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On the Diophantine equations $x^2 - xy - y^2 \pm lx = 0$ and $x^2 - 3xy + y^2 \pm lx = 0$

Yaowaluk Alibaud, Supawadee Prugsapitak*, Wuttichon Aukkhosuwan

Algebra and Applications Research Unit, Department of Mathematics and Statistics, Faculty of Science, Prince of Songkla University, Songkhla 90110 Thailand

*Corresponding author, e-mail: supawadee.p@psu.ac.th

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ABSTRACT: This study determines all positive integer solutions of the Diophantine equations of the form $x^2 - xy - y^2 \pm lx = 0$ and $x^2 - 3xy + y^2 \pm lx = 0$, where l is of the form $m\prod_{i=1}^{n} p_i^{a_i}$, p_i is a prime congruent to 2 or 3 modulo 5, and m is either 1 or 5. In addition, we also provide all positive integer solutions of $x^2 - xy - y^2 \pm ly = 0$.

KEYWORDS: Fibonacci, Lucas, Diophantine

MSC2010: 11D09 11D45

INTRODUCTION

Marlewski and Zarzycki [1] proved that the equation

$$x^2 - kxy + y^2 + x = 0, \tag{1}$$

where k is a positive integer, has an infinite number of positive integer solutions x and y if and only if $k = 3$. Yuan and Hu [2] proved that the Diophantine equation $x^2 - kxy + y^2 + lx = 0$, where $l = 1, 2, 4$, has an infinite number of positive integer solutions x and y if and only if $(k, l) = (3, 1), (3, 2), (4, 2), (3, 4), (4, 4), (6, 4)$. Moreover, they also proved that the Diophantine equation $x^2 - kxy + y^2 + x = 0$ has an infinite number of integer solutions x and y if and only if $k \neq 0, \pm 1$, which generally characterizes integer solutions of (1). Keskin [3] demonstrated that when $k > 3$, $x^2 - kxy + y^2 + x = 0$ has no positive integer solutions, but the equation $x^2 - kxy + y^2 - x = 0$ does have positive integer solutions. Moreover, the equations $x^2 - kxy - y^2 \pm x = 0$ and $x^2 - kxy - y^2 \pm y = 0$ are shown to have positive integer solutions when $k \geqslant 1$. Karaatli and Şiar [4] studied the Diophantine equation $x^2 - kxy + ky^2 + ly = 0$, $l \in \{1, 2, 4, 8\}$ and determined the values of k when the given equations has infinitely many positive integer solutions x and y. Feng et al [5] showed that for any given positive integer l, there are only finitely many integers k such that $x^2 - kxy + y^2 + lx = 0$ has an infinite number of positive integer solutions. Moreover, they determined all integers k such that the Diophantine equation $x^2 - kxy + y^2 + lx = 0$, $1 \leqslant l \leqslant 33$, has an infinite number of positive integer solutions. Cipu [6] determined values of a such that for fixed positive integer k, the equation $x^2 - axy + y^2 + kx = 0$ has infinitely many positive integer solutions. Özdemir and Keskin [7] gave all positive integer solutions of the equations $x^2 - 3xy + y^2 \pm x = 0$ and showed that $x^2 - 7xy + y^2 + x = 0$ has no positive integer solutions.

In this study, we examine the Diophantine equations of the form $x^2 - kxy - y^2 \pm lx = 0$, $x^2 - kxy - y^2 \pm ly = 0$, and $x^2 - kxy + y^2 \pm lx = 0$, where k is either 1 or 3, l is of the form $m\prod_{i=1}^{n} p_i^{a_i}$, p_i is a prime congruent to 2 or 3 modulo 5, and m is either 1 or 5. We prove that these equations have an infinite number of solutions and explicitly provide their positive integer solutions.

PRELIMINARY

This section presents some theorems that are later used in our results section. The proofs can be found in [3]. Before we state the theorems, we first review the famous Fibonacci and Lucas sequences. The Fibonacci sequence is defined by

$$F_0 = 0, F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}$$

for $n \geqslant 1$, and the Lucas sequence is defined by

$$L_0 = 2, L_1 = 1 \quad \text{and} \quad L_{n+1} = L_n + L_{n-1}$$

for $n \geqslant 1$.

Theorem 1 *The positive integer solutions of equations $x^2 - xy - y^2 \pm x = 0$ or $x^2 - xy - y^2 \pm y = 0$ are given as follows:*

(a) *All positive integer solutions of equation* $x^2 - xy - y^2 + x = 0$ *are given by* $(x,y) = (F_{2n}^2, F_{2n}F_{2n-1})$ *with* $n \geqslant 1$.

(b) *All positive integer solutions of equation* $x^2 - xy - y^2 - x = 0$ *are given by* $(x,y) = (F_{2n+1}^2, F_{2n+1}F_{2n})$ *with* $n \geqslant 1$.

(c) *All positive integer solutions of equation* $x^2 - xy - y^2 + y = 0$ *are given by* $(x,y) = (F_{2n}F_{2n-1}, F_{2n-1}^2)$ *with* $n \geqslant 1$.

(d) *All positive integer solutions of equation* $x^2 - xy - y^2 - y = 0$ *are given by* $(x,y) = (F_{2n+1}F_{2n}, F_{2n+1}^2)$ *with* $n \geqslant 1$.

Theorem 2 *The positive integer solutions of equations* $x^2 - xy - y^2 \pm 5x = 0$ *or* $x^2 - xy - y^2 \pm 5y = 0$ *are given as follows:*

(a) *All positive integer solutions of equation* $x^2 - xy - y^2 + 5x = 0$ *are given by* $(x,y) = (5F_{2n}^2, 5F_{2n}F_{2n-1})$ *with* $n \geqslant 1$ *or* $(x,y) = (L_{2n+1}^2, L_{2n+1}L_{2n})$ *with* $n \geqslant 0$.

(b) *All positive integer solutions of equation* $x^2 - xy - y^2 - 5x = 0$ *are given by* $(x,y) = (5F_{2n+1}^2, 5F_{2n+1}F_{2n})$ *with* $n \geqslant 1$ *or* $(x,y) = (L_{2n}^2, L_{2n}L_{2n-1})$ *with* $n \geqslant 1$.

(c) *All positive integer solutions of equation* $x^2 - xy - y^2 + 5y = 0$ *are given by* $(x,y) = (5F_{2n}F_{2n-1}, 5F_{2n-1}^2)$ *with* $n \geqslant 1$ *or* $(x,y) = (L_{2n}L_{2n+1}, L_{2n+1}^2)$ *with* $n \geqslant 0$.

(d) *All positive integer solutions of equation* $x^2 - xy - y^2 - 5y = 0$ *are given by* $(x,y) = (5F_{2n+1}F_{2n}, 5F_{2n}^2)$ *or* $(x,y) = (L_{2n}L_{2n-1}, L_{2n-1}^2)$ *with* $n \geqslant 1$.

Theorem 3 *The positive integer solutions of equations* $x^2 - 3xy + y^2 \pm x = 0$ *are given as follows:*

(a) *All positive integer solutions of equation* $x^2 - 3xy + y^2 + x = 0$ *are given by* $(x,y) = (F_{2n+1}^2, F_{2n+1}F_{2n-1})$ *or* $(x,y) = (F_{2n-1}^2, F_{2n-1}F_{2n+1})$ *with* $n \geqslant 1$.

(b) *All positive integer solutions of equation* $x^2 - 3xy + y^2 - x = 0$ *are given by* $(x,y) = (F_{2n}^2, F_{2n}F_{2n+2})$ *or* $(x,y) = (F_{2n+2}^2, F_{2n+2}F_{2n})$ *with* $n \geqslant 1$.

Theorem 4 *The positive integer solutions of equations* $x^2 - 3xy + y^2 \pm 5x = 0$ *are given as follows:*

(a) *All positive integer solutions of equation* $x^2 - 3xy + y^2 + 5x = 0$ *are given by* $(x,y) = (5F_{2n+1}^2, 5F_{2n+1}F_{2n-1})$ *with* $n \geqslant 1$ *or* $(x,y) = (5F_{2n-1}^2, 5F_{2n-1}F_{2n+1})$ *with* $n \geqslant 1$ *or* $(x,y) = (L_{2n+2}^2, L_{2n+2}L_{2n})$ *with* $n \geqslant 0$ *or* $(x,y) = (L_{2n}^2, L_{2n}L_{2n+2})$ *with* $n \geqslant 0$.

(b) *All positive integer solutions of equation* $x^2 - 3xy + y^2 - 5x = 0$ *are given by* $(x,y) = (5F_{2n+2}^2, 5F_{2n+2}F_{2n})$ *with* $n \geqslant 1$ *or* $(x,y) = (5F_{2n}^2, 5F_{2n}F_{2n+2})$ *with* $n \geqslant 1$ *or* $(x,y) = (L_{2n+1}^2, L_{2n+1}L_{2n-1})$ *with* $n \geqslant 1$ *or* $(x,y) = (L_{2n-1}^2, L_{2n-1}L_{2n+1})$ *with* $n \geqslant 1$.

MAIN RESULTS

We first prove the following lemma.

Lemma 1 *Let* x, y, *and* k *be integers, and let* p *be an odd prime number. Let the Legendre symbol* $(\frac{k^2+4}{p}) = -1$. *Then*

$$p \mid y \text{ and } p \mid x \text{ if and only if } p \mid (x-y)^2 - k(x-y)y - y^2.$$

Proof: If $p \mid y$ and $p \mid x$, then it is clear that $p \mid (x-y)^2 - k(x-y)y - y^2$. Now suppose that $p \mid (x-y)^2 - k(x-y)y - y^2$. We claim that $p \mid y$. On the contrary, assume that $p \nmid y$. Let $a = \frac{p-1}{2}$. Then $2a \equiv -1 \pmod p$. We get

$$\begin{aligned} 0 &\equiv (x-y)^2 - k(x-y)y - y^2 \pmod p \\ &\equiv (x-y)^2 + (p-1)k(x-y)y - y^2 \pmod p \\ &\equiv [x-y+aky]^2 - [(ak)^2+1]y^2 \pmod p. \end{aligned}$$

As $p \nmid y$, $y^{-1} \pmod p$ exists and it is seen that

$$0 \equiv (y^{-1}[x-y+aky])^2 - [(ak)^2+1] \pmod p$$

and so

$$(2y^{-1}[x-y+aky])^2 \equiv 4(ak)^2 + 4 \pmod p.$$

Letting $b = 2y^{-1}[x-y+aky]$, it follows that

$$b^2 \equiv 4a^2k^2 + 4 \equiv (2a)^2k^2 + 4 \equiv k^2 + 4 \pmod p.$$

This contradicts our assumption that $(\frac{k^2+4}{p}) = -1$. Therefore $p \mid y$. Since $p \mid (x-y)^2 - k(x-y)y - y^2$, we get $p \mid x^2$, and so $p \mid x$. □

Lemma 2 *Let* $k = \pm 1$ *and* n *be a positive integer of the form* $\prod_{i=1}^r p_i^{a_i}$, *where* $p_i \equiv 2, 3 \pmod 5$ *is a prime number, and* $a_i \geqslant 0$ *for* $1 \leqslant i \leqslant r$. *If positive integers* x *and* y *satisfy the equation* $x^2 - (k+2)xy + ky^2 \pm nx = 0$, *then* $n \mid x$ *and* $n \mid y$.

