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Normal families of meromorphic functions which share a set

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ABSTRACT: In this paper, by using the Nevanlinna's value distribution theory and the method of Zalcman-Pang, it investigates the normality of a family of meromorphic functions, denoted by $\mathscr{F}$, defined in a domain D, which concerns the conditions for each $f \in \mathscr{F}$: (i) $\overline{E}(S_1, f) = \overline{E}(S_2, (f^{(k)})^q)$; (ii) both zeros and poles of $f - a$ have multiplicities at least k (> 2 or $\geqslant 2$) and $k+1$, respectively, or $k\,(\geqslant 4)$ and $k-1$, respectively, where k and q are positive integers, a is any finite complex number, $S_1 = \{a_1, a_2, a_3\}$ and $S_2 = \{b_1, b_2, b_3\}$ are made up of finite complex numbers. The conclusion still holds if condition (ii) is replaced by the assumption that zeros of $f - a_i$ have multiplicities at least k, where $k \geqslant 1$ and $i = 1, 2, 3$.

KEYWORDS: meromorphic functions, normality, shared set

MSC2010: 30D35 30D45

INTRODUCTION

In this paper, we use the standard notations and definitions of values distribution theory such as $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ and $S(r, f)$, see [1, 2].

Let D be a domain in $\mathbb{C}$ and let $\mathscr{F}$ be a family of meromorphic functions defined in D. $\mathscr{F}$ is said to be normal in D in the sense of Montel if each sequence $\{f_n\} \subset \mathscr{F}$ contains a subsequence that converges spherically locally uniformly in D, to a meromorphic function or ∞, see [3, 4].

Let f and g be two meromorphic functions in a domain D, and let a be any complex number in $\mathbb{C}$. If the zeros of $f - a$ are the same as the zeros of $g - a$, we say f and g share a IM, see [5, 6].

Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, if there exists a neighborhood (denoted by $\Delta(z_0)$) of point z_0 such that $\mathscr{F}$ is normal in $\Delta(z_0)$, $\mathscr{F}$ is said to be normal at $z_0 \in D$,see [7, 8].

Schwick [9] was the first one who gave a connection between normality and shared values, and proved the following result.

Theorem A ([9]) *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let a_1, a_2 and a_3 be three distinct finite complex numbers. If f and f' share a_i IM for each $f \in \mathscr{F}$, where $i = 1, 2, 3$, then $\mathscr{F}$ is normal in D.*

Fang [10] generalized Theorem A as follows, and proposed the concept of shared set.

Let f and g be two meromorphic functions in a domain D, and let $S = \{a_1, a_2, a_3\}$, where S consists of finite complex numbers. Denote $\overline{E}(S, f) = \bigcup_{a_i \in S}\{z \in D : f(z) - a_i = 0\}$, where $i = 1, 2, 3$. If $\overline{E}(S, f) = \overline{E}(S, g)$, we say that f and g share S.

Theorem B ([10]) *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, let a_1, a_2 and a_3 be three distinct finite complex numbers. If $\overline{E}(S, f) = \overline{E}(S, f')$ for any $f \in \mathscr{F}$, where $S = \{a_1, a_2, a_3\}$, then $\mathscr{F}$ is normal in D.*

Since then, many results of normality criteria concerning sharing values have been obtained, for example in [11, 12]. It is natural to ask whether the result is valid or not if a_1, a_2 and a_3 (Theorem A) are replaced by a set $S := \{a_1, a_2, a_3\}$. In this direction, Liu and Pang [13] proved the following result.

Theorem C ([13]) *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let a_1, a_2 and a_3 be three distinct finite complex numbers. If $\overline{E}(S, f) = \overline{E}(S, f')$ for any $f \in \mathscr{F}$, where $S = \{a_1, a_2, a_3\}$, then $\mathscr{F}$ is normal in D.*

Zhang et al [14] showed the following result.

Theorem D ([14]) *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let a_1, a_2 and a_3 be three distinct finite complex numbers, let $k > 2$ be a positive integer, let a be any finite complex number, and let $S = \{a_1, a_2, a_3\}$. If for each $f \in \mathscr{F}$,*

(i) $\overline{E}(S,f)=\overline{E}(S,f^{(k)})$;
(ii) *both zeros and poles of* $f-a$ *are of multiplicities at least* k,
then $\mathscr{F}$ *is normal in* D.

We generalize Theorem D as follows.

Theorem 1 *Let* $\mathscr{F}$ *be a family of meromorphic functions in a domain* D, *let* $S_1=\{a_1,a_2,a_3\}$, $S_2=\{b_1,b_2,b_3\}$, *where* S_1 *and* S_2 *are made up of finite complex numbers, let* $k>2$ *and* q *be two positive integers, and let* a *be any finite complex number. Suppose that for each* $f\in\mathscr{F}$,
(i) $\overline{E}(S_1,f)=\overline{E}(S_2,(f^{(k)})^q)$;
(ii) *both zeros and poles of* $f-a$ *have multiplicities at least* k,
then $\mathscr{F}$ *is normal in* D.

In Theorem 1, if condition (ii) is replaced by $f-a_i$ $(i=1,2,3)$ has zeros with multiplicities at least $k\geqslant 1$ or $f-a$ has zeros and poles with multiplicities at least $k\geqslant 2$ and $k+1$, respectively, or $f-a$ has zeros and poles with multiplicities at least $k\geqslant 4$ and $k-1$, respectively, the conclusion still holds. We get the following results.

Theorem 2 *Let* $\mathscr{F}$ *be a family of meromorphic functions in a domain* D, *let* $S_1=\{a_1,a_2,a_3\}$, $S_2=\{b_1,b_2,b_3\}$, *where* S_1 *and* S_2 *consist of finite complex numbers, and let* k *and* q *be two positive integers. If for all* $f\in\mathscr{F}$,
(i) $\overline{E}(S_1,f)=\overline{E}(S_2,(f^{(k)})^q)$;
(ii) $f-a_i$, $i=1,2,3$, *has zeros with multiplicities at least* k,
then $\mathscr{F}$ *is normal in* D.

Remark 1 Theorem C is a corollary of Theorem 2.

