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New Hermite-Hadamard-Fejér type inequalities for (η_1, η_2)-convex functions via fractional calculus

Sikander Mehmood*, **Fiza Zafar**, **Nusrat Yasmin**

Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60800 Pakistan

*Corresponding author, e-mail: sikander.mehmood@yahoo.com

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ABSTRACT: In this paper, we present some new Hermite-Hadamard-Fejér type inequality for fractional integrals for (η_1, η_2)-convex functions. Our results give some new error bounds for the weighted trapezoidal and weighted mid-point rules in fractional domain. The results presented here are noteworthy extensions of earlier works.

KEYWORDS: Hermite-Hadamard-inequality, Hermite-Hadamard-Fejér inequality, Riemann-Liouville fractional integral, convex function

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INTRODUCTION

Recently, fractional calculus has proved to be a powerful tool for study of interesting facts in different fields of sciences. It plays an important role in many physical phenomenon including nonlinear oscillations of earthquakes, and flow of seepage in the porous medium. It is also used in modeling in fluid dynamics.

Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $x_1, x_2 \in I$ with $x_1 < x_2$, then

$$g\left(\frac{x_1+x_2}{2}\right) \leqslant \frac{1}{x_2-x_1}\int_{x_1}^{x_2} g(x)\,\mathrm{d}x \leqslant \frac{g(x_1)+g(x_2)}{2}. \quad (1)$$

The inequality is known as Hermite-Hadamard inequality for convex functions. Fejér [1] gave a weighted generalization of (1) as follows:

$$g\left(\frac{x_1+x_2}{2}\right)\int_{x_1}^{x_2} w(x)\,\mathrm{d}x \leqslant \int_{x_1}^{x_2} w(x)g(x)\,\mathrm{d}x \leqslant \frac{g(x_1)+g(x_2)}{2}\int_{x_1}^{x_2} w(x)\,\mathrm{d}x, \quad (2)$$

where $w : [x_1; x_2] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = (x_1 + x_2)/2$. The concept of invex sets was given by Antczak [2].

Definition 1 A set $H \subseteq \mathbb{R}^n$ is invex with respect to the map $\eta : H \times H \to \mathbb{R}^n$ if for every $x_1, x_2 \in H$ and $t \in [0, 1]$, $x_2 + t\eta(x_1, x_2) \in H$.

The invex set is also called an η-connected set. Every convex set is an invex set but its converse is not necessarily true. Weir and Mond [3], defined preinvex functions as a generalization of convex functions.

Definition 2 Let $H \subseteq \mathbb{R}^n$ be an invex set, a function $g : H \to \mathbb{R}$ is said to be preinvex with respect to η if for all $x_1, x_2 \in H$,

$$g(x_2 + t\eta(x_1, x_2)) \leqslant t g(x_1) + (1-t)g(x_2).$$

If $\eta(x_1, x_2) = x_1 - x_2$, then the preinvex function becomes a convex functions in the classical sense. Because of the wide application of fractional calculus and Hermite-Hadamard inequalities, researchers have extended their work of Hermite-Hadamard inequalities in fractional domain. Hermