RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2020.045
ScienceAsia 46 (2020): **361–367**

Uniqueness problems on difference operators of meromorphic functions

Chuangxin Chen[a], Ranran Zhang[b,*], Zhibo Huang[c], Shuangting Lan[d]

[a] College of Computational Sciences, Zhongkai University of Agriculture and Engineering, Guangzhou 510225 China
[b] Department of Mathematics, Guangdong University of Education, Guangzhou 510303 China
[c] School of Mathematical Sciences, South China Normal University, Guangzhou 510631 China
[d] School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou 510665 China

*Corresponding author, e-mail: zhangranran@gdei.edu.cn

Received 19 Nov 2019
Accepted 9 Jun 2020

ABSTRACT: In this paper, we study the shared-value problem of forward differences $\Delta_c^2 f(z)$ and $\Delta_c f(z)$ of a meromorphic function $f(z)$. For an entire function $f(z)$ with a Borel exceptional small function, we give the specific expression of $f(z)$ when $\Delta_c^2 f(z)$ and $\Delta_c f(z)$ share a small function CM. For a meromorphic function $f(z)$ with a small deficient function, we obtain the relationship of $\Delta_c^2 f(z)$ and $\Delta_c f(z)$ when they share a small function and ∞ CM.

KEYWORDS: difference operators, meromorphic function, sharing value

MSC2010: 30D35 39A10

INTRODUCTION AND RESULTS

In this paper, we shall use the standard notations of Nevanlinna's value distribution theory such as $T(r,f), m(r,f), N(r,f)$ and $S(r,f)$ [1–3]. In addition, We use the notation $S(f)$ to denote the set of small functions of $f(z)$.

Uniqueness theory of meromorphic functions is an important part of complex analysis. As for the standard notations, let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a(z) \in S(f) \cap S(g)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (IM), if $f-a$ and $g-a$ have the same zeros counting multiplicities (ignoring multiplicities). We say that $f(z)$ and $g(z)$ share the value ∞ CM (IM), if f and g have the same poles counting multiplicities (ignoring multiplicities). The classical results in the uniqueness theory are five-point, respectively, four-point, theorems [4–7].

An active subject in the uniqueness theory is the investigation on the uniqueness of the meromorphic function sharing values with its derivatives, which was initiated by Rubel et al [8]. We first recall the following result by Jank et al [9].

Theorem 1 ([9]) *Let $f(z)$ be a nonconstant meromorphic function, and let $a \neq 0$ be a finite constant. If $f(z)$, $f'(z)$ and $f''(z)$ share the value a CM, then $f(z) \equiv f'(z)$.*

Recently, difference analogues of meromorphic functions have become an interest subject, and many results were expeditiously obtained [10–14]. In particular, some authors considered the uniqueness of meromorphic functions sharing small functions with their difference operators. The difference operators are defined by $\Delta_c f(z) = f(z+c) - f(z)$ and $\Delta_c^k f(z) = \Delta_c(\Delta_c^{k-1} f(z))$, $k \in \mathbb{N}$, $k \geqslant 2$. Chen et al [15] considered the difference analogue of Theorem 1, and obtained the following result.

Theorem 2 ([15]) *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z) \not\equiv 0 \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c^2 f(z) \equiv \Delta_c f(z)$.*

In [15], the authors gave the following example to show that the conclusion of Theorem 2 can occur.

Example 1 Let $f(z) = e^{z \ln 2}$ and $c = 1$. Then, for any $a \in \mathbb{C}$, we notice that $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share a CM and we can easily see that $\Delta_c^2 f(z) \equiv \Delta_c f(z)$.

In fact, Example 1 implies that $\Delta_c^2 f(z) \equiv \Delta_c f(z) \equiv f(z)$. Farissi et al [16] further studied the above problem and found that the claim $\Delta_c^2 f(z) \equiv \Delta_c f(z)$ in Theorem 2 can be replaced by $\Delta_c f(z) \equiv f(z)$. They obtained the following theorem.

Theorem 3 ([16]) *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z) \not\equiv 0 \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f(z) \equiv f(z)$.*

Example 2 Let $f(z) = e^{z\ln 2} + 1$ and $c = 1$. By calculation, we see that $\Delta_c f(z) = e^{z\ln 2}$ and $\Delta_c^2 f(z) = e^{z\ln 2}$ share every finite value b CM and can easily see that $\Delta_c^2 f(z) \equiv \Delta_c f(z)$, but cannot obtain $\Delta_c^2 f(z) \equiv \Delta_c f(z) \equiv f(z)$.

From Example 2, we find that $f(z)$ does not satisfy the condition "$f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM", but we still have the conclusion "$\Delta_c^2 f(z) \equiv \Delta_c f(z)$". We also find that the condition "$f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM" is relatively strong. Noting that the conclusion of Theorem 2 is "$\Delta_c^2 f(z) \equiv \Delta_c f(z)$", which does not involve $f(z)$, we pose the following questions.

Question 1. What will happen if we replace the condition "$f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM" by "$\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM" in Theorem 2?

Question 2. Can we get rid of the condition "$a(z)(\not\equiv 0)$ is a periodic entire function with period c" and only retain "$a(z)$ is an entire function" in Theorem 2 and Theorem 3?

In fact, we find that the entire functions in Example 1 and Example 2 both have a finite Borel exceptional value. Hence, we answer the above questions partly from the point of view of Borel exceptional values. In fact, we prove the following theorem and give the precise expression of $f(z)$, which is more profound than the conclusion in Theorem 2 and Theorem 3. The method we used is completely different from that used in Theorem 2 and Theorem 3, and basically comes from [10].

In the following, the notations $\rho(f)$ and $\rho_2(f)$ are used to denote the order and the hyper-order of a meromorphic function $f(z)$, respectively. The notation $\lambda(f)$ is used to denote the exponent of convergence of the zeros of $f(z)$. The deficiency of $a(z) \in S(f)$ is defined by

$$\delta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

If $\delta(a, f) > 0$, then $a(z)$ is called a small deficient function of $f(z)$.

We now answer the above questions and obtain:

Theorem 4 *Let $f(z)$ be a finite order transcendental entire function such that $\lambda(f - a) < \sigma(f)$, where $a(z)(\in S(f))$ is an entire function and satisfies $\rho(a) < 1$, let $c(\in \mathbb{C})$ be a constant such that $\Delta_c^2 f(z) \not\equiv 0$. If $\Delta_c^2 f(z)$ and $\Delta_c f(z)$ share the entire function $b(z)(\in S(f))$ CM, where $b(z) \not\equiv \Delta_c a(z)$ and $\rho(b) < 1$, then*

$$f(z) = a(z) + B\,e^{Az},$$

where A and B are two nonzero constants.

Remark 1 We see that Example 2 satisfies Theorem 4.

Noting that if $a(z)$ is a constant, then $\Delta_c a(z) = 0$. So by Theorem 4, we get the following corollary.

Corollary 1 *Let $f(z)$ be a finite order transcendental entire function with a finite Borel exceptional value a, and let $c(\in \mathbb{C})$ be a constant such that $\Delta_c^2 f(z) \not\equiv 0$. If $\Delta_c^2 f(z)$ and $\Delta_c f(z)$ share a finite value $b(\neq 0)$ CM, then*

$$f(z) = a + B\,e^{Az},$$

where A, B are two nonzero constants.

Remark 2 The conclusion of Corollary 1 implies that $\Delta_c^2 f(z) \equiv \Delta_c f(z) (\equiv B\,e^{Az})$ provided $Ac = \ln 2$. But generally, we cannot get $f(z) = a + B\,e^{Az}$ from $\Delta_c^2 f(z) \equiv \Delta_c f(z)$. So the conclusion of Corollary 1 is more specific than the conclusion of Theorem 2.

The condition "$f(z)$ is a finite order transcendental entire function with $\lambda(f - a) < \sigma(f)$" in Theorem 4 implies $\delta(a, f) = 1$. A natural question is: What can be said if we relax the restriction? For example, replace $\delta(a, f) = 1$ with $\delta(a, f) > 0$, or let $f(z)$ be a transcendental meromorphic function, or let the order of $f(z)$ be infinite. Next, we consider this question and obtain the following theorem.

Theorem 5 *Let $f(z)$ be a transcendental meromorphic function with $\rho_2(f) < 1$, and let $b(z), a(z) \in S(f)$ such that $b(z) \not\equiv a(z)$, $b(z) \not\equiv \Delta_c^i a(z)\,(i = 1, 2)$ and $\max\{\rho(b), \rho(a)\} < 1$. If $\Delta_c^2 f(z)$ and $\Delta_c f(z)$ share $b(z)$, ∞ CM and $\delta(a, f) > 0$, then*

$$\frac{\Delta_c^2 f(z) - b(z)}{\Delta_c f(z) - b(z)} = D$$

for some nonzero constant D. In particular, if the deficient function $a(z) \equiv 0$, then $\Delta_c^2 f(z) \equiv \Delta_c f(z)$.