Theorem 3 *Let* $\mathscr{F}$ *be a family of meromorphic functions in a domain* D, *let* $S_1=\{a_1,a_2,a_3\}$, $S_2=\{b_1,b_2,b_3\}$, *where* S_1 *and* S_2 *consist of finite complex numbers, let* $k\geqslant 2$ *and* q *be two positive integers, and let* a *be any finite complex number. If for each* $f\in\mathscr{F}$,
(i) $\overline{E}(S_1,f)=\overline{E}(S_2,(f^{(k)})^q)$;
(ii) $f-a$ *has zeros and poles with multiplicities at least* k *and* $k+1$, *respectively,*
then $\mathscr{F}$ *is normal in* D.

Theorem 4 *Let* $\mathscr{F}$ *be a family of meromorphic functions in a domain* D, *let* $S_1=\{a_1,a_2,a_3\}$, $S_2=\{b_1,b_2,b_3\}$, *where* S_1 *and* S_2 *consist of finite complex numbers, let* $k\geqslant 4$ *and* q *be two positive integers, and let* a *be any finite complex number. Suppose that for each* $f\in\mathscr{F}$,
(i) $\overline{E}(S_1,f)=\overline{E}(S_2,(f^{(k)})^q)$;
(ii) $f-a$ *has zeros and poles with multiplicities at least* k *and* $k-1$, *respectively,*
then $\mathscr{F}$ *is normal in* D.

The following example due to Zhang et al [14] illustrates that the condition that the zeros of $f-a$ have multiplicities at least $k>2$ is necessary in Theorem 1.

Example 1 We denote the unit disk $\{z\in\mathbb{C}\colon |z|<1\}$ by $\triangle$, let $\mathscr{F}=\{f_n\}$ be a family of meromorphic functions in $\triangle$, where $f_n=n(e^{w_1z}-e^{w_2z})$, $n=1,2,\dots,w_1\neq w_2$, $w_1^k=w_2^k=1$, and $k\geqslant 2$ be a positive integer. After a simple calculation, we obtain $f=f^{(k)}$ for each f, getting $|f'(0)|/(1+|f(0)|^2)=n(w_1-w_2)\to\infty$ for large n. According to Marty's normality criteria, we know $\mathscr{F}$ is not normal in $\triangle$.

LEMMAS

In order to prove our results, we need the following lemmas.

Lemma 1 ([15]) *Let* $\mathscr{F}$ *be a family of meromorphic functions in the unit disk* Δ *with the property that for each* $f\in\mathscr{F}$, *both zeros and poles of* $f-a$ *are of multiplicities at least* k *and* p *respectively. Suppose that there exists a positive number* $A>1$ *such that* $|f^{(k)}(z)|\leqslant A$ *whenever* $f(z)=0$. *If* $\mathscr{F}$ *is not normal in* Δ, *then for any real number* α, $p<\alpha\leqslant k$, *there exists*
(i) *a sequence of complex numbers* $z_n\to z_0$, $|z_n|<r<1$,
(ii) *a sequence of functions* $f_n\in\mathscr{F}$,
(iii) *a sequence of positive numbers* $\rho_n\to 0$,
(iv) *a real number* r, $0<r<1$,
such that $g_n(\xi)=\rho_n^{-\alpha}f_n(z_n+\rho_n\xi)$ *converges spherically locally uniformly to a non-constant meromorphic function* $g(\xi)$ *in* $\mathbb{C}$ *such that* $g^{\#}(\xi)=\frac{|g'(\xi)|}{1+|g(\xi)|^2}\leqslant g^{\#}(0)=kA+1$, *where* $g(\xi)$ *has zeros and poles with multiplicities at least* k *and* p, *respectively. Moreover,* $g(\xi)$ *is of order at most two.*

Remark 2 If $\mathscr{F}$ is not normal at $z_0\in\Delta$, the above conclusion still holds when there exists points z_n such that $z_n\to z_0$ for large n.

Lemma 2 *Suppose that* f *is a meromorphic function of finite order,* a_1,a_2 *and* a_3 *are three distinct finite complex numbers. If the number of zeros of* f *is finite in* $\mathbb{C}$ *and* $(f^{(k)})^q\in S$ *implies* $f(z)=0$, *where* $S=\{a_1,a_2,a_3\}$, *then* f *is a rational function.*

Proof: By the second Nevanlinna fundamental theorem, we have

$$T(r,(f^{(k)})^q) \leqslant \sum_{i=1}^{3} \overline{N}\Big(r, \frac{1}{(f^{(k)})^q - a_i}\Big) + S(r,(f^{(k)})^q)$$
$$\leqslant \overline{N}(r, \frac{1}{f}) + S(r,(f^{(k)})^q).$$

Obviously,

$$T(r,(f^{(k)})^q) \leqslant qT(r,f^{(k)}) \leqslant q(k+1)T(r,f) + qS(r,f). \quad (1)$$

According to (1) and the assumption that f is a meromorphic function of finite order, we see that $(f^{(k)})^q$ is also a meromorphic function with finite order, and get

$$S(r,(f^{(k)})^q) = O(\log r). \quad (2)$$

By the condition that the number of zeros of $f(z)$ is finite in $\mathbb{C}$, we have

$$\overline{N}\Big(r, \tfrac{1}{f}\Big) = O(\log r). \quad (3)$$

It follows from (2) and (3) that

$$T(r,(f^{(k)})^q) \leqslant O(\log r),$$

which implies that f is a rational function. □

Lemma 3 ([3]) *Let f be a non-constant meromorphic function on the open complex plane, k be a positive integer, then we get*

$$N(r, \frac{1}{f^{(k)}}) \leqslant T(r,f^{(k)}) - T(r,f) + N(r, \frac{1}{f}) + S(r,f),$$
$$N(r, \frac{1}{f^{(k)}}) \leqslant N(r, \frac{1}{f}) + k\overline{N}(r,f) + S(r,f).$$

PROOF OF Theorem 1

Proof: Suppose that $\mathscr{F}$ is not normal in a domain D, without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0 \in D$. Then we consider two cases.

Case 1. $a \in S_1$. Suppose that $a = a_1$, by Lemma 1, there exists $z_n \to z_0$, $f_n \in \mathscr{F}$ and $\rho_n \to 0$ satisfying

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) - a_1 \to g(\zeta)$$

uniformly on compact subsets of $\mathbb{C}$, where $g(\zeta)$ is a non-constant meromorphic function. Moreover, both zeros and poles of $g(\zeta)$ have multiplicities at least $k > 2$.

Subcase 1.1 There exists $\zeta_0 \in \mathbb{C}$ such that $g(\zeta_0) = 0$.

Let $G_n(\zeta) = \rho_n^{-k}(f_n(z_n + \rho_n\zeta) - a_1)$. In the view of the condition, we know that zeros of $G_n(\zeta)$ have multiplicities at least k. Obviously, there is a neighbourhood $|\zeta - \zeta_0| < \delta(\zeta_0)$ such that $G_n(\zeta)$ is a holomorphic function. Now we claim that $\{G_n(\zeta)\}$ is not normal at ζ_0. Indeed, if $\{G_n(\zeta)\}$ is normal at ζ_0, we have $G_n(\zeta_n) \to G(\zeta)$ uniformly in $|\zeta - \zeta_0| < \delta(\zeta_0)$ as $n \to \infty$, where $G(\zeta)$ is a holomorphic function. We have $g(\zeta) \not\equiv 0$ by the condition that $g(\zeta)$ is non-constant. By Hurwitz's theorem, we choose a sequence of points $\zeta_n \to \zeta_0$ such that $g_n(\zeta_n) = f_n(z_n + \rho_n\zeta_n) - a_1 = 0$ and $G_n(\zeta_n) = 0$ for sufficiently large n. Clearly, there exists a deleted neighborhood (denoted by $\triangle'(\zeta_0, \delta(\zeta_0)) = \{\zeta : 0 < |\zeta - \zeta_0| < \delta(\zeta_0)\}$) of point ζ_0 such that $g(\zeta) \neq 0$ for all $\zeta \in \triangle'$, that is to say, $f_n(z_n + \rho_n\zeta) \neq a_1$ in $\triangle'$, without loss of generality, suppose that $f_n(z_n + \rho_n\zeta) = a_2$ in $\triangle'$, then we know that $G_n(\zeta) = \infty$ in $\triangle'$ as $n \to \infty$, which is conflicting.

$G_n(\zeta) = 0$ is equal to $f_n(z_n + \rho_n\zeta) = a_1$. Based on the condition, we have $(f_n^{(k)}(z_n + \rho_n\zeta))^q = (G_n^{(k)}(\zeta))^q \in S_2$ and choose $b_i \in S_2$ such that $(G_n^{(k)}(\zeta))^q = b_i$, from which we know that $|G_n^{(k)}(\zeta)| \leqslant \sum_{i=1}^{3} \sqrt[q]{|b_i|} + k$ whenever $G_n(\zeta) = 0$, where $i = 1,2,3$. Suppose that $A = \sum_{i=1}^{3} \sqrt[q]{|b_i|} + k$.

Combining with Lemma 1, there exists $\zeta_n \to \zeta_0$, $\eta_n \to 0$ and a subsequence of functions $\{G_n\}$ such that

$$F_n(\xi) = \eta_n^{-k} G_n(\zeta_n + \eta_n\xi) \to F(\xi)$$

uniformly on any compact subset of $\mathbb{C}$. $F(\xi)$ is a non-constant meromorphic function such that $F^{\#}(\xi) = |F'(\xi)|/(1 + |F(\xi)|^2) \leqslant F^{\#}(0) = kA + 1$. Moreover, $F(\xi)$ has zeros with multiplicities at least k and $F(\xi)$ is of order at most 2. We claim that

(a) The number of distinct zeros of $F(\xi)$ is finite in $\mathbb{C}$;

(b) $F(\xi) = 0 \iff (F^{(k)}(\xi))^q \in S_2$.

Let ζ_0 be a zero of $g(\zeta)$ with multiplicity m, we prove that $F(\xi)$ has at most m different zeros. Otherwise, if $F(\xi)$ has $m+1$ distinct points $\xi_1, \xi_2, \ldots, \xi_{m+1}$ in $\mathbb{C}$ such that $F(\xi_j) = 0$ for $1 \leqslant j \leqslant m+1$. From Hurwitz's theorem and the fact that $F(\xi) \not\equiv 0$, there exists a sequence of points $\xi_{n_j} \to \xi_j$ satisfying $F_n(\xi_{n_j}) = 0$, then we know that $f_n(z_n + \rho_n(\zeta_n + \eta_n\xi_{n_j})) - a_1 = 0$ when $\zeta_n + \eta_n\xi_{n_j} \to \zeta_0$ for large n, where $j = 1,2,\ldots,m+1$. We deduce that ζ_0 is a zero of $g(\zeta)$ with multiplicity $m+1$

by Hurwitz's theorem, which contradicts the above hypothesis. Thus the proof of claim (a) is complete.

If $F(\xi_0)=0$, by Hurwitz's theorem and the condition $F(\xi)\not\equiv 0$, we can find points $\xi_n \to \xi_0$ such that $F_n(\xi_n)=0$, so is $f_n(z_n+\rho_n(\zeta_n+\eta_n\xi_n))=a_1$. By the assumption that $\overline{E}(S_1,f)=\overline{E}(S_2,(f^{(k)})^q)$, we know that $(F_n^{(k)}(\xi_n))^q=(f_n^{(k)}(z_n+\rho_n(\zeta_n+\eta_n\xi_n)))^q\in S_2$ and $\lim_{n\to\infty}(F_n^{(k)}(\xi_n))^q=(F^{(k)}(\xi_0))^q\in S_2$.

