_1(t)|, \quad V_4(t) = D^+|u_2(t) - \bar{u}_2(t)|.
\]
By using the mean value theorem, it yields from system (7) that

\[ V_1(t) = \text{sgn}(N_1(t) - \tilde{N}_1(t))(N_1(t) - \tilde{N}_1(t))' \]
\[ \leq -[\alpha^t e^{|N_1|} - c^M H |N_1(t) - \tilde{N}_1(t)| + c^M H |N_2(t) - \tilde{N}_2(t)| + d^M |u_1(t) - \tilde{u}_1(t)|, \]
\[ V_2(t) = \text{sgn}(N_2(t) - \tilde{N}_2(t))(N_2(t) - \tilde{N}_2(t))' \]
\[ \leq -f^I \ e^{N_2_1 - N_1_1} |N_2(t) - \tilde{N}_2(t)| + f^M e^{N_2_1 - N_1_1} |N_1(t) - \tilde{N}_1(t)| + p^M |u_2(t) - \tilde{u}_2(t)|, \]
and, for \( i = 1, 2 \),
\[ V_i(t) = \text{sgn}(u_i(t) - \tilde{u}_i(t))(u_i(t) - \tilde{u}_i(t))' \]
\[ \leq -\beta^I_i |u_i(t) - \tilde{u}_i(t)| + \gamma^M_i e^{N_1} |N_i(t) - \tilde{N}_i(t)|. \]

Therefore, for \( t \in \mathbb{T}_1 \),

\[ D^+ V^A(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \]
\[ \leq -[\alpha^t e^{|N_1|} - c^M H |N_1(t) - \tilde{N}_1(t)| - f^I \ e^{N_2_1 - N_1_1} |N_2(t) - \tilde{N}_2(t)| - \beta^I_i |u_i(t) - \tilde{u}_i(t)| + \gamma^M_i e^{N_1} |N_i(t) - \tilde{N}_i(t)|] |u_i(t) - \tilde{u}_i(t)| \]
\[ -[\alpha^t e^{|N_1|} - c^M H |N_1(t) - \tilde{N}_1(t)| - f^I \ e^{N_2_1 - N_1_1} |N_2(t) - \tilde{N}_2(t)| - \beta^I_i |u_i(t) - \tilde{u}_i(t)| + \gamma^M_i e^{N_1} |N_i(t) - \tilde{N}_i(t)|] |u_i(t) - \tilde{u}_i(t)| \]
\[ \leq -\Xi_3 V(t). \]

Integrating (8) from 0 to \( t \) leads to

\[ V(t) + \Xi_3 \int_0^t \left[ \sum_{i=1}^2 |N_i(s) - \tilde{N}_i(s)| + \sum_{i=1}^2 |u_i(s) - \tilde{u}_i(s)| \right] ds \leq V(0) < \infty, \]
for all \( t > 0 \), that is,
\[ \int_0^t \left[ \sum_{i=1}^2 |N_i(s) - \tilde{N}_i(s)| + \sum_{i=1}^2 |u_i(s) - \tilde{u}_i(s)| \right] ds < \infty, \]
which implies
\[ \lim_{s \to \infty} \left[ \sum_{i=1}^2 |N_i(s) - \tilde{N}_i(s)| + \sum_{i=1}^2 |u_i(s) - \tilde{u}_i(s)| \right] = 0. \]

Thus, system (3) is uniformly asymptotically stable.

**Case 2:** \( \mathbb{T}_1 = \mathbb{T} \), i.e., \( \sigma(t) > t \), \( \forall t \in \mathbb{T} \). In view of system (6), we get

\[ \begin{cases} \text{\( N_1(\sigma(t)) \rightarrow \tilde{N}_1(\sigma(t)) \)} = N_1(t) - \tilde{N}_1(t) - \mu(t)d(t)(u(t) - \tilde{u}(t)) \\
- \mu(t)a(t)(\exp(N_1(t)) - \exp(\tilde{N}_1(t))) \\
- \mu(t)c(t)(\exp(N_2(t)) - \exp(\tilde{N}_2(t))) \\
- \mu(t)e(t)(\exp(N_1(t) + N_2(t))) \\
+ \mu(t)\gamma_1(t)(\exp(N_1(t)) - \exp(\tilde{N}_1(t))) \end{cases} \]
\[ N_2(\sigma(t)) \rightarrow \tilde{N}_2(\sigma(t)) = N_2(t) - \tilde{N}_2(t) - \mu(t)p(t)(u(t) - \tilde{u}(t)) \\
- \mu(t)f(t)(\exp(N_1(t)) - \exp(\tilde{N}_1(t))) \\
- \mu(t)\gamma_2(t)(\exp(N_2(t)) - \exp(\tilde{N}_2(t))) \]
\[ u_i(\sigma(t)) - \tilde{u}_i(\sigma(t)) = (1 - \mu(t))\beta_i(t)(u(t) - \tilde{u}(t)) \]
\[ + \mu(t)\gamma_i(t)(\exp(N(t)) - \exp(\tilde{N}(t))), \quad i = 1, 2. \]

Calculating the right derivative \( D^+ V^A \) of \( V \) along the solution of system (9) leads to

\[ D^+ V^A(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t), \]
where
\[ \dot{V}_1(t) = \frac{|N_1(t) - \tilde{N}_1(t)|}{\mu(t)} \]
\[ \dot{V}_2(t) = \frac{|N_2(t) - \tilde{N}_2(t)|}{\mu(t)} \]
\[ \dot{V}_3(t) = \frac{|u_1(t) - \tilde{u}_1(t)|}{\mu(t)} \]
\[ \dot{V}_4(t) = \frac{|u_2(t) - \tilde{u}_2(t)|}{\mu(t)}. \]