LEMMAS

Lemma 1 ([14]) *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$W(z,f)P(z,f) = Q(z,f),$$

where $W(z,f)$, $P(z,f)$, $Q(z,f)$ are difference polynomials with small meromorphic coefficients, the degrees $\deg W(z,f) = n$, and $\deg Q(z,f) \leqslant n$. Moreover, we assume that $W(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then, for each $\varepsilon > 0$,

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2 ([18]) *Let $P_n(z), \dots, P_0(z)$ be polynomials such that $P_nP_0 \not\equiv 0$ and satisfy*

$$P_n(z) + \cdots + P_0(z) \not\equiv 0. \tag{1}$$

Then every finite order transcendental meromorphic solution $g(z)(\not\equiv 0)$ of the equation

$$P_n(z)g(z+n) + \cdots + P_0(z)g(z) = 0 \tag{2}$$

satisfies $\rho(g) \geqslant 1$, and $g(z)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often and $\lambda(g-a) = \rho(g)$.

Remark 3 It is easy to show that if $g(z+i)(i = 0,1,\dots,n)$ is replaced by $g(z+ic)(i = 0,1,\dots,n)$, where $c \neq 0$, the conclusion in Lemma 2 is still valid.

Lemma 3 ([19]) *Let g be a function transcendental and meromorphic in the plane of order < 1. Let $h > 0$. Then there exists an ε-set E such that*

$$g(z+c) - g(z) = cg'(z)(1+o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C}\backslash E,$$

uniformly in c for $|c| \leqslant h$.

Lemma 4 *Suppose that n is a positive integer, $f(z)$ is a finite order transcendental entire function such that $\lambda(f-a) < \rho(f)$, where $a(z)\,(\in S(f))$ is an entire function and satisfies $\rho(a) < 1$. If $\Delta_c^2 f(z) \not\equiv 0\ (c \in \mathbb{C})$ and*

$$\frac{\Delta_c^2 f(z) - b(z)}{\Delta_c f(z) - b(z)} = D, \tag{3}$$

where D is a nonzero constant and $b(z)\,(\not\equiv \Delta a(z))$ is an entire function with $\rho(b) < 1$, then

$$f(z) = a(z) + B\,\mathrm{e}^{Az} \quad \text{and} \quad D = \frac{\Delta_c^2 a(z) - b(z)}{\Delta_c a(z) - b(z)},$$

where A and B are two nonzero constants.

Proof: By the assumptions and Hadamard's factorization theory, $f(z)$ can be written as

$$f(z) = a(z) + B(z)\,\mathrm{e}^{h(z)}, \tag{4}$$

where $B(z)(\not\equiv 0)$ is an entire function, $h(z)$ is a polynomial of degree $\deg h(z) = k\,(k \geqslant 1)$, $B(z)$ and $a(z)$ satisfy

$$\lambda(B) = \rho(B) = \lambda(f-a) = \rho_1 < \rho(f) = \deg h. \tag{5}$$

Substituting (4) into (3), we can conclude that

$$\frac{B(z+2c)\,\mathrm{e}^{h(z+2c)} - 2B(z+c)\,\mathrm{e}^{h(z+c)} + B(z)\,\mathrm{e}^{h(z)} + u_2(z)}{B(z+c)\,\mathrm{e}^{h(z+c)} - B(z)\,\mathrm{e}^{h(z)} + u_1(z)} = D, \tag{6}$$

where $u_2(z) = \Delta_c^2 a(z) - b(z)$, $u_1(z) = \Delta_c a(z) - b(z)$. It is easy to see that, for $j = 1,2$,

$$\rho(u_j(z)) \leqslant \max\{\rho(\Delta_c^j a(z)), \rho(b(z))\} < 1. \tag{7}$$

We rewrite (6) in the form

$$B(z+2c)\,\mathrm{e}^{h(z+2c)-h(z)} - (2+D)B(z+c)\,\mathrm{e}^{h(z+c)-h(z)} + (1+D)B(z) = [Du_1(z) - u_2(z)]\,\mathrm{e}^{-h(z)}. \tag{8}$$

Firstly, we observe that $Du_1(z) - u_2(z) \equiv 0$. On the contrary, if $Du_1(z) - u_2(z) \not\equiv 0$, then (7) gives that $\rho(Du_1(z) - u_2(z)) < 1 \leqslant k$. From $\rho(B) < \deg h(z) = k$ and $\deg(h(z+jc) - h(z)) = k-1\ (j = 1,2)$, a contradiction is derived by comparing the orders of both sides of (8). So $Du_1(z) - u_2(z) \equiv 0$, that is

$$D = \frac{u_2(z)}{u_1(z)} = \frac{\Delta_c^2 a(z) - b(z)}{\Delta_c a(z) - b(z)}. \tag{9}$$

Thus, (8) can be written as

$$B(z+2c)\,\mathrm{e}^{h(z+2c)-h(z)} - (2+D)B(z+c)\,\mathrm{e}^{h(z+c)-h(z)} + (1+D)B(z) = 0. \tag{10}$$

Secondly, we prove that $\rho(f) = \deg h = 1$. Indeed, if $\rho(f) = k \geqslant 2$, we will deduce a contradiction from the following two cases.