Thus we have proved $F(\xi)=0\to(F^{(k)}(\xi))^q\in S_2$. On the other hand, if $(F^{(k)}(\xi_0))^q\in S_2$, for a given point $\xi_0\in\mathbb{C}$, there exists $(F^{(k)}(\xi_0))^q=b_i$, where $i=1,2,3$. First of all, we prove $(F^{(k)}(\xi))^q\not\equiv b_i$. On the contrary, we suppose that $(F^{(k)}(\xi))^q\equiv b_i$, from which we obtain that $F(\xi)$ is a polynomial with degree at most k, according to the fact that each zero of $F(\xi)$ is of multiplicity at least k, we claim that $F(\xi)$ is a polynomial of degree k such that $F(\xi)=(k!)^{-1}\sqrt[q]{b_i}(\xi-\xi_0)^k$. If $\sqrt[q]{b_i}\neq 0$, we have

$$F^{\#}(0)=\frac{|F'(0)|}{1+|F(0)|^2}=\frac{\frac{\sqrt[q]{|b_i|}}{(k-1)!}|\xi_0|^{k-1}}{1+\frac{(\sqrt[q]{|b_i|})^2}{(k!)^2}|\xi_0|^{2k}},$$

$$F^{\#}(0)\leqslant\begin{cases}\frac{\sqrt[q]{|b_i|}}{(k-1)!}|\xi_0|^{k-1}\leqslant\frac{\sqrt[q]{|b_i|}}{2}, & |\xi_0|<1\\ \frac{\frac{\sqrt[q]{|b_i|}}{(k-1)!}|\xi_0|^{k-1}}{2\frac{\sqrt[q]{|b_i|}}{k!}|\xi_0|^{k}}\leqslant\frac{k}{2}, & |\xi_0|\geqslant 1.\end{cases}$$

All in all, $F^{\#}(0)\leqslant\frac{\sqrt[q]{|b_i|}}{2}+\frac{k}{2}$. If $\sqrt[q]{|b_i|}=0$, the above inequality also holds, which contradicts the fact that $F^{\#}(0)=kA+1$. By Hurwitz's theorem and the condition that $(F^{(k)}(\xi))^q\not\equiv b_i$, we choose a sequence of points ξ_n satisfying $\xi_n\to\xi_0$, such that

$$(F_n^{(k)}(\xi_n))^q=(f_n^{(k)}(z_n+\rho_n(\zeta_n+\eta_n\xi_n)))^q=b_i.$$

Combining with the assumption

$$\overline{E}(S_1,f)=\overline{E}(S_2,(f^{(k)})^q),$$

we obtain $f_n(z_n+\rho_n(\zeta_n+\eta_n\xi_n))\in S_1$. Next we prove that there exists a subsequence of f_n such that $f_n(z_n+\rho_n(\zeta_n+\eta_n\xi_n))=a_1$ if n is sufficiently large. Otherwise, without loss of generality, suppose that $f_n(z_n+\rho_n(\zeta_n+\eta_n\xi_n))=a_2$ for large n, we obtain $F(\xi_0)=\lim_{n\to\infty}\rho_n^{-k}\eta_n^{-k}(f_n(z_n+\rho_n(\zeta_n+\eta_n\xi_n))-a_1)=\lim_{n\to\infty}\rho_n^{-k}\eta_n^{-k}(a_2-a_1)=\infty$. This is inconsistent with the fact that $(F^{(k)}(\xi_0))^q=b_i$. Hence, we get $F(\xi_0)=\lim_{n\to\infty}\rho_n^{-k}\eta_n^{-k}(f_n(z_n+\rho_n(\zeta_n+\eta_n\xi_n))-a_1)=0$. This illustrates that $F(\xi)=0$ whenever $F^{(k)}(\xi))^q\in S_2$, thus claim (b) is proved.

By claims (a) and (b), and Lemma 2, we claim that $F(\xi)$ is a rational function. Next we prove that $F(\xi)$ is a polynomial.

Suppose that $F(\xi)$ is a non-constant rational function, but $F(\xi)$ is not a polynomial. Set $F=\frac{H}{Q}$, both H and Q are relatively prime polynomials. Zeros of H are $\alpha_1,\dots,\alpha_s$ with multiplicities $m_1,\dots,m_s$ respectively. Zeros of Q are $\beta_1,\dots,\beta_t$ with multiplicities $n_1,\dots,n_t$, respectively. We get $m_i\geqslant k$ by the hypothesis that zeros of F have multiplicities at least k, where $i=1,2,\dots,s$. Let $M=\deg H=\sum_{i=1}^s m_i$ and $N=\deg Q=\sum_{j=1}^t n_j$. Write $(F^{(k)})^q=\frac{(H_k)^q}{(Q_k)^q}$. After a simple calculation, we get $\deg((Q_k)^q)=q(N+kt)$, thus

$$2q(N+kt)\log r\leqslant 2T(r,(F^{(k)})^q).$$

According to the second Nevanlinna fundamental theorem, we obtain

$$\begin{aligned}2T(r,(F^{(k)})^q)&\leqslant\sum_{i=1}^{3}\overline{N}\Big(r,\tfrac{1}{(F^{(k)})^q-b_i}\Big)+\overline{N}(r,F)+O(1)\\&\leqslant\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,F)+O(1)\\&\leqslant(t+s)\log r+O(1).\end{aligned}$$

We find that $2q(N+kt)\log r\leqslant(t+s)\log r+O(1)$ and $s\geqslant 2qN+1$. By the condition that zeros of F have multiplicities at least k, we have $M\geqslant ks\geqslant 2qNk+k>2N$. After a simple calculation, we obtain

$$\deg((H_k)^q)-\deg((Q_k)^q)=q(M-N-k)>0. \quad (4)$$

From (4), we have $\deg((H_k)^q)=q((M-N-k)+N+kt)=q(M+k(t-1))\geqslant M$ and get

$$\begin{aligned}2M\log r&\leqslant(t+s)\log r+O(1)\\&\leqslant(M+N)\log r+O(1).\end{aligned}\quad (5)$$

It follows from (5) that $M\leqslant N$, which is impossible.

Suppose that $F(\xi)=C_p\xi^p+C_{p-1}\xi^{p-1}+\cdots+C_k\xi^k+C_{k-1}\xi^{k-1}+\cdots+C_0$, where C_i, $i=0,1,\dots,p$, is a complex number, and $C_p\neq 0$. Combining with the condition that zeros of F have multiplicities at least k, we obtain $p\geqslant k$.

By the second Nevanlinna fundamental theorem, we have

$$\begin{aligned}2T(r,(F^{(k)})^q)&\leqslant\sum_{i=1}^{3}\overline{N}\Big(r,\tfrac{1}{(F^{(k)})^q-b_i}\Big)+\overline{N}(r,F)+O(1)\\&\leqslant N(r,\tfrac{1}{F})+O(1)\\&=T(r,F)+O(1),\end{aligned}$$

from which we obtain $2q(p-k)\log r\leqslant p\log r+O(1)$ and $k\leqslant p\leqslant(1+\frac{1}{2q-1})k\leqslant 2k$.