By the mean value theorem, we have from system (9) that
\[ |N_1(\sigma(t)) - \tilde{N}_1(\sigma(t))| \leq \mu(t)(a^t + c^M H)|N_1(t) - \tilde{N}_1(t)| \]
\[ + \mu(t)^2 e^{N_1} |N_2(t) - \tilde{N}_2(t)| \]
\[ + \mu(t)d^M |u_1(t) - \tilde{u}_1(t)|, \]
\[ |N_2(\sigma(t)) - \tilde{N}_2(\sigma(t))| \leq \mu(t)f^*|N_2(t) - \tilde{N}_2(t)| \]
\[ + \mu(t)^2 e^{N_2} |N_1(t) - \tilde{N}_1(t)| \]
\[ + \mu(t)p^M |u_2(t) - \tilde{u}_2(t)|, \]
\[ |u_1(\sigma(t)) - \tilde{u}_1(\sigma(t))| \leq \mu(t)^2 \beta_i(t)^* |u_i(t) - \tilde{u}_i(t)| \]
\[ + \mu(t)\gamma_i(t)^* |u_i(t) - \tilde{u}_i(t)|, \quad i = 1, 2. \]
Thus, system (3) is uniformly asymptotically stable. This completes the proof.

Remark 2 In Theorem 2, condition (H₃) corresponds to $\mathbb{T} = \mathbb{R}$, condition (H₄) corresponds to $\mathbb{T} = \mathbb{Z}$ and condition (H₅) corresponds to $\mathbb{T} \neq \mathbb{R}$ and $\mathbb{T} \neq \mathbb{Z}$. When $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, Theorem 2 gives a decision theorem for uniform asymptotical stability of continuous or discrete Leslie predator-prey model with feedback controls, respectively. These results were obtained in [11, 15]. Therefore, our work in this paper extends the corresponding results in [11, 15].

**BIOLOGICAL MEANINGS**

Theorems 1 and 2 imply the following biological indications:

1. The prey and predator populations have the upper bounds if the growth rate of prey exceeds the prey death rate, and the product of the growth rate of predator and the maximum upper bounds of prey exceed the maximum value $f$ which represents per capita reduction rate of prey $N₁$.

2. The prey and predator populations have the lower bounds if the growth rate of prey (i.e., $b$) exceeds the feedback control $d₁$ and the growth rate of predator (i.e., $g$) exceeds the feedback control $p₂$.

3. The maximum upper bounds of prey and predator populations are $(b^M - d^1) / a^1$ and $(g^M - p^2) / N^2₁$, respectively. It is clear that predator population grows as prey population grows.

4. From the definitions of $N^*₁$ and $N^*₂$, the growth rates of prey and predator, the prey death rate and the maximum value $f$ which represents per capita reduction rate of prey $N₁$ could effectively regulate the upper bounds of prey and predator populations, respectively. The upper bounds of prey and predator populations grow as the growth rates of prey and predator grow, and the prey death rate and the maximum value $f$ reduce, respectively.

5. The minimum lower bounds of prey and predator populations are $\ln\left(\frac{(b^₁ - d^M u^₁)}{(a^M + \frac{u^M}{N^1₁})}\right)$ and $\ln\left(\frac{(g^₁ - p^M u^₂)}{N^2₁}\right)$, respectively. It is easy to observe that the lower bound of prey population grows as the upper bound of predator population grows. Meanwhile, the lower bound of predator population grows as the lower bound of prey population grows.

6. From conditions (H₅)-(H₇), the uniform asymptotical stability of system (3) is influenced by the coefficients of the system. If the maximum value $c$ of the attainable per capita reduction rate of $N₁$, the interference parameter $h$ and the feedback control coefficients (i.e., $d, p, γ₁, γ₂$) are small enough, then system (3) is uniformly asymptotically stable.
NUMERICAL ILLUSTRATIONS

Example 1 Regarding the following Leslie-Gower system with feedback controls:

\[
\begin{aligned}
N_1^\Delta(t) &= 4 + |t - 4.5| \exp\{N_1(t)\} - 0.002u_1(t) \\
&\quad - 0.01\exp\{2N_2(t)\} + \exp\{2N_1(t)\}, \\
N_2^\Delta(t) &= 0.3 + 0.1\cos t - 0.2\exp\{N_1(t) - N_2(t)\} \quad (14) \\
&\quad - 0.01u_2(t), \\
&\quad t \in T = \{0.1k : k \in \mathbb{Z}, t = 1, 2, \\}
\end{aligned}
\]

Corresponding to system (3), Theorems 1 and 2, \( N_1^* = 0.11, N_2^* = 1.2321, u_1^* = 0.0558, u_2^* = 0.1714, N_1 = -0.5130, N_2 = -0.1075, u_1 = 0.0299, u_2 = 0.0449 \). Clearly, \((H_1)-(H_2)\) in Theorem 1 are satisfied. By Theorem 1, system (14) is permanent, which can be seen in Fig. 1.

Further, \( a^* = 7.3058, f^* = 9.8391, \beta_1^* = \beta_2^* = 9.8, H = 35.2780, \Xi_2 = \min(1.5082, 0.0913, 0.1980, 0.1990) = 0.0913 > 0 \). So \((H_4)\) in Theorem 2 holds. By Theorem 2, system (14) is uniformly asymptotically stable, as shown in Fig. 2 and Fig. 3.

Acknowledgements: This work is supported by National Nature Science Foundation of China (Nos. 11461082, 61903323 and 11961078), the Natural Scientific Research Fund Project of Yunnan Province (Grant No. 2018FH001-012) and the Construction Plan of Key Laboratory of Institutions of Higher Education in Yunnan Province.
REFERENCES