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Hyers-Ulam stability for C^1 solution of series-like iterative equation with variable coefficients

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ABSTRACT: Hyers-Ulam stability is a basic sense of stability for functional equations. In the present paper we discuss the Hyers-Ulam stability of series-like iterative equations with variable coefficients. By the construction of a uniformly convergent sequence of functions we prove that if we can find a C^1 approximate solution of such an equation, then there must be a unique C^1 solution of this equation which is close to the C^1 approximate solution.

KEYWORDS: Hyers-Ulam stability, iterative equation, variable coefficients

MSC2010: 39B82 39B12 47H99

INTRODUCTION

It is well known that the study of functional equations' stability originated from a question of Ulam [1], concerning the stability of group homomorphisms. The problem for Cauchy equation, in Banach spaces, was answered affirmatively by Hyers [2] in 1941. Due to the problem of Ulam and the answer of Hyers, such stability of functional equations is called after their names. We prefer the result of Ciepliński [3] because it has a detailed survey of this topic.

Let f^m denote the m-th iterate of a mapping f, i.e., $f^m = f \circ f^{m-1}$, $f^0 = \text{id}$. Equations having iteration as their main operation, that is, including iterates of the unknown mapping, are called iterative equations. The following equation is a natural generalization of the problem for iterative roots.

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \qquad (1)$$

for $x \in I = [a,b]$, which is known as polynomial-like iterative equation, where $n \geqslant 2$ is an integer, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $F : I \to \mathbb{R}$ is a given mapping and $f : I \to I$ is unknown. Polynomial-like iterative equations are important not only in the study of functional equations but also in the study of dynamical systems [4].

In 1986, Zhang [5] constructed an interesting operator called "structural operator" for (1) and used the fixed point theory in Banach spaces to get the solutions of (1). Employing this idea, Kulczycki and Tabor discussed the iterative functional equations in the class of Lipschitz functions [6]. The case of all constant λ_i's was considered in [5, 7, 8]. In 2000, Zhang and Baker first discussed the continuous solutions of such an iterative equation with variable coefficients $\lambda_i = \lambda_i(x)$ which are all continuous in interval $[a,b]$. In 2001, Si and Wang furthermore gave the continuously differentiable solution of such equation in the same conditions [9]. In this paper, we consider the series-like iterative equation with variable coefficient

$$\sum_{i=1}^{\infty} \lambda_i(x) f^i(x) = F(x), \quad x \in I = [a,b], \qquad (2)$$

where $\lambda_i(x) : I \to [0,1]$ are given continuous functions and $\sum_{i=1}^{\infty} \lambda_i(x) = 1$, $\lambda_1(x) \geqslant c > 0$ $(\forall x \in I)$, $\max_{x \in I} \lambda_i(x) = c_i$.

As mentioned in [10], most known results on stability of functional equations are given for equations in several variables and there are much less results of stability for functional equations in single variable, so the authors give a survey on this topic. They clarified the relation between Hyers-Ulam stability and other senses of stability which are used for functional equations, such as iterative stability, continuous dependence and robust stability. Hyers-Ulam stability is also discussed for a general form of iterative equations which include the polynomial-like iterative equation with variable coefficients.

It should be pointed that, in 2002, Xu and Zhang [11] discussed Hyers-Ulam stability for C^0 solution of a nonlinear iterative equation. Motivated and inspired by the above results, in this paper, we discuss Hyers-Ulam stability for C^1 solution of (2).

PRELIMINARIES

Let $C^0(I,\mathbb{R}) = \{f : I \to \mathbb{R}, f \text{ is continuous}\}$, clearly $(C^0(I,\mathbb{R}), \|\cdot\|_{c^0})$ is a Banach space, where $\|f\|_{c^0} = \max_{x\in I} |f(x)|$, for f in $C^0(I,\mathbb{R})$.

Let $C^1(I,\mathbb{R}) = \{f : I \to \mathbb{R}, f \text{ is continuous and continuously differentiable}\}$, then $C^1(I,\mathbb{R})$ is a Banach space with norm $\|f\|_{c^1}$, where $\|f\|_{c^1} = \|f\|_{c^0} + \|f'\|_{c^0}$, where $f' = \frac{\mathrm{d}}{\mathrm{d}x} f$.

Being a closed subset, $C^1(I,I)$ defined by

$$C^1(I,I) = \left\{f \in C^1(I,\mathbb{R}), f(I) \subset I, \forall x \in I\right\}$$

is a complete space.

For given constants $M_1 > 0$ and $M_2 > 0$, let

$$\begin{aligned}\mathscr{A}(M_1,M_2) = \big\{&\varphi \in C^1(I,I) : |\varphi'(x)| \leqslant M_1, \forall x \in I,\\ &|\varphi'(x_1)-\varphi'(x_2)| \leqslant M_2|x_1-x_2|, \forall x_1,x_2 \in I\big\}. \quad (3)\end{aligned}$$

The following lemmas are useful, for more details one can refer to [12]. The methods of proof are similar to those of [12], but the conditions are weaker than those of [7].

Lemma 1 ([12]) *Suppose that $\varphi \in C^1(I,I)$ and*

$$|\varphi'(x)| \leqslant M, \quad x \in I, \tag{4}$$

$$|\varphi'(x_1)-\varphi'(x_2)| \leqslant M'|x_1-x_2|, \quad \forall x_1,x_2 \in I, \tag{5}$$

where M and M' are positive constants. Then

$$|(\varphi^n(x_1))' - (\varphi^n(x_2))'| \leqslant M'\left(\sum_{t=n-1}^{2n-2} M^t\right)|x_1-x_2|,$$

for any $x_1,x_2 \in I$, where $(\varphi^n)'$ denotes $\frac{\mathrm{d}}{\mathrm{d}x}\varphi^n$.

Lemma 2 ([12]) *Suppose that $\varphi_1,\varphi_2 \in C^1(I,I)$ satisfy (4). Then*

$$\left\|\varphi_1^n - \varphi_2^n\right\|_{c^0} \leqslant \left(\sum_{t=1}^{n} M^t\right)\|\varphi_1-\varphi_2\|_{c^0}$$

Lemma 3 ([12]) *Suppose that $\varphi_1,\varphi_2 \in C^1(I,I)$ satisfy (4) and (5). Then*

$$\begin{aligned}\left\|(\varphi_1^{k+1})' - (\varphi_2^{k+1})'\right\|_{c^0} &\leqslant (k+1)M^k\left\|\varphi_1'-\varphi_2'\right\|_{c^0}\\ &+ Q(k+1)M'\left(\sum_{i=1}^{k}(k-i+1)M^{k+i-1}\right)\|\varphi_1-\varphi_2\|_{c^0}\end{aligned}$$

for $k = 0,1,2,\ldots$, $Q(1) = 0$ and $Q(s) = 1$ for $s \geqslant 2$.

Similar to Definition 8 in [13] and Definition 2.9 in [14], we give the definition of Hyers-Ulam stability for C^1 solution of (2).

Definition 1 Let M_1 and M_2 be given positive constants, and $F \in \mathscr{A}(M_1,M_2)$ be a given function. For $g \in \mathscr{A}(N_1,N_2)$ with constants $N_1 \geqslant 1$ and $N_2 > 0$ satisfying

$$\left\|F - \sum_{i=1}^{\infty}\lambda_i(x)g^i\right\|_{c^1} \leqslant \delta,$$

if there is a solution $f \in \mathscr{A}(N_1,N_2)$ of (2) such that $\|f-g\|_{c^1} \leqslant K\delta$, where $K > 0$ is a constant, then (2) is said to have a Hyers-Ulam stability and g is called a C^1 approximate solution of (2).

MAIN RESULTS

Theorem 1 *Given positive constants M_1 and M_2 and $F \in \mathscr{A}(M_1,M_2)$, if there exists constants $N_1 \geqslant 1$ and $N_2 > 0$, such that*

(P1) $c - \sum_{i=2}^{\infty} c_i N_1^{i-1} \geqslant \dfrac{M_1}{N_1}$,

(P2) $c - \sum_{i=2}^{\infty} c_i \Big(\sum_{j=t-1}^{2i-2} N_1^j\Big) \geqslant \dfrac{M_2}{N_2}$,

(P3) $\sum_{i=2}^{\infty} c_i \Big[\sum_{j=1}^{i} N_1^{j-1} + Q(i)N_2 \sum_{j=1}^{i-1}(i-j)N_1^{i+j-2}\Big] < c$,

(P4) $\sum_{i=2}^{\infty} i N_1^{i-1} < c$,

then for any C^1 function $g \in \mathscr{A}(N_1,N_2)$ with

$$\left\|F - \sum_{i=1}^{\infty}\lambda_i(x)g^i\right\|_{c^1} \leqslant \delta,$$

where $\delta > 0$, there exist a constant K independent of δ and a unique C^1 solution $f \in \mathscr{A}(N_1,N_2)$ of (2) such that

$$\|f-g\|_{c^1} \leqslant K\delta. \tag{6}$$

Proof: For convenience, let $d = \max\{|a|,|b|\}$. Define $\mathscr{K} : \mathscr{A}(N_1,N_2) \to C^1(I,I)$ such that

$$\mathscr{K}(\varphi(t)) = \sum_{i=1}^{\infty}\lambda_i(x)\varphi^i(t), \quad \forall x,t \in I.$$

Since $\varphi \in \mathscr{A}(N_1,N_2)$, it is easy to see that $|\varphi^i(t)| \leqslant d$ for all $t \in I$, and $|\lambda_i(x)\varphi^i(t)| \leqslant d|\lambda_i(x)|$ for all $x,t \in I$. It follows from $\sum_{i=1}^{\infty}\lambda_i(x) = 1$ that $\sum_{i=1}^{\infty}\lambda_i(x)\varphi^i(t)$ is uniformly convergent. Then $\mathscr{K}(\varphi(t))$ is continuous for all $t \in I$. As $\lambda_i(x) \geqslant 0$, $\lambda_i(x)a \leqslant \lambda_i(x)\varphi^i(x) \leqslant \lambda_i(x)b$, for $i = 1,2,\ldots$. This implies that

$$a = \sum_{i=1}^{\infty}\lambda_i(x)a \leqslant \sum_{i=1}^{\infty}\lambda_i(x)\varphi^i(t) \leqslant \sum_{i=1}^{\infty}\lambda_i(x)b = b,$$

thus $\mathscr{K}(\varphi)\in C^0(I,I)$.

For any $\varphi\in\mathscr{A}(N_1,N_2)$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\lambda_i(x)(\varphi^i(t)))\right| = \lambda_i(x)\left|\varphi'(\varphi^{i-1})(\varphi^{i-1}(t))'\right| \leqslant c_i N_1^i.$$

By condition (P1), we see that $\sum_{i=1}^{\infty} c_i N_1^i$ is convergent, therefore $\sum_{i=1}^{\infty} c_i(\varphi^i(t))'$ is uniformly convergent for $t\in I$, this implies that $\mathscr{K}(\varphi(t))$ is continuously differentiable for $t\in I$. Moreover

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{K}(\varphi(t))\right| \leqslant \sum_{i=1}^{\infty}\lambda_i(x)|(\varphi^i(t))'| \leqslant \sum_{i=1}^{\infty} c_i N_1^i := \mu_1.$$

By Lemma 1,

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{K}(\varphi(t_1))) - \frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{K}(\varphi(t_2)))\right| &\leqslant \sum_{i=1}^{\infty}\lambda_i(x)|(\varphi^i(t_1))' - (\varphi^i(t_2))'| \\ &\leqslant \sum_{i=1}^{\infty} c_i\Big(N_2\sum_{j=i-1}^{2i-2} N_1^j\Big)|t_1-t_2| := \mu_2|t_1-t_2|.\end{aligned}$$

Thus $\mathscr{K}(\varphi)\in\mathscr{A}(\mu_1,\mu_2)$.

We can construct a sequence $\{f_k\}$ of functions as follows. We take $f_0=g$ and define for $k\geqslant 1$, $t,x\in I$,

$$f_k(t) = \tfrac{1}{\lambda_1(x)}F(t) - \tfrac{1}{\lambda_1(x)}\mathscr{K}(f_{k-1}(t)) + f_{k-1}(t).$$

Because $\mathscr{K}(\varphi)$, F, and $f_0=g$ are continuously differentiable for all $t\in I$, f_k is continuously differentiable for all $t\in I, k\geqslant 1$. By conditions (P1) and (P2), we will prove $|\frac{\mathrm{d}}{\mathrm{d}t}f_k(t)|\leqslant N_1$ and $|\frac{\mathrm{d}}{\mathrm{d}t}f_k(t_1) - \frac{\mathrm{d}}{\mathrm{d}t}f_k(t_2)| \leqslant N_2|t_1-t_2|$ by induction. First, of course $|\frac{\mathrm{d}}{\mathrm{d}t}f_0(t)| = |\frac{\mathrm{d}}{\mathrm{d}t}g(t)|\leqslant N_1$. Suppose this is true for a positive integer k, then we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}f_{k+1}(t)\right| &\leqslant \tfrac{1}{\lambda_1(x)}|F'(t)| + \tfrac{1}{\lambda_1(x)}\sum_{i=2}^{\infty}\lambda_i(x)|(f_k^i(t))'| \\ &\leqslant \tfrac{1}{c}M_1 + \tfrac{1}{c}\sum_{i=2}^{\infty}c_i N_1^i \\ &\leqslant \tfrac{1}{c}M_1 + \tfrac{1}{c}(cN_1 - M_1) = N_1.\end{aligned}$$

We have $|\frac{\mathrm{d}}{\mathrm{d}t}f_0(t_1) - \frac{\mathrm{d}}{\mathrm{d}t}f_0(t_2)| \leqslant N_1|t_1-t_2|$. Suppose this is true for a positive integer k, then

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}f_{k+1}(t_1) - \frac{\mathrm{d}}{\mathrm{d}t}f_{k+1}(t_2)\right| &\leqslant \tfrac{1}{\lambda_1(x)}\left|F'(t_1)-F'(t_2)\right| \\ &\quad + \tfrac{1}{\lambda_1(x)}\sum_{i=2}^{\infty}\lambda_i(x)\left|(f_k^i(t_1))' - (f_k^i(t_2))'\right| \\ &\leqslant \tfrac{1}{c}M_1|t_1-t_2| + \tfrac{1}{c}\sum_{i=2}^{\infty}c_i N_1^i|t_1-t_2| \\ &\leqslant \tfrac{1}{c}M_1|t_1-t_2| + \tfrac{1}{c}(cN_1-M_1)|t_1-t_2| = N_1|t_1-t_2|.\end{aligned}$$

Thus for any $k=0,1,2,\dots$, we have $f_k\in\mathscr{A}(N_1,N_2)$. Now we claim that

$$\|f_k - f_{k-1}\|_{c^1} \leqslant \tfrac{1}{c}M^k\delta \tag{7}$$

holds for $k=1,2,\dots$, where $M=\max\{E_1,E_2\}$, $E_1 = \frac{1}{c}\sum_{i=2}c_i\left(\sum_{j=1}^{i}N_1^{j-1} + Q(i)N_2\sum_{j=1}^{i-1}(i-j)N_1^{i+j-2}\right)$, $E_2 = \frac{1}{c}\sum_{c_1} iN_1^{i-1}$. We will prove inequality (7) inductively. First,

$$\|f_1 - f_0\|_{c^1} = \left\|\tfrac{1}{\lambda_1(x)}F(t) - \tfrac{1}{\lambda_1(x)}\mathscr{K}(g(t))\right\|_{c^1} \leqslant \tfrac{1}{c}\delta.$$

Assume that this inequality is true for an integer k. Then by the same arguments, we can get that

$$\begin{aligned}\|f_{k+1}-f_k\|_{c^0} &= \max_{t\in I}\Big|\tfrac{1}{\lambda_1(x)}\mathscr{K}(f_k(t)) \\ &\qquad - \tfrac{1}{\lambda_1(x)}\mathscr{K}(f_k(t)) + f_k - f_{k-1}\Big| \\ &\leqslant \tfrac{1}{c}\max_{t\in I}\left|\sum_{i=2}^{\infty}\lambda_i f_k^i(t) - \sum_{i=2}^{\infty}\lambda_i f_{k-1}^i(t)\right| \\ &\leqslant \tfrac{1}{c}\sum_{i=2}^{\infty}c_i\left\|f_k^i - f_{k-1}^i\right\|_{c^0} \\ &\leqslant \tfrac{1}{c}\sum_{i=2}^{\infty}c_i\Big(\sum_{j=1}^{i}N_1^{j-1}\Big)\|f_k - f_{k-1}\|_{c^0}\end{aligned} \tag{8}$$

and

$$\begin{aligned}\left\|\frac{\mathrm{d}}{\mathrm{d}t}f_{k+1} - \frac{\mathrm{d}}{\mathrm{d}t}f_k\right\|_{c^0} &= \max_{t\in I}\Big|\tfrac{1}{\lambda_1(x)}\mathscr{K}(f_k(t))' \\ &\qquad - \tfrac{1}{\lambda_1(x)}\mathscr{K}(f_k(t))' + f_k' - f_{k-1}'\Big| \\ &\leqslant \tfrac{1}{c}\max_{t\in I}\left|\sum_{i=2}^{\infty}\lambda_i(x)(f_k^i(t))' - \sum_{i=2}^{\infty}\lambda_i(x)(f_{k-1}^i(t))'\right| \\ &\leqslant \tfrac{1}{c}\sum_{i=2}^{\infty}c_i\left\|(f_k^i)' - (f_{k-1}^i)'\right\|_{c^0} \\ &\leqslant \tfrac{1}{c}\sum_{i=2}^{\infty}c_t\Big[iN_1^{i-1}\left\|f_k' - f_{k-1}'\right\|_{c^0} \\ &\qquad + Q(i)N_2\Big(\sum_{j=1}^{i-1}(i-j)N_1^{i+j-2}\Big)\|f_k - f_{k-1}\|_{c^0}\Big].\end{aligned} \tag{9}$$

Then we have

$$\begin{aligned}\|f_{k+1}-f_k\|_{c^1} &= \|f_{k+1}-f_k\|_{c^0} + \left\|(f_{k+1})'-(f_k)'\right\|_{c^0}\\ &\leqslant E_1\|f_k-f_{k-1}\|_{c^0} + E_2\left\|(f_k)'-(f_{k-1})'\right\|_{c^0}\\ &\leqslant M\|f_k-f_{k-1}\|_{c^1}\\ &\leqslant M\tfrac{1}{c}M^{k-1}\delta \leqslant \tfrac{1}{c}M^k\delta.\end{aligned}$$

For any positive integers k and s with $k>s$, we have

$$\begin{aligned}\|f_k-f_s\|_{c^1} &\leqslant \|f_k-f_{k-1}\|_{c^1}+\cdots+\|f_{s+1}-f_s\|_{c^1}\\ &\leqslant \tfrac{1}{c}(M^{k-1}\delta+M^{k-2}\delta+\cdots+M^{s-1}\delta)\\ &= \frac{\delta}{c}\frac{M^s-M^k}{1-M}.\end{aligned}$$

By conditions (P3) and (P4), $M<1$, it follows that

$$\|f_k-f_s\|_{c^1}\to 0 \text{ as } k>s\to\infty.$$

As a Cauchy sequence $\{f_k\}_{k=0}^{\infty}$ converges uniformly in $\mathscr{A}(N_1,N_2)$. Let

$$\lim_{k\to\infty} f_k = f.$$

Clearly, $f\in\mathscr{A}(N_1,N_2)$. From

$$\begin{aligned}\|F-\mathscr{K}(f)\|_{c^1} &= \lim_{k\to\infty}\|F-\mathscr{K}(f_k)\|_{c^1}\\ &= \lim_{k\to\infty}\|\lambda_1(x)f_{k+1}-\lambda_1(x)f_k\|_{c^1}\\ &\leqslant \lim_{k\to\infty} M^k\delta = 0,\end{aligned}$$

we know that f is a solution of (2). Furthermore,

$$\begin{aligned}\|f-g\|_{c^1} &= \lim_{k\to\infty}\|f_k-f_0\|_{c^1}\\ &\leqslant \lim_{k\to\infty}\{\|f_k-f_{k-1}\|_{c^1}+\cdots+\|f_1-f_0\|_{c^1}\}\\ &\leqslant \lim_{k\to\infty}\tfrac{1}{c}\{M^{k-1}+M^{k-2}+\cdots+1\}\delta = 0\\ &= \tfrac{1}{c-cM}\delta.\end{aligned}$$

This proves (6), where $K=1/(c-cM)$.

Concerning uniqueness, we assume that there is another continuous solution $\phi\in\mathscr{A}(N_1,N_2)$ for (2), such that $\|\phi-g\|\leqslant\varepsilon$, where $\varepsilon>0$ only depends on δ. It follows that

$$\begin{aligned}\|f-\phi\|_{c^1} &= \left\|\tfrac{1}{\lambda_1(x)}\sum_{i=2}^{\infty}\lambda_i(x)f^i-\tfrac{1}{\lambda_1(x)}\sum_{i=2}^{\infty}\lambda_i(x)\phi^i\right\|_{c^1}\\ &\leqslant \tfrac{1}{c}\sum_{i=2}^{\infty}c_i\left\|f^i-\phi^i\right\|_{c^1}.\end{aligned}$$

By Lemma 2, we have

$$\left\|f^i-\phi^i\right\|_{c^0}\leqslant\sum_{j=1}^{i}N_1^{j-1}\|f-\phi\|_{c^0}.$$

By Lemma 3, we have

$$\begin{aligned}\left\|(f^i)'-(\phi^i)'\right\|_{c^0} &\leqslant iN_1^{i-1}\left\|f'-\phi'\right\|_{c^0}\\ &\quad+Q(i)N_2\Big(\sum_{j=1}^{i-1}(i-j)N_1^{i+j-2}\Big)\|-\phi\|_{c^0}.\end{aligned}$$

Those imply that

$$\begin{aligned}\|f-\phi\|_{c^1} &\leqslant \tfrac{1}{c}\sum_{i=2}^{\infty}c_i\Big(\sum_{j=1}^{i}N_1^{j-1}\Big)\|f-\phi\|_{c^0}\\ &\quad+\tfrac{1}{c}\sum_{i=2}^{\infty}c_i\Big[iN_1^{i-1}\left\|f'-\phi'\right\|_{c^0}\\ &\quad+Q(i)N_2\Big(\sum_{j=1}^{i-1}(i-j)N_1^{i+j-2}\Big)\|f-\phi\|_{c^0}\Big]\\ &\leqslant M\|f-\phi\|_{c^1},\end{aligned}$$

which means

$$(1-M)\,\|\,f-\phi\,\|_{c^1}\leqslant 0.$$

However, $M<1$, which implies that $\|f-\phi\|_{c^1}=0$, i.e., $f=\phi$. □

Example 1 We consider the equation

$$\sum_{i=1}^{\infty}\lambda_i(x)f^i(x)=\tfrac{1}{4}x^2,\quad x\in I=[-1,1],\quad (10)$$

where $\lambda_1(x)=\frac{15}{16}+\frac{1}{32}\cos^2(\frac{\pi x}{2})$ and $\lambda_i(x)=\frac{1}{16\cdot 2^i}(1+\sin^2(\frac{\pi x}{2}))$, for $i\geqslant 2$. It is easy to see that $0\leqslant\lambda_i(x)<1$ and $\sum_i^{\infty}\lambda_i(x)=1$, $c_1=\frac{15}{16}$, $c_i=\frac{1}{8\cdot 2^i}$, for all $i\geqslant 2$. For any $x,y\in[-1,1]$,

$$\begin{aligned}&|F'(x)|=|0.5x|\leqslant 0.5,\\ &|F'(x)-F'(y)|\leqslant|0.5x|+|0.5y|\leqslant 1,\end{aligned}$$

thus $F\in\mathscr{A}(0.5,1)$, which means $M_1=0.5$, $M_2=1$. Take $N_1=1.1$, $N_2=1.5$, we have

$$\begin{aligned}c-\sum_{i=2}^{\infty}c_iN_1^{i-1} &= \frac{15}{16}-\sum_{i=2}^{\infty}\frac{1}{8\cdot 2^i}\cdot\left(\frac{11}{10}\right)^{i-1}\\ &= \frac{31}{36}>\frac{5}{11}=\frac{M_1}{N_1},\end{aligned}$$

thus (P1) is satisfied.

$$\begin{aligned}&c-\sum_{i=2}^{\infty}c_i\Big(\sum_{j=i-1}^{2i-2}N_1^j\Big)\\ &=\frac{15}{16}-\sum_{i=2}^{\infty}10\cdot\frac{1}{8\cdot 2^i}\left[\left(\frac{11}{10}\right)^{2i-1}-\left(\frac{11}{10}\right)^{i-1}\right]\\ &=\frac{15}{16}-\frac{121\cdot 10}{16\cdot 79}+\frac{55}{72}>\frac{2}{3}=\frac{M_2}{N_2},\end{aligned}$$

thus (P2) is satisfied.

$$\begin{aligned}
&\sum_{i=2} c_i\Big(\sum_{j=1}^{i} N_1^{j-1} + Q(i)N_2\sum_{j=1}^{i-1}(i-j)N_1^{i+k-2}\Big)\\
&= \sum_{i=2}\frac{1}{8\cdot 2^i}\Big(\sum_{j=1}^{i}\left(\frac{11}{10}\right)^{j-1} + \frac{3}{2}\sum_{j=1}^{i-1}(i-j)\left(\frac{11}{10}\right)^{i+j-2}\Big)\\
&= \frac{31}{144} + \frac{3}{2}\bigg[\frac{21}{16}\left(\frac{242}{79} + \frac{14641}{15800}\right)\\
&\quad - \frac{0.605(0.605^2 - 1.815 + 4)}{0.395^3\cdot 16} - \frac{5}{4}\cdot\frac{561}{180}\bigg]\\
&< \frac{15}{16} = c,
\end{aligned}$$

thus (P3) is satisfied.

$$\sum_{i=2}^{\infty} c_i i N_1^{i-1} = \frac{319}{2880} < \frac{15}{16} = c,$$

thus (P4) is satisfied.

By Theorem 1, The equation (10) has Hyers-Ulam stability in $\mathscr{A}(1.1, 1.5)$.

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