Subcase 1.1.1 $p=k$. Then F is a polynomial of degree k, so that $(F^{(k)}(\xi))^q\equiv d$, where d is a constant. If $d \in S_2$, we get $F(\xi) \equiv 0$ by claim (b), which is invalid. If $d \notin S_2$, we know that $F(\xi)$ is a non-zero constant from the fact that $F(\xi)$ is a polynomial, which contradicts the condition that F is a non-constant meromorphic function.

Subcase 1.1.2 $p = 2k$. If $p = 2k$, we know that F is a polynomial with degree $2k$. We claim that F has only two distinct zeros or only one zero by the condition that zeros of F have multiplicity at least k.

Subcase 1.1.2.1 $F(\xi)$ has only one zero. From the second Nevanlinna fundamental theorem and claim (b),

$$\begin{aligned}2T(r,(F^{(k)})^q) &\leqslant \sum_{i=1}^{3}\overline{N}\Big(r,\tfrac{1}{(F^{(k)})^q-b_i}\Big)\\ &\quad+\overline{N}(r,F)+O(1)\\ &\leqslant N(r,\tfrac{1}{(F^{(k)})^q})+O(1)\\ &= T(r,(F^{(k)})^q)+O(1),\end{aligned}$$

which is conflicting.

Subcase 1.1.2.2 $F(\xi)$ has only two distinct zeros. Write $F(\xi) = m(\xi-\xi_0)^k(\xi-\xi_1)^k$, where $m \neq 0$, ξ_0 and ξ_1 are distinct finite complex numbers. For any $i = 1,2,3$, $(F^{(k)}(\xi))^q - b_i$ has at least one zero, then we know that $F(\xi)$ has at least three distinct zeros by the fact that $(F^{(k)}(\xi))^q \in S_2 \to F(\xi) = 0$, which is impossible.

Subcase 1.2 $g(\zeta) \neq 0$. Suppose that there exists ζ_0 satisfying $g(\zeta_0) = a_2 - a_1$. Note that $g(\zeta) \not\equiv a_2 - a_1$, by Hurwitz's theorem, we find a sequence ζ_n such that $g_n(\zeta_n) = a_2 - a_1$ when $\zeta_n \to \zeta_0$, an equivalent statement is that $f_n(z_n + \rho_n\zeta_n) = a_2$. Since $\overline{E}(S_1, f) \subseteq \overline{E}(S_2,(f^{(k)})^q)$, we obtain $(f_n^{(k)}(z_n + \rho_n\zeta_n))^q \in S_2$ and find a subsequence of $\{f_n\}$ such that $(f_n^{(k)}(z_n + \rho_n\zeta_n))^q = s$, and therefore $(g^{(k)}(\zeta_0))^q = \lim_{n\to\infty}(g_n^{(k)}(\zeta_n))^q = \lim_{n\to\infty}(\rho_n^k f_n^{(k)}(z_n + \rho_n\zeta_n))^q = \lim_{n\to\infty}\rho^{kq}s^q = 0$, where $s \in S_2$. This illustrates that $\overline{E}(a_2 - a_1, g) \subseteq \overline{E}(0,(g^{(k)})^q)$. In a similar fashion, we can prove $\overline{E}(a_3 - a_1, g) \subseteq \overline{E}(0,(g^{(k)})^q)$. By the second Nevanlinna fundamental theorem and the hypothesis that poles of g are of multiplicities at least k and Lemma 3,

$$\begin{aligned}2T(r,g) &\leqslant \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})\\ &\quad+\sum_{i=2}^{3}\overline{N}\Big(r,\tfrac{1}{g-(a_i-a_1)}\Big)+S(r,g)\\ &\leqslant \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{(g^{(k)})^q})+S(r,g)\\ &= \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g^{(k)}})+S(r,g),\end{aligned}$$

which gives

$$\begin{aligned}2T(r,g) &\leqslant \tfrac{1}{k}N(r,g)+N(r,\tfrac{1}{g})+k\overline{N}(r,g)+S(r,g)\\ &\leqslant (\tfrac{1}{k}+1)T(r,g)+S(r,g),\end{aligned}$$

which contradicts the fact that $k > 2$.

Case 2. $a \notin S_1$. By Lemma 1, there exists $z_n \to z_0$, $\rho_n \to 0$ and a subsequence of $f_n \in \mathscr{F}$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) - a \to g(\zeta)$$

uniformly on any compact subset of $\mathbb{C}$, where $g(\zeta)$ is a non-constant meromorphic function such that both zeros and poles of $g(\zeta)$ have multiplicities at least k. Similar to Subcase 1.2, we have $\overline{E}(a_i - a, g) \subseteq \overline{E}(0,(g^{(k)})^q)$, $i = 1,2,3$. By the second Nevanlinna fundamental theorem and the fact that both zeros and poles of g are of multiplicities at least k and Lemma 3, we get

$$\begin{aligned}&3T(r,g)\\ &\leqslant \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+\sum_{i=1}^{3}\overline{N}(r,\tfrac{1}{g-(a_i-a)})+S(rg)\\ &= \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+\overline{N}(r,\tfrac{1}{(g^{(k)})^q})+S(r,g)\\ &= \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+\overline{N}(r,\tfrac{1}{g^{(k)}})+S(r,g)\\ &\leqslant \tfrac{1}{k}N(r,g)+\tfrac{1}{k}N(r,\tfrac{1}{g})+N(r,\tfrac{1}{g})+N(r,g)+S(r,g)\\ &\leqslant (\tfrac{2}{k}+2)T(r,g)+S(r,g),\end{aligned}$$

which contradicts the fact that $k > 2$. Theorem 1 is completely proved. □

PROOF OF Theorem 2

Proof: Assume that $\mathscr{F}$ is not normal in a domain D, without loss of generality, we suppose that $\mathscr{F}$ is not normal at $z_0 \in D$. By Lemma 1, we choose $z_n \to z_0$, $f_n \in \mathscr{F}$ and $\rho_n \to 0$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) \to g(\zeta)$$

uniformly on compact subsets of $\mathbb{C}$. We know that at least one of $g(\zeta) - a_i$ must have zeros by the condition that $g(\zeta)$ is non-constant and Picard theorem, where $i = 1,2,3$. Let ζ_0 be a zero of $g(\zeta) - a_1$ with multiplicity k.

Write $G_n(\zeta) = \rho_n^{-k}(f_n(z_n+\rho_n\zeta)-a_1)$. According to the assumption, we claim that zeros of $G_n(\zeta)$ have multiplicities at least k. In the same manner as in the proof of Subcase 1.1 in Theorem 1, we can prove that $\{G_n(\zeta)\}$ is not normal at ζ_0. Combining with the assumption that $\overline{E}(S_1, f) = \overline{E}(S_2,(f^{(k)})^q)$, we know that $|G_n^{(k)}(\zeta)| \leqslant \sum_{i=1}^{3}\sqrt[q]{|b_i|} + k$ whenever $G_n(\zeta) = 0$. Suppose that $A = \sum_{i=1}^{3}\sqrt[q]{|b_i|} + k$.

By Lemma 1, there exists $\zeta_n \to \zeta_0$, $\eta_n \to 0$ and a sequence of $\{G_n\}$ such that $F_n(\xi) = \eta_n^{-k} G_n(\zeta_n + \eta_n \xi)$ which converges spherically locally uniformly to a non-constant meromorphic function $F(\xi)$ satisfying $F^{\#}(\xi) = |F'(\xi)|/(1+|F(\xi)|^2) \leqslant F^{\#}(0) = kA+1$ on $\mathbb{C}$; moreover, zeros of $F(\xi)$ are of multiplicities at least k, and $F(\xi)$ is of order at most 2.

Also as in the proof of Subcase 1.1 in Theorem 1, we claim that

(c) The number of distinct zeros of $F(\xi)$ is finite in $\mathbb{C}$;

(d) $F(\xi) = 0 \iff (F^{(k)}(\xi))^q \in S_2$.

According to claims (c) and (d), and Lemma 2, we know that $F(\xi)$ is a rational function. In the similar fashion to the proof of Subcase 1.1 in Theorem 1, we claim that $F(\xi)$ is a polynomial.

Set $F(\xi) = C_p \xi^p + C_{p-1}\xi^{p-1} + \cdots + C_k \xi^k + C_{k-1}\xi^{k-1} + \cdots + C_0$, where $C_i\,(i = 0, 1, \ldots, p)$ is a complex number, and $C_p \neq 0$. We get $p \geqslant k$ by the fact that zeros of $F(\xi)$ have multiplicities $\geqslant k$.

Next we can prove Theorem 2 by using the same argument as in Subcase 1.1 of Theorem 1. □

PROOF OF Theorem 3

Proof: Suppose that $\mathscr{F}$ is not normal in a domain D, without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0 \in D$. Then we consider two cases.

Case 1. $a \in S_1$. Suppose that $a = a_1$, by Lemma 1, there exists

(i) a sequence of complex numbers $z_n \to z_0$, $|z_n| < r < 1$;
(ii) a sequence of functions $f_n \in \mathscr{F}$;
(iii) a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\zeta) = f_n(z_n + \rho_n \zeta) - a_1$ converges uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\zeta)$ in $\mathbb{C}$. Moreover, $g(\zeta)$ is of order at most 2, all of whose zeros and poles have multiplicities at least $k \geqslant 2$ and $k+1$, respectively.

Subcase 1.1 There exists $\zeta_0 \in \mathbb{C}$ such that $g(\zeta_0) = 0$. Set $G_n(\zeta) = \rho_n^{-k}(f_n(z_n + \rho_n\zeta) - a_1)$. By the condition, we know that zeros of $G_n(\zeta)$ have multiplicities at least $k \geqslant 2$. Obviously, there is a neighbourhood $|\zeta - \zeta_0| < \delta(\zeta_0)$ such that $G_n(\zeta)$ is a holomorphic function.

Also as in the proof of Subcase 1.1 in Theorem 1, we claim that $\{G_n(\zeta)\}$ is not normal at ζ_0 and $|G_n^{(k)}(\zeta)| \leqslant \sum_{i=1}^{3} \sqrt[q]{|b_i|} + k$ whenever $G_n(\zeta) = 0$, $i = 1, 2, 3$. Suppose that $A = \sum_{i=1}^{3} \sqrt[q]{|b_i|} + k$.

Combining with Lemma 1, there exists $\zeta_n \to \zeta_0$, $\eta_n \to 0$ and a subsequence of functions $\{G_n\}$ such that $F_n(\xi) = \eta_n^{-k} G_n(\zeta_n + \eta_n \xi)$ converges locally uniformly to a non-constant meromorphic function $F(\xi)$ such that $F^{\#}(\xi) = |F'(\xi)|/(1+|F(\xi)|^2) \leqslant F^{\#}(0) = kA+1$ on any compact subset of $\mathbb{C}$, where $F(\xi)$ has zeros with multiplicities at least k and $F(\xi)$ is of order at most 2. We claim that

(e) The number of distinct zeros of $F(\xi)$ is finite in $\mathbb{C}$;

(f) $F(\xi) = 0 \iff (F^{(k)}(\xi))^q \in S_2$.

By claims (e) and (f) and, Lemma 2, we claim that $F(\xi)$ is a rational function. In a similar fashion to the proof of Subcase 1.1 in Theorem 1, we claim that $F(\xi)$ is a polynomial.

Next, the proof of Subcase 1.1 in Theorem 3 is completely similar with the proof of Subcase 1.1 in Theorem 1.

Subcase 1.2 $g(\zeta) \neq 0$. If there exists ζ_0 such that $g(\zeta_0) = a_2 - a_1$. By the fact that $g(\zeta) \not\equiv a_2 - a_1$ and Hurwitz's theorem, we find a sequence ζ_n such that $g_n(\zeta_n) = a_2 - a_1$ when $\zeta_n \to \zeta_0$, which means that $f_n(z_n + \rho_n\zeta_n) = a_2$. We obtain $(f_n^{(k)}(z_n + \rho_n\zeta_n))^q \in S_2$ and find a subsequence of $\{f_n\}$ such that $(f_n^{(k)}(z_n + \rho_n\zeta_n))^q = s$ by the condition that $\overline{E}(S_1, f) \subseteq \overline{E}(S_2, (f^{(k)})^q)$, and therefore $(g^{(k)}(\zeta_0))^q = \lim_{n\to\infty}(g_n^{(k)}(\zeta_n))^q = \lim_{n\to\infty} \rho_n^k f_n^{(k)}(z_n + \rho_n\zeta_n))^q = \lim_{n\to\infty} \rho^{kq} s^q = 0$, where $s \in S_2$. This illustrates that $\overline{E}(a_2 - a_1, g) \subseteq \overline{E}(0, (g^{(k)})^q)$. In a similar fashion, we can prove $\overline{E}(a_3 - a_1 g) \subseteq \overline{E}(0, (g^{(k)})^q)$. By the second Nevanlinna fundamental theorem and the hypothesis that poles of g are of multiplicities at least $k+1$, and Lemma 3, it is known that

$$\begin{aligned}
2T(r,g) &\leqslant \overline{N}(r,g) + \overline{N}(r, \tfrac{1}{g}) \\
&\quad + \sum_{i=2}^{3} \overline{N}(r, \tfrac{1}{g-(a_i-a_1)}) + S(r,g) \\
&\leqslant \overline{N}(r,g) + \overline{N}(r, \tfrac{1}{(g^{(k)})^q}) + S(r,g) \\
&= \overline{N}(r,g) + \overline{N}(r, \tfrac{1}{g^{(k)}}) + S(r,g) \\
&\leqslant \tfrac{1}{k+1} N(r,g) + N(r, \tfrac{1}{g}) + k\overline{N}(r,g) + S(r,g) \\
&\leqslant (\tfrac{1}{k+1} + \tfrac{k}{k+1}) T(r,g) + S(r,g) \\
&= T(r,g) + S(r,g),
\end{aligned}$$

which is conflicting.

Case 2. $a \notin S_1$. By Lemma 1, there exists $z_n \to z_0$, $\rho_n \to 0$ and a subsequence of $f_n \in \mathscr{F}$ such that $g_n(\zeta) = f_n(z_n + \rho_n\zeta) - a$ converges locally uniformly to a non-constant meromorphic function $g(\zeta)$ on any compact subset of $\mathbb{C}$, where both zeros and poles of $g(\zeta)$ have multiplicities at least k and

$k+1$, respectively. It is similar to Subcase 1.2, we have $\overline{E}(a_i-a,g)\subseteq\overline{E}(0,(g^{(k)})^q)$, $i=1,2,3$.

By the second Nevanlinna fundamental theorem and the fact that both zeros and poles of g are of multiplicities at least k and $k+1$, respectively, and Lemma 3, we get

$$\begin{aligned}3T(r,g)&\leqslant\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})\\&\quad+\sum_{i=1}^{3}\overline{N}(r,\tfrac{1}{g-(a_i-a)})+S(r,g)\\&=\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+\overline{N}(r,\tfrac{1}{(g^{(k)})^q})+S(r,g)\\&=\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+\overline{N}(r,\tfrac{1}{g^{(k)}})+S(r,g)\\&\leqslant\tfrac{1}{k+1}N(r,g)+\tfrac{1}{k}N(r,\tfrac{1}{g})+N(r,\tfrac{1}{g})\\&\quad+\tfrac{k}{k+1}N(r,g)+S(r,g)\\&\leqslant(\tfrac{1}{k}+2)T(r,g)+S(r,g),\end{aligned}$$

which contradicts the fact that $k\geqslant 2$. Theorem 3 is proved completely. □

PROOF OF Theorem 4

Proof: If $\mathscr{F}$ is not normal in a domain D, without loss of generality, we suppose that $\mathscr{F}$ is not normal at $z_0\in D$. Next, we consider two cases.

Case 1. $a\in S_1$. Assume that $a=a_1$, by Lemma 1, there exists $z_n\to z_0$, $f_n\in\mathscr{F}$ and $\rho_n\to 0$ satisfying

$$g_n(\zeta)=f_n(z_n+\rho_n\zeta)-a_1\to g(\zeta)$$

uniformly on compact subsets of $\mathbb{C}$, where $g(\zeta)$ has zeros and poles with multiplicities at least $k\geqslant 4$ and $k-1$, respectively, $g(\zeta)$ is a non-constant meromorphic function of order at most 2.

Subcase 1.1 There exists $\zeta_0\in\mathbb{C}$ such that $g(\zeta_0)=0$. Write $G_n(\zeta)=\rho_n^{-k}(f_n(z_n+\rho_n\zeta)-a_1)$. We know that $G_n(\zeta)$ has zeros with multiplicities at least $k\geqslant 4$ by the condition. It is clear to find a neighbourhood $|\zeta-\zeta_0|<\delta(\zeta_0)$ such that $G_n(\zeta)$ is a holomorphic function.

In a similar fashion to the proof of Subcase 1.1 in Theorem 1, we claim that $\{G_n(\zeta)\}$ is not normal at ζ_0 and $|G_n^{(k)}(\zeta)|\leqslant\sum_{i=1}^{3}\sqrt[q]{|b_i|}+k$ whenever $G_n(\zeta)=0$, $i=1,2,3$. Suppose that $A=\sum_{i=1}^{3}\sqrt[q]{|b_i|}+k$.

By Lemma 1, there exists $\zeta_n\to\zeta_0$, $\eta_n\to 0$ and a subsequence of functions $\{G_n\}$ such that

$$F_n(\xi)=\eta_n^{-k}G_n(\zeta_n+\eta_n\xi)\to F(\xi)$$

uniformly on any compact subset of $\mathbb{C}$. $F(\xi)$, whose zeros have multiplicities at least $k\geqslant 4$, is a non-constant meromorphic function such that $F^{\#}(\xi)=|F'(\xi)|/(1+|F(\xi)|^2)\leqslant F^{\#}(0)=kA+1$, and $F(\xi)$ is of order at most 2. We claim that

(g) The number of distinct zeros of $F(\xi)$ is finite in $\mathbb{C}$;

(h) $F(\xi)=0\iff(F^{(k)}(\xi))^q\in S_2$.

We claim that $F(\xi)$ is a rational function according to claims (g) and (h), and Lemma 2. In a similar fashion to the proof of Subcase 1.1 in Theorem 1, we claim that $F(\xi)$ is a polynomial.

Next, also as in the proof of Subcase 1.1 in Theorem 1, we can prove Subcase 1.1 in Theorem 4.