Proof: Let $d = \gcd(x, y, n)$. Then $x = dx_1$, $y = dy_1$, and $n = dn_1$ for some positive integers x_1, y_1, and n_1 with $\gcd(x_1, y_1, n_1) = 1$. As $x^2-(k+2)xy+ky^2 \pm nx = 0$, we get

$$x_1^2-(k+2)x_1y_1+ky_1^2 \pm n_1x_1 = 0.$$

Assume that $n_1 > 1$. Then there is a prime p dividing n_1. If $p = 2$, then $2 \mid x_1^2-(k+2)x_1y_1+ky_1^2$; that is $2 \mid x_1^2 - x_1y_1 - y_1^2$ or $2 \mid x_1^2 - 3x_1y_1 + y_1^2$. It follows that $2 \mid x_1$ and $2 \mid y_1$ in each case, which is impossible since $\gcd(x_1, y_1, n_1) = 1$. Assume that $p > 2$. Then $p \mid x_1^2-(k+2)x_1y_1+ky_1^2$ since $p \mid n_1$. As $p \equiv 2, 3 \pmod 5$, it is seen that $(\frac{k^2+4}{p}) = -1$ for $k = \pm 1$. Then, by Lemma 1, we get $p \mid x_1$ and $p \mid y_1$ since $p \mid (x_1-y_1)^2 - k(x_1-y_1)y_1 - y_1^2$. Also $p \mid n_1$ implies that $p \mid \gcd(x_1, y_1, n_1)$. This is impossible because $\gcd(x_1, y_1, n_1) = 1$. Therefore $n_1 = 1$. This shows that $d = n$, which yields $n \mid x$ and $n \mid y$. □

Lemma 3 below can be proved similarly, so we will omit its proof.

Lemma 3 *Let $k = \pm 1$ and n be a positive integer of the form $\prod_{i=1}^{r} p_i^{a_i}$ where $p_i \equiv 2, 3 \pmod 5$ is a prime number, and $a_i \geqslant 0$ for $1 \leqslant i \leqslant r$. If positive integers x and y satisfy the equation $x^2-(k+2)xy+ky^2 \pm ny = 0$, then $n \mid x$ and $n \mid y$.*

From now on, we will assume that n is of the form stated in Lemma 2.

Theorem 5 *Let n be an integer as defined in Lemma 2. Then all positive integer solutions of equation $x^2 - xy - y^2 + nx = 0$ are given by*

$$(x, y) = (nF_{2m}^2, nF_{2m}F_{2m-1}), \qquad m \geqslant 1.$$

Proof: Let x and y be positive integers such that $x^2 - xy - y^2 + nx = 0$. Then by Lemma 2, we have $x = nx'$, $y = ny'$, where x' and y' are positive solutions of

$$x^2 - xy - y^2 + x = 0.$$

By Theorem 1(a), $x' = F_{2m}^2$ and $y' = F_{2m}F_{2m-1}$. Hence, $x = nF_{2m}^2$ and $y = nF_{2m}F_{2m-1}$. □

Theorem 6 *Let n be an integer as defined in Lemma 2. Then all positive integer solutions of equation $x^2 - 3xy + y^2 + nx = 0$ are given by*

$$(x, y) = (nF_{2m+1}^2, nF_{2m+1}F_{2m-1}) \quad \text{or}$$
$$(x, y) = (nF_{2m-1}^2, nF_{2m-1}F_{2m+1})$$

with $m \geqslant 1$.

Proof: Let x and y be positive integers such that $x^2 - 3xy - y^2 + nx = 0$. Then by Lemma 2, we have $x = nx'$, $y = ny'$, where x' and y' are positive solutions of

$$x^2 - 3xy - y^2 + x = 0.$$

By Theorem 3(a), $(x', y') = (F_{2m+1}^2, F_{2m+1}F_{2m-1})$ or $(x', y') = (F_{2m-1}^2, F_{2m-1}F_{2m+1})$. Hence, $(x, y) = (nF_{2m+1}^2, nF_{2m+1}F_{2m-1})$ or $(x, y) = (nF_{2m-1}^2, nF_{2m-1}F_{2m+1})$. □

Since the proofs of the following corollaries can be done similarly, we will omit their proofs. Lemma 2, Lemma 3 and some preliminary results may be useful in the proofs of the corollaries. Please note that in Corollary 2 and Corollary 4, the integer n in Lemma 2 and Lemma 3 should be replaced with $5n$.

Corollary 1 *The positive integer solutions of equations $x^2-xy-y^2-nx = 0$ and $x^2-xy-y^2 \pm ny = 0$ are given as follows:*

(a) All positive integer solutions of equation $x^2 - xy - y^2 - nx = 0$ are given by $(x, y) = (nF_{2m+1}^2, nF_{2m+1}F_{2m})$ with $m \geqslant 1$.

