Improved Young's inequalities for positive linear operators

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ABSTRACT: Some improved Young's inequalities for scalars were first presented. Then, based on them, we have obtained new operator inequalities in Hilbert spaces.

KEYWORDS: Young's inequality, operator inequality, positive linear operators

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INTRODUCTION

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H equipped with the operator norm $S(H)$, the set of all bounded self-adjoint operators, and $\mathbb{P} = \mathbb{P}(H)$, the open convex cone of all positive invertible operators. For $X, Y \in S(H)$, we write $X \leqslant Y$ if $Y - X$ is positive, and $X < Y$ if $Y - X$ is positive invertible.

The classical Young's inequality says that if $a, b \geqslant 0$ and $0 \leqslant v \leqslant 1$, then

$$a^v b^{1-v} \leqslant va + (1-v)b \qquad (1)$$

with equality if and only if $a = b$.

This inequality has been studied, generalized, and refined in various directions. It is worth mentioning that Kittaneh et al [1] obtained an improved version, which can be stated as follows:

$$a^v b^{1-v} + r_0(\sqrt{a} - \sqrt{b})^2 \leqslant va + (1-v)b, \qquad (2)$$

where $r_0 = \min\{v, 1-v\}$.

In [2], an even more refined version was presented, which can be stated as follows:
(i) if $0 < v \leqslant 1/2$, then

$$a^{1-v} b^v + v(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{a})^2 \\ \leqslant (1-v)a + vb, \quad (3)$$

(ii) if $1/2 < v < 1$, then

$$a^{1-v} b^v + (1-v)(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{b})^2 \\ \leqslant (1-v)a + vb, \quad (4)$$

where $r = \min\{v, 1-v\}$ and $r_1 = \min\{2r, 1-2r\}$.

Let $A, B \in B(H)$ be two positive operators, $v \in [0, 1]$. The v-weighted arithmetic mean of A and B, denoted by $A\nabla_v B$, is defined as

$$A\nabla_v B = (1-v)A + vB.$$

If A is invertible, v-geometric mean of of A and B, denoted by $A\sharp_v B$, is defined as

$$A\sharp_v B = A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2}.$$

For more details, please see [3].

When $v = 1/2$, we may write $A\nabla B$ and $A\sharp B$ for brevity. It is well known that if A and B are positive invertible operators, then

$$A\nabla_v B \geqslant A\sharp_v B, \qquad v \in [0, 1].$$

Furuichi [3] gave a refined version as follows:

$$A\nabla_v B \geqslant A\sharp_v B + 2r(A\nabla B - A\sharp B) \geqslant A\sharp_v B.$$

Zhao and Wu [2] presented the following improved inequalities:
(i) if $0 < v \leqslant 1/2$, then

$$r_0(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A) + 2v(A\nabla B - A\sharp B) + A\sharp_v B \\ \leqslant A\nabla_v B, \quad (5)$$

(ii) If $1/2 < v < 1$, then

$$r_0(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B) + 2(1-v)(A\nabla B - A\sharp B) + A\sharp_v B \\ \leqslant A\nabla_v B, \quad (6)$$

where $r = \min\{v, 1-v\}$ and $r_0 = \min\{2r, 1-2r\}$.

Since then, many researchers have tried to give new refinements and generalizations of these inequalities and have obtained a series of improvements, one can refer to [4–7].

In this paper, some Young's inequalities for scalars were presented as improvements of (3) and (4). Then based on them, we have obtained new operator inequalities in Hilbert spaces as refinements of (5) and (6).

REFINEMENTS OF THE YOUNG'S INEQUALITY FOR SCALARS

In this section, some improved Young's inequalities for scalars were presented.

Theorem 1 *Let* a, b *be two nonnegative real numbers, and let* $v \in (0, 1]$.
(i) *If* $0 < v \leqslant 1/4$, *then*

$$a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2 + 2v(\sqrt[4]{ab} - \sqrt{a})^2$$
$$+ R(\sqrt[8]{a^3b} - \sqrt{a})^2 \leqslant (1-v)a + vb; \quad (7)$$

(ii) *if* $1/4 < v \leqslant 1/2$, *then*

$$a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2 + (1-2v)(\sqrt[4]{ab} - \sqrt{a})^2$$
$$+ R(\sqrt[8]{a^3b} - \sqrt[4]{ab})^2 \leqslant (1-v)a + vb; \quad (8)$$

(iii) *if* $1/2 < v \leqslant 3/4$, *then*

$$a^{1-v}b^v + (1-v)(\sqrt{a} - \sqrt{b})^2 + (2v-1)(\sqrt[4]{ab} - \sqrt{b})^2$$
$$+ R(\sqrt[8]{ab^3} - \sqrt[4]{ab})^2 \leqslant (1-v)a + vb; \quad (9)$$

(iv) *if* $3/4 < v \leqslant 1$, *then*

$$a^{1-v}b^v + (1-v)(\sqrt{a} - \sqrt{b})^2 + (2-2v)(\sqrt[4]{ab} - \sqrt{b})^2$$
$$+ R(\sqrt[8]{ab^3} - \sqrt{b})^2 \leqslant (1-v)a + vb, \quad (10)$$

where $r = \min\{v, 1-v\}$, $t = \min\{2r, 1-2r\}$, *and* $R = \min\{2t, 1-2t\}$.