Subcase 1.2 $g(\zeta)\neq 0$. Without loss of generality, we assume that there exists ζ_0 satisfying $g(\zeta_0)=a_2-a_1$. Combining with Hurwitz's theorem and the fact that $g(\zeta)$ is non-constant, there exists a sequence ζ_n such that $g_n(\zeta_n)=a_2-a_1$ and $f_n(z_n+\rho_n\zeta_n)=a_2$ for $\zeta_n\to\zeta_0$. We get $(f_n^{(k)}(z_n+\rho_n\zeta_n))^q\in S_2$ and choose a subsequence of $\{f_n\}$ such that $(f_n^{(k)}(z_n+\rho_n\zeta_n))^q=s$ by the condition, so $(g^{(k)}(\zeta_0))^q=\lim_{n\to\infty}(g_n^{(k)}(\zeta_n))^q=\lim_{n\to\infty}(\rho_n^k f_n^{(k)}(z_n+\rho_n\zeta_n))^q=\lim_{n\to\infty}\rho^{kq}s^q=0$, where $s\in S_2$. This illustrates that $\overline{E}(a_2-a_1,g)\subseteq\overline{E}(0,(g^{(k)})^q)$. In a similar fashion, we can prove $\overline{E}(a_3-a_1,g)\subseteq\overline{E}(0,(g^{(k)})^q)$. By the second Nevanlinna fundamental theorem and the hypothesis that poles of g are of multiplicities at least $k-1$ and Lemma 3, it is known that

$$\begin{aligned}2T(r,g)&\leqslant\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})\\&\quad+\sum_{i=2}^{3}\overline{N}(r,\tfrac{1}{g-(a_i-a_1)})+S(r,g)\\&\leqslant\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{(g^{(k)})^q})+S(r,g)\\&=\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g^{(k)}})+S(r,g)\\&\leqslant\tfrac{1}{k-1}N(rg)+N(r,\tfrac{1}{g})+\tfrac{k}{k-1}N(r,g)+S(r,g)\\&=(\tfrac{2}{k-1}+1)T(r,g)+S(r,g),\end{aligned}$$

which contradicts the fact that $k\geqslant 4$.

Case 2. $a\notin S_1$. According to Lemma 1, there exists $z_n\to z_0$, $\rho_n\to 0$ and a subsequence of $f_n\in\mathscr{F}$ such that

$$g_n(\zeta)=f_n(z_n+\rho_n\zeta)-a\to g(\zeta)$$

uniformly on any compact subset of $\mathbb{C}$, where $g(\zeta)$ is a non-constant meromorphic function such that both zeros and poles of $g(\zeta)$ have multiplicities at least $k\geqslant 4$ and $k-1$, respectively. By using the same argument as in Subcase 1.2, we have $\overline{E}(a_i-a,g)\subseteq$

$\overline{E}(0,(g^{(k)})^q)$, $i=1,2,3$. By the second Nevanlinna fundamental theorem and the fact that both zeros and poles of g are of multiplicities at least k and $k-1$, respectively, and Lemma 3, we get

$$\begin{aligned}
3T(r,g) &\leqslant \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g}) \\
&\quad +\sum_{i=1}^{3}\overline{N}(r,\tfrac{1}{g-(a_i-a)})+S(r,g) \\
&=\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+\overline{N}(r,\tfrac{1}{(g^{(k)})^q})+S(r,g) \\
&=\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+\overline{N}(r,\tfrac{1}{g^{(k)}})+S(r,g) \\
&\leqslant \tfrac{1}{k-1}N(r,g)+\tfrac{1}{k}N(r,\tfrac{1}{g})+N(r,\tfrac{1}{g}) \\
&\quad +\tfrac{k}{k-1}N(r,g)+S(r,g) \\
&< (\tfrac{2+k}{k-1}+1)T(r,g)+S(r,g) \\
&\leqslant 3T(r,g)+S(r,g)
\end{aligned}$$

which contradicts the fact that $k\geqslant 4$. Theorem 4 is proved completely. □

OPEN QUESTION

In Theorem 1, if S_1 and S_2 consist of holomorphic functions, and we replace a with $a(z)$ being any holomorphic function, it is natural to ask:

Open question: Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let $S_1=\{a_1(z),a_2(z),a_3(z)\}$, $S_2=\{b_1(z),b_2(z),b_3(z)\}$, where S_1 and S_2 consist of holomorphic functions, let $k>2$ and q be two positive integers, and let $a(z)$ be any holomorphic function. Suppose that for each $f\in\mathscr{F}$,

(i) $\overline{E}(S_1,f)=\overline{E}(S_2,(f^{(k)})^q)$;

(ii) both zeros and poles of $f-a(z)$ have multiplicities at least k,

then $\mathscr{F}$ is normal in D.

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REFERENCES

1. Laine I (1993) *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin.
2. Yang L (1993) *Value Distribution Theory*, Science Press, Beijing.
3. Gu YX, Pang XC, Fang ML (2007) *Theory of Normal Families and Its Applications*, Science Press, Beijing.
4. Ahlfors L (1979) *Complex Analysis*, 3rd edn, McGraw-Hill, New York.
5. Schiff J (1993) *Normal Families*, Springer-Verlag, Berlin.
6. Pang XC (2000) Normality of a family of meromorphic functions and normal function. *Chin Annal Math* **21**, 601–604.
7. Yi HX, Yang CC (2003) *The Uniqueness Theory of Meromorphic Function*, Kluwer Academic Publishers, Dordrecht.
8. Hayman WK (1964) *Meromorphic Functions*, Clarendon Press, Oxford.
9. Schwick W (1992) Sharing values and normality. *Arch Math* **59**, 50–54.
10. Fang ML (1996) A note on sharing values and normality. *Math Res* **29**, 29–32.
11. Fang ML, Zalcman L (2004) A note on normality and shared values. *Aust Math Soc* **76**, 141–150.
12. Pang XC, Zalcman L (2000) Normality and shared values. *Ark Math* **38**, 171–182.
13. Liu XJ, Pang XC (2007) Shared values and normality. *Acta Math Sin* **52**, 409–412.
14. Zhang H, Xie D, Zhang QD (2011) Normal criteria concerning shared values of meromorphic function. *Acta Math Sci* **31**, 1290–1294.
15. Pang XC, Zalcman L (2000) Normal families and shared values. *Bull London Math Soc* **32**, 325–331.