Case 1. Suppose that $D = -1$. Then (10) gives

$$\mathrm{e}^{h(z+2c)-h(z+c)} = \frac{B(z+c)}{B(z+2c)}. \tag{11}$$

So $R_1(z) := \frac{B(z+c)}{B(z+2c)}$ is a nonconstant entire function. By a version of the difference analogue of the logarithmic derivative lemma in [11], for each $\varepsilon_1 (0 < 4\varepsilon_1 < k - \rho_1)$, we have

$$T(r, R_1(z)) = m(r, R_1(z)) = O(r^{\rho_1 - 1 + \varepsilon_1}),$$

which gives $\rho(R_1(z)) \leqslant \rho_1 - 1 + \varepsilon_1 < k-1$. We get a contradiction by comparing the orders of both sides of (11).

Case 2. Suppose that $D \neq -1$. Then we deduce from (10) that

$$W_2(z, R_2(z)) \cdot R_2(z) = -(1+D), \qquad (12)$$

where

$$R_2(z) = e^{h(z+c)-h(z)},$$
$$W_2(z, R_2(z)) = \frac{B(z+2c)}{B(z)} R_2(z+c) - (2+D)\frac{B(z+c)}{B(z)}.$$

Since $R_2(z)$ is of regular growth, for any given $\varepsilon_2\,(0 < 4\varepsilon_2 < k - \rho_1)$ and all $r > r_0\,(>0)$, we have $T(r, R_2(z)) > r^{k-1-\varepsilon_2}$. On the other hand, the difference analogue of the logarithmic derivative in [11] gives $m\left(r, \frac{B(z+jc)}{B(z)}\right) = O(r^{\rho_1-1+\varepsilon_2})$ $(j = 1, 2)$. So

$$m\left(r, \frac{B(z+jc)}{B(z)}\right) = S(r, R_2) \quad (j = 1, 2).$$

Although the coefficients of $W_2(z, R_2(z))$, namely $m\left(r, \frac{B(z+jc)}{B(z)}\right)$, satisfy $m\left(r, \frac{B(z+jc)}{B(z)}\right) = S(r, R_2)$ instead of $T\left(r, \frac{B(z+jc)}{B(z)}\right) = S(r, R_2)$, we may however apply the method of proof of Lemma 1 for (12) to obtain

$$m(r, R_2) = S(r, R_2).$$

Since $R_2(z)$ is an entire function, this is impossible.

Therefore, we obtain $\rho(f) = \deg h(z) = 1$. Together with (4) and (5), we have

$$f(z) = a(z) + B(z)e^{Az+A_0} = a(z) + B_*(z)e^{Az}, \quad (13)$$

where $A(\neq 0), A_0$ are two constants and $B_*(z) = B(z)e^{A_0}\,(\not\equiv 0)$ is an entire function such that

$$\rho(B_*) = \lambda(B_*) = \lambda(f-a) < \rho(f) = 1.$$

At last, we prove that $B_*(z)\,(\not\equiv 0)$ is a constant. To this end, we only need to prove $B_*'(z) \equiv 0$. Substituting (13) into (3) and noting (9), we obtain

$$e^{2Ac}B_*(z+2c) - (2+D)e^{Ac}B_*(z+c) + (1+D)B_*(z) = 0. \quad (14)$$

We assert that the sum of all coefficients of equation (14) is equal to zero, that is,

$$e^{2Ac} - (2+D)e^{Ac} + (1+D) = 0. \qquad (15)$$

If $B_*(z)$ is a polynomial, we suppose that $B_*(z) = c_k z^k + c_{k-1}z^{k-1} + \cdots + c_0$ $(k \geqslant 0,\ c_k \neq 0)$. Substituting this into (14), we get for $k \geqslant 1$,

$$c_k\left(e^{2Ac} - (2+D)e^{Ac} + (1+D)\right)z^k + O(z^{k-1}) \equiv 0, \quad (16)$$

or for $k = 0$,

$$c_0\left(e^{2Ac} - (2+D)e^{Ac} + (1+D)\right) \equiv 0. \qquad (17)$$

We deduce from (16) (or (17)) that (15) holds.

If $B_*(z)$ is a transcendental entire function and $e^{2Ac} - (2+D)e^{Ac} + (1+D) \neq 0$, then we deduce from Lemma 2 and Remark 3 that $\rho(B_*) \geqslant 1$, a contradiction. So (15) always holds.