Proof: For (i), if $0 < v \leqslant 1/4$, then by (2), we have

$$(1-v)a + vb - a^{1-v}b^v - v(\sqrt{a} - \sqrt{b})^2 - 2v(\sqrt[4]{ab} - \sqrt{a})^2$$
$$= (1-4v)a + 4v\sqrt[4]{a^3b} - a^{1-v}b^v$$
$$\geqslant a^{1-4v}(\sqrt[4]{a^3b})^{4v} + R(\sqrt[8]{a^3b} - \sqrt{a})^2 - a^{1-v}b^v$$
$$= R(\sqrt[8]{a^3b} - \sqrt{a})^2.$$

For (ii), if $1/4 < v \leqslant 1/2$, then by (2), we have

$$(1-v)a + vb - a^{1-v}b^v - v(\sqrt{a} - \sqrt{b})^2$$
$$- (1-2v)(\sqrt[4]{ab} - \sqrt{a})^2$$
$$= (4v-1)\sqrt{ab} + (2-4v)\sqrt[4]{a^3b} - a^{1-v}b^v$$
$$\geqslant (\sqrt{ab})^{4v-1}(\sqrt[4]{a^3b})^{2-4v} + R(\sqrt[8]{a^3b} - \sqrt[4]{ab})^2$$
$$- a^{1-v}b^v$$
$$= R(\sqrt[8]{a^3b} - \sqrt[4]{ab})^2.$$

For (iii), if $1/2 < v \leqslant 3/4$, then by (2), we have

$$(1-v)a + vb - a^{1-v}b^v - (1-v)(\sqrt{a} - \sqrt{b})^2$$
$$- (2v-1)(\sqrt[4]{ab} - \sqrt{b})^2$$
$$= (3-4v)\sqrt{ab} + (4v-2)\sqrt[4]{ab^3} - a^{1-v}b^v$$
$$\geqslant (\sqrt{ab})^{3-4v}(\sqrt[4]{ab^3})^{4v-2} + R(\sqrt[8]{ab^3} - \sqrt[4]{ab})^2$$
$$- a^{1-v}b^v$$
$$= R(\sqrt[8]{ab^3} - \sqrt[4]{ab})^2.$$

For (iv), if $3/4 < v \leqslant 1$, then by (2), we have

$$(1-v)a + vb - a^{1-v}b^v - (1-v)(\sqrt{a} - \sqrt{b})^2$$
$$- (2-2v)(\sqrt[4]{ab} - \sqrt{b})^2$$
$$= (4v-3)b + (4-4v)\sqrt[4]{ab^3} - a^{1-v}b^v$$
$$\geqslant b^{4v-3}(\sqrt[4]{ab^3})^{4-4v} + R(\sqrt[8]{ab^3} - \sqrt{b})^2 - a^{1-v}b^v$$
$$= R(\sqrt[8]{ab^3} - \sqrt{b})^2.$$

$$\square$$

Remark 1 Since $R \geqslant 0$, the inequalities (7)–(10) are the improvements of the scalar Young's inequalities (3) and (4).

Replacing a by a^2, b by b^2, respectively, the following corollary can be obtained.

Corollary 1 *Let* a, b *be two nonnegative real numbers, and let* $v \in (0, 1]$.
(i) *If* $0 < v \leqslant 1/4$, *then*

$$(a^{1-v}b^v)^2 + v(a-b)^2 + 2v(\sqrt{ab} - a)^2$$
$$+ R(\sqrt[4]{a^3b} - a)^2 \leqslant (1-v)a^2 + vb^2; \quad (11)$$

(ii) *if* $1/4 < v \leqslant 1/2$, *then*

$$(a^{1-v}b^v)^2 + v(a-b)^2 + (1-2v)(\sqrt{ab} - a)^2$$
$$+ R(\sqrt[4]{a^3b} - \sqrt{ab})^2 \leqslant (1-v)a^2 + vb^2; \quad (12)$$

(iii) *if* $1/2 < \nu \leqslant 3/4$, *then*

$$(a^{1-\nu}b^{\nu})^2 + (1-\nu)(a-b)^2 + (2\nu-1)(\sqrt{ab}-b)^2$$
$$+ R(\sqrt[4]{ab^3} - \sqrt{ab})^2 \leqslant (1-\nu)a^2 + \nu b^2; \quad (13)$$

(iv) *If* $3/4 < \nu \leqslant 1$, *then*

$$(a^{1-\nu}b^{\nu})^2 + (1-\nu)(a-b)^2 + (2-2\nu)(\sqrt{ab}-b)^2$$
$$+ R(\sqrt[4]{ab^3} - b)^2 \leqslant (1-\nu)a^2 + \nu b^2, \quad (14)$$

where $r = \min\{\nu, 1-\nu\}$, $t = \min\{2r, 1-2r\}$, *and* $R = \min\{2t, 1-2t\}$.