(b) All positive integer solutions of equation $x^2 - xy - y^2 + ny = 0$ are given by $(x, y) = (nF_{2m}F_{2m-1}, nF_{2m-1}^2)$ with $m \geqslant 1$.

(c) All positive integer solutions of equation $x^2 - xy - y^2 - ny = 0$ are given by $(x, y) = (nF_{2m+1}F_{2m}, nF_{2m+1}^2)$ with $m \geqslant 1$.

Corollary 2 *The positive integer solutions of equations $x^2-xy-y^2 \pm 5nx = 0$ and $x^2-xy-y^2 \pm 5ny = 0$ are given as follows:*

(a) All positive integer solutions of equation $x^2 - xy - y^2 + 5nx = 0$ are given by $(x, y) = (5nF_{2m}^2, 5nF_{2m}F_{2m-1})$ with $m \geqslant 1$ or $(x, y) = (nL_{2m+1}^2, nL_{2m+1}L_{2m})$ with $m \geqslant 0$.

(b) All positive integer solutions of equation $x^2 - xy - y^2 - 5nx = 0$ are given by $(x, y) = (5nF_{2m+1}^2, 5nF_{2m+1}F_{2m})$ with $m \geqslant 1$ or $(x, y) = (nL_{2m}^2, nL_{2m}L_{2m-1})$ with $m \geqslant 1$.

(c) All positive integer solutions of equation $x^2 - xy - y^2 + 5ny = 0$ are given by $(x, y) = (5nF_{2m}F_{2m-1}, 5nF_{2m-1}^2)$ with $m \geqslant 1$ or $(x, y) = (nL_{2m}L_{2m+1}, nL_{2m+1}^2)$ with $m \geqslant 0$.

(d) All positive integer solutions of equation $x^2 - xy - y^2 - 5ny = 0$ are given by $(x, y) = (5nF_{2m+1}F_{2m}, 5nF_{2m}^2)$ or $(x, y) = (nL_{2m}L_{2m-1}, nL_{2m-1}^2)$ with $m \geqslant 1$.

Corollary 3 *All positive integer solutions of equation* $x^2 - 3xy + y^2 - nx = 0$ *are given by* $(x, y) = (nF_{2m}^2, nF_{2m}F_{2m+2})$ *or* $(x, y) = (nF_{2m+2}^2, nF_{2m+2}F_{2m})$ *with* $m \geqslant 1$.

Corollary 4 *The positive integer solutions of equations* $x^2 - 3xy + y^2 \pm 5nx = 0$ *are given as follows:*

(a) *All positive integer solutions of equation* $x^2 - 3xy + y^2 + 5nx = 0$ *are given by* $(x, y) = (5nF_{2m+1}^2, 5nF_{2m+1}F_{2m-1})$ *with* $m \geqslant 1$ *or* $(x, y) = (5nF_{2m-1}^2, 5nF_{2m-1}F_{2m+1})$ *with* $m \geqslant 1$ *or* $(x, y) = (nL_{2m+2}^2, nL_{2m+2}L_{2m})$ *with* $m \geqslant 0$ *or* $(x, y) = (nL_{2m}^2, nL_{2m}L_{2m+2})$ *with* $m \geqslant 0$.

(b) *All positive integer solutions of equation* $x^2 - 3xy + y^2 - 5nx = 0$ *are given by* $(x, y) = (5nF_{2m+2}^2, 5nF_{2m+2}F_{2m})$ *with* $m \geqslant 1$ *or* $(x, y) = (5nF_{2m}^2, 5nF_{2m}F_{2m+2})$ *with* $m \geqslant 1$ *or* $(x, y) = (nL_{2m+1}^2, nL_{2m+1}L_{2m-1})$ *with* $m \geqslant 1$ *or* $(x, y) = (nL_{2m-1}^2, nL_{2m-1}L_{2m+1})$ *with* $m \geqslant 1$.

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