By (14) and (15), we have

$$e^{Ac}[B_*(z+2c) - B_*(z)] \\ - (2+D)[B_*(z+c) - B_*(z)] = 0. \quad (18)$$

We see from Lemma 3 that there exist two ε-sets E_j^* such that for $j = 1, 2$, as $z \to \infty$ in $\mathbb{C}\backslash E_j^*$,

$$B_*(z+jc) - B_*(z) = jcB_*'(z)(1 + o_j(1)).$$

Together with (18), we obtain as $z \to \infty$ in $\mathbb{C}\backslash E$,

$$B_*'(z)K + B_*'(z)K \cdot o(1) = 0, \qquad (19)$$

where $E = E_1^* \cup E_2^*$ and $K = c\,e^{Ac}[2e^{Ac} - (2+D)]$. We can derive that $K \neq 0$, and so (19) implies that $B_*'(z) \equiv 0$. □

Lemma 5 ([20, 21]) *Suppose that $n \geqslant 2$ and let $f_1(z), \dots, f_n(z)$ be meromorphic functions and $g_1(z), \dots, g_n(z)$ be entire functions such that*

(i) $\Sigma_{j=1}^n f_j(z)\exp\{g_j(z)\} = 0$;

(ii) *when $1 \leqslant j < k \leqslant n$, $g_j(z) - g_k(z)$ is not constant;*

(iii) *when $1 \leqslant j \leqslant n$, $1 \leqslant h < k \leqslant n$,*

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\}\ (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ has finite linear measure or logarithmic measure.

Then $f_j(z) \equiv 0, \quad j = 1, \dots, n.$

Lemma 6 ([22]) *Suppose that h is a nonconstant meromorphic function satisfying*

$$\overline{N}(r, h) + \overline{N}(r, 1/h) = S(r, h).$$

Let $f = a_0h^p + a_1h^{p-1} + \cdots + a_p$ and $g = b_0h^q + b_1h^{q-1} + \cdots + b_q$ be polynomials in h with coefficients $a_0, a_1, \dots, a_p, b_0, b_1, \dots, b_q$, being small functions of h and $a_0b_0a_p \not\equiv 0$. If $q \leqslant p$, then $m(r, g/f) = S(r, h)$.

Lemma 7 ([19]) *Let g be a transcendental function of order less than 1, and let h be a positive constant. Then there exists an ε—set E such that*

$$\frac{g'(z+c)}{g(z+c)} \to 0, \ \frac{g(z+c)}{g(z)} \to 1, \text{ as } z \to \infty \text{ in } \mathbb{C}\backslash E.$$

uniformly in c for $|c| \leqslant h$. Further, the set E may be chosen so that for large $|z| \notin E$, the function g has no zeros or poles in $|\zeta - z| \leqslant h$.

PROOF OF Theorem 4

Proof: By the hypotheses of Theorem 4, we see that (4) and (5) still hold. Since $\Delta_c^2 f(z)$ and $\Delta_c f(z)$ share $b(z)(\not\equiv \Delta_c a(z))$ CM, we conclude that

$$\frac{\Delta_c^2 f(z)-b(z)}{\Delta_c f(z)-b(z)} = \frac{B(z+2c)\,e^{h(z+2c)}-2B(z+c)\,e^{h(z+c)}+B(z)\,e^{h(z)}+u_2(z)}{B(z+c)\,e^{h(z+c)}-B(z)\,e^{h(z)}+u_1(z)} = e^{Q(z)}, \quad (20)$$

where $Q(z)$ is a polynomial, $u_j(z)=\Delta_c^j a(z)-b(z)\,(j=1,2)$ and $\rho(u_j(z))<1\,(j=1,2)$.

Since Lemma 4 holds, in order to prove Theorem 4, we only need to prove

$$\frac{\Delta_c^2 f(z)-b(z)}{\Delta_c f(z)-b(z)} = D, \quad (21)$$

where D is a nonzero constant.

If $Q(z)\equiv 0$, then (21) obviously holds by (20). So we only need to suppose that $Q(z)\not\equiv 0$ and prove that $\deg Q(z)=s=0$. Set

$$h(z)=a_k z^k+a_{k-1}z^{k-1}+\cdots+a_0,$$
$$Q(z)=b_s z^s+b_{s-1}z^{s-1}+\cdots+b_0,$$

where $k=\rho(f)\geqslant 1$, $a_k(\neq 0), a_{k-1},\dots,a_0$, $b_s(\neq 0), b_{s-1},\dots,b_0$ are constants. From (20),

$$0\leqslant \deg Q=s\leqslant \deg h=k.$$

Next we prove that neither $1\leqslant s<k$ nor $1\leqslant s=k$ holds.