OPERATOR INEQUALITIES FOR THE IMPROVED YOUNG'S INEQUALITIES

Based on the improvements of the scalar Young's inequalities (7)–(10), we present corresponding operator inequalities.

Lemma 1 *Let* $X \in B(H)$ *be self-adjoint, and let* f *and* g *be continuous real functions such that* $f(t) \geqslant g(t)$ *for all* $t \in Sp(X)$ *(the spectrum of* X*). Then* $f(X) \geqslant g(X)$.

Theorem 2 *Let* $A, B \in B(H)$ *be two positive invertible operators, and let* $\nu \in (0, 1]$.
(i) *If* $0 < \nu \leqslant 1/4$, *then*

$$A\sharp_\nu B + 2\nu(A\nabla B - A\sharp B) + 2\nu(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A)$$
$$+ R(A\sharp_{\frac{1}{4}}B - 2A\sharp_{\frac{1}{8}}B + A) \leqslant A\nabla_\nu B; \quad (15)$$

(ii) *if* $1/4 < \nu \leqslant 1/2$, *then*

$$A\sharp_\nu B + 2\nu(A\nabla B - A\sharp B) + (1-2\nu)(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A)$$
$$+ R(A\sharp_{\frac{1}{4}}B - 2A\sharp_{\frac{3}{8}}B + A\sharp B) \leqslant A\nabla_\nu B; \quad (16)$$

(iii) *if* $1/2 < \nu \leqslant 3/4$, *then*

$$A\sharp_\nu B + 2(1-\nu)(A\nabla B - A\sharp B) + (2\nu-1)(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B)$$
$$+ R(A\sharp_{\frac{3}{4}}B - 2A\sharp_{\frac{5}{8}}B + A\sharp B) \leqslant A\nabla_\nu B; \quad (17)$$

(iv) *if* $3/4 < \nu \leqslant 1$, *then*

$$A\sharp_\nu B + 2(1-\nu)(A\nabla B - A\sharp B) + (2-2\nu)(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B)$$
$$+ R(A\sharp_{\frac{3}{4}}B - 2A\sharp_{\frac{7}{8}}B + B) \leqslant A\nabla_\nu B, \quad (18)$$

where $r = \min\{\nu, 1-\nu\}$, $t = \min\{2r, 1-2r\}$ *and* $R = \min\{2t, 1-2t\}$.

Proof: For (i), if $0 < \nu \leqslant 1/4$, then by (7), we have

$$b^\nu + \nu(1-\sqrt{b})^2 + 2\nu(\sqrt[4]{b}-1)^2 + R(\sqrt[8]{b}-1)^2 \leqslant (1-\nu) + \nu b$$

for any $b > 0$, which for $X = A^{-1/2}BA^{-1/2}$ with $Sp(X) \subseteq (0, \infty)$, then provides

$$X^\nu + \nu(X - 2X^{\frac{1}{2}} + I) + 2\nu(X^{\frac{1}{2}} - 2X^{\frac{1}{4}} + I)$$
$$+ R(X^{\frac{1}{4}} - 2X^{\frac{1}{8}} + I) \leqslant (1-\nu)I + \nu X. \quad (19)$$

Multiplying both sides of (19) by $A^{1/2}$, we get

$$A\sharp_\nu B + \nu(A + B - 2A\sharp B) + 2\nu(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A)$$
$$+ R(A\sharp_{\frac{1}{4}}B - 2A\sharp_{\frac{1}{8}}B + A) \leqslant A\nabla_\nu B. \quad (20)$$

This leads to (15),

$$A\sharp_\nu B + 2\nu(A\nabla B - A\sharp B) + 2\nu(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A)$$
$$+ R(A\sharp_{\frac{1}{4}}B - 2A\sharp_{\frac{1}{8}}B + A) \leqslant A\nabla_\nu B.$$

The proof of inequalities (16)–(18) are similar to that of (15). $\qquad\square$

Remark 2 Since $f(t) = (\sqrt[8]{t} - 1)^2$ is a continuous function on $(0, \infty)$ and $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is a positive operator, we have $Sp(f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})) \subseteq [0, \infty)$. Then $A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = A\sharp_{\frac{1}{4}}B - 2A\sharp_{\frac{1}{8}}B + A$ is a positive operator. Similarly, we can obtain that the operators $A\sharp_{\frac{1}{4}}B - 2A\sharp_{\frac{3}{8}}B + A\sharp B$, $A\sharp_{\frac{3}{4}}B - 2A\sharp_{\frac{5}{8}}B + A\sharp B$ and $A\sharp_{\frac{3}{4}}B - 2A\sharp_{\frac{7}{8}}B + B$ are positive operators. So the inequalities (15)–(18) are the refinements of (5) and (6).

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