Firstly, we prove that $1\leqslant s<k$ cannot hold. By (20), we have

$$B(z+2c)\,e^{h(z+2c)-h(z)}-2B(z+c)\,e^{h(z+c)-h(z)} +B(z)-\big[B(z+c)\,e^{h(z+c)-h(z)}-B(z)\big]\,e^{Q(z)} = \big[u_1(z)\,e^{Q(z)}-u_2(z)\big]\,e^{-h(z)}.$$

Comparing the orders of both sides of the above equality, we obtain a contradiction.

Secondly, we prove that $1\leqslant s=k$ cannot hold. To this end, we consider the following three cases.

Case 1. $b_k\neq \pm a_k$. Rewrite (20) in the form

$$G_{11}(z)\,e^{Q(z)}+G_{12}\,e^{Q(z)-h(z)}+G_{13}\,e^{-h(z)}+G_{14}\,e^{h_0(z)}=0, \quad (22)$$

where $h_0(z)\equiv 0$ and

$$G_{11}(z)=B(z+c)\,e^{h(z+c)-h(z)}-B(z);$$
$$G_{12}(z)=u_1(z);\quad G_{13}(z)=-u_2(z);$$
$$G_{14}(z)=-\big[B(z+2c)\,e^{h(z+2c)-h(z)} -2B(z+c)\,e^{h(z+c)-h(z)}+B(z)\big].$$

We deduce from $\rho(B)<k$ and $\deg(h(z+jc)-h(z))=k-1\,(j=1,2)$ that

$$\rho(G_{1m}(z))<k \qquad (m=1,\,2,\,3,\,4),$$
$$\deg(Q\pm h)=\deg(Q-h_0)=\deg(-h-h_0)=k.$$

So, for $m=1$, 2, 3, 4,

$$T(r,G_{1m})=o\left(T\left(r,e^{Q\pm h}\right)\right);$$
$$T(r,G_{1m})=o\left(T\left(r,e^{Q}\right)\right);$$
$$T(r,G_{1m})=o\left(T\left(r,e^{-h}\right)\right).$$

By (22) and Lemma 5, we get $G_{1m}(z)\equiv 0$, $m=1$, 2, 3, 4. Thus, $G_{12}(z)=u_1(z)=\Delta_c a(z)-b(z)\equiv 0$, which contradicts the assumption $b(z)\not\equiv\Delta_c a(z)$.

Case 2. $b_k=a_k$. Rewrite (20) in the form

$$G_{21}(z)\,e^{Q(z)}+G_{22}\,e^{-h(z)}+G_{23}\,e^{h_0(z)}=0,$$

where $h_0(z)\equiv 0$ and

$$G_{21}(z)=B(z+c)\,e^{h(z+c)-h(z)}-B(z);$$
$$G_{22}(z)=-u_2(z);$$
$$G_{23}(z)=u_1(z)\,e^{Q(z)-h(z)}-\big[B(z+2c)\,e^{h(z+2c)-h(z)} -2B(z+c)\,e^{h(z+c)-h(z)}+B(z)\big].$$

Using a proof similar to that of Case 1, we can obtain $G_{2m}(z)\equiv 0\,(m=1,\,2,\,3)$. From $G_{21}(z)=0$, we get $B(z+c)\,e^{h(z+c)}\equiv B(z)\,e^{h(z)}$. Combining this with $f(z)=a(z)+B(z)\,e^{h(z)}$, we have $\Delta_c f(z)=\Delta_c a(z)$, which implies $\Delta_c^2 f(z)=\Delta_c^2 a(z)$. So by $G_{22}(z)=-u_2(z)\equiv 0$, we obtain $\Delta_c^2 f(z)-b(z)=\Delta_c^2 a(z)-b(z)=u_2(z)\equiv 0$, which is impossible by (20).

Case 3. $b_k=-a_k$. Rewrite (20) in the form

$$G_{31}(z)\,e^{Q(z)}+G_{32}\,e^{Q(z)-h(z)}+G_{33}\,e^{h_0(z)}=0,$$

where $h_0(z)\equiv 0$ and

$$G_{31}(z)=B(z+c)\,e^{h(z+c)-h(z)}-B(z)-u_2(z)\,e^{-Q(z)-h(z)};$$
$$G_{32}(z)=u_1(z);$$
$$G_{33}(z)=-\big[B(z+2c)\,e^{h(z+2c)-h(z)} -2B(z+c)\,e^{h(z+c)-h(z)}+B(z)\big].$$

Using a proof similar to that of Case 1, we can obtain $G_{3m}(z)\equiv 0\,(m=1,\,2,\,3)$. So $G_{32}(z)=u_1(z)=\Delta_c a(z)-b(z)\equiv 0$, which contradicts $b(z)\not\equiv\Delta_c a(z)$. □

PROOF OF Theorem 5

Proof: Since $\Delta_c^2 f(z)$ and $\Delta_c f(z)$ share $b(z)$ and ∞ CM, we have

$$\frac{\Delta_c^2 f(z)-b(z)}{\Delta_c f(z)-b(z)} = e^{P(z)}, \quad (23)$$

where $P(z)$ is an entire function. By (23) we have

$$T(r, e^{P(z)}) = O(T(r, f)),$$

and so

$$S(r, e^{P(z)}) = S(r, f).$$

Since $T(r, \Delta_c^j f(z)) = O(T(r, f))$, we have

$$S(r, \Delta_c^j f(z)) = S(r, f), \quad (j = 1, 2).$$

Now we prove that $P(z)$ is a constant. Suppose that, on the contrary, $P(z)$ is not a constant. Since $\max\{\rho(b), \rho(a)\} < 1$, $\max\{\rho(\Delta_c^j b(z)), \rho(\Delta_c^j a(z))\} \leqslant \max\{\rho(b), \rho(a)\} < 1$ $(j = 1, 2)$ and $e^{P(z)}$ is of regular growth with $\rho(e^P) \geqslant 1$, we have

$$\begin{aligned} \max\{T(r, b(z)), T(r, a(z))\} &= o\left(T(r, e^P)\right); \\ \max\{T(r, \Delta_c^j b(z)), T(r, \Delta_c^j a(z))\} &= o\left(T(r, e^P)\right). \end{aligned} \tag{24}$$

From (23), we get

$$\begin{aligned} &\Delta_c^2(f(z) - a(z)) - e^{P(z)} \Delta_c(f(z) - a(z)) \\ &\quad = b(z) - \Delta_c^2 a(z) + (\Delta_c a(z) - b(z))\, e^{P(z)}. \end{aligned} \tag{25}$$

We assert that $b(z) - \Delta_c^2 a(z) + (\Delta_c a(z) - b(z))\, e^{P(z)} \not\equiv 0$. Otherwise, we have

$$e^{P(z)} = \frac{\Delta_c^2 a(z) - b(z)}{\Delta_c a(z) - b(z)},$$

which implies that

$$\begin{aligned} \rho(e^{P(z)}) &\leqslant \max\{\rho(\Delta_c^2 a),\, \rho(\Delta_c a),\, \rho(b)\} \\ &\leqslant \max\{\rho(b),\, \rho(a)\} < 1. \end{aligned}$$

This contradicts $\rho(e^P) \geqslant 1$. Hence $b(z) - \Delta_c^2 a(z) + (\Delta_c a(z) - b(z))\, e^{P(z)} \not\equiv 0$.

Dividing both sides of (25) by $(b(z) - \Delta_c^2 a(z) + (\Delta_c a(z) - b(z))\, e^{P(z)})(f(z) - a(z))$, we obtain

$$\frac{\frac{\Delta_c^2(f(z)-a(z))}{f(z)-a(z)} - \frac{\Delta_c(f(z)-a(z))}{f(z)-a(z)}\, e^{P(z)}}{b(z) - \Delta_c^2 a(z) + (\Delta_c a(z) - b(z))\, e^{P(z)}} = \frac{1}{f(z) - a(z)}. \tag{26}$$

Since $b(z) - \Delta_c^j a(z) \not\equiv 0\, (j = 1, 2)$, we deduce from Lemma 6 and (24) that

$$m\left(r, \frac{1}{b(z) - \Delta_c^2 a(z) + (\Delta_c a(z) - b(z))\, e^{P(z)}}\right) = S(r, e^P),$$

$$m\left(r, \frac{e^{P(z)}}{b(z) - \Delta_c^2 a(z) + (\Delta_c a(z) - b(z))\, e^{P(z)}}\right) = S(r, e^P).$$

Furthermore, by a version of the difference analogue of the logarithmic derivative in [23], we get

$$m\left(r, \frac{\Delta_c^j(f(z) - a(z))}{f(z) - a(z)}\right) = S(r, f), \quad j = 1, 2.$$

So, by (26), we obtain

$$m\left(r, \frac{1}{f(z) - a(z)}\right) = S(r, e^P) + S(r, f) = S(r, f),$$

which gives $\delta(a, f) = 0$, contradicting $\delta(a, f) > 0$. Hence we have proved that $P(z)$ is a constant. Setting $e^{P(z)} = D$, we have

$$\frac{\Delta_c^2 f(z) - b(z)}{\Delta_c f(z) - b(z)} = D. \tag{27}$$

Next we consider the case $a(z) \equiv 0$ and $b(z) \not\equiv 0$. By (27), we get

$$\Delta_c^2 f(z) - D \Delta_c f(z) = (1 - D) b(z).$$

If $D \neq 1$, then dividing the above equality by $(1 - D) b(z) f(z)$, we obtain

$$\frac{1}{(1-D)b(z)} \frac{\Delta_c^2 f(z)}{f(z)} - \frac{D}{(1-D)b(z)} \frac{\Delta_c f(z)}{f(z)} = \frac{1}{f(z)}.$$

So by (24) and the difference analogue of the logarithmic derivative in [23], we get

$$m\left(r, \frac{1}{f(z)}\right) = S(r, f),$$

which gives $\delta(0, f) = 0$, contradicting $\delta(0, f) > 0$. Hence $D = 1$ and $\Delta_c^2 f(z) \equiv \Delta_c f(z)$. □

Acknowledgements: This research was supported by the National Natural Science Foundation of China (Nos. 11801093, 11801110, 11871260), the Natural Science Foundation of Guangdong Province (2018A030313508), Guangdong Young Innovative Talents Project (2018KQNCX117) and Characteristic Innovation Project of Guangdong Province(2019KTSCX119).

REFERENCES

1. Hayman WK (1964) *Meromorphic Functions*, Clarendon Press, Oxford.
2. Laine I (1993) *Nevanlinna Theory and Complex Differential Equations*, De Gruyter, Berlin.
3. Yang L (1993) *Value Distribution Theory*, Science Press, Beijing.
4. Gundersen G (1983) Meromorphic functions that share four values. *Trans Amer Math Soc* **277**, 545–567.

5. Gundersen G (1987) Correction to meromorphic functions that share four values. *Trans Amer Math Soc* **304**, 847–850.
6. Mues E (1989) Meromorphic functions sharing four values. *Complex Var Elliptic Equ* **12**, 169–179.
7. Nevanlinna R (1926) Einige eindentigkeitssätze in der theorie der meromorphen funktionen. *Acta Math* **48**, 367–391.
8. Rubel LA, Yang CC (1977) Values shared by an entire function and its derivative. In: Buckholtz JD, Suffridge TJ (eds) *Complex Analysis*, Lecture Notes in Mathematics, Vol 599, Springer, Berlin, Heidelberg.
9. Jank G, Mues E, Volkman L (1986) Meromorphe funktionen, die mit ihrerersten und zweiten ableitung einen wertteilen. *Complex Var Theory Appl* **6**, 51–71.
10. Chen ZX, Yi HX (2013) On sharing values of meromorphic functions and their difference. *Results Math* **63**, 557–565.
11. Chiang YM, Feng SJ (2008) On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. *Ramanujan J* **16**, 105–129.
12. Halburd RG, Korhonen R (2006) Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. *J Math Anal Appl* **314**, 477–487.
13. Laine I, Yang CC (2007) Value distribution of difference polynomials. *Proc Japan Acad Ser A Math Sci* **83**, 148–151.
14. Laine I, Yang CC (2007) Clunie theorems for difference and q-difference polynomials. *J Lond Math Soc* **76**, 556–566.
15. Chen BQ, Chen ZX, Li S (2012) Uniqueness theorems on entire functions and their difference operators and shifts. *Abstr Appl Anal* **2012**, ID 906893.
16. El Farissi A, Latreuch Z, Asiri A (2016) On the uniqueness theory of entire functions and their difference operators. *Complex Anal Oper Theory* **10**, 1317–1327.
17. Zhang J, Kang HY, Liao LW (2015) Entire functions sharing a small entire function with their difference operators. *Bull Iranian Math Soc* **41**, 1121–1129.
18. Chen ZX (2013) On growth of meromorphic solution for linear difference equations. *Abstr Appl Anal* **2013**, ID 619296.
19. Bergweiler W, Langley JK (2007) Zeros of differences of meromorphic functions. *Math Proc Camb Phil Soc* **142**, 133–147.
20. Gross F (1972) *Factorization of Meromorphic Functions*, US Government Printing Office, Washington.
21. Yang CC, Yi HX (2003) *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers Group, Dordrecht.
22. Li P, Wang WJ (2007) Entire functions that share a small function with its derivative. *J Math Anal Appl* **328**, 743–751.
23. Halburd RG, Korhonen R, Tohge K (2014) Holomorphic curves with shift-invariant hyper-plane preimages. *Trans Amer Math Soc* **366**, 4267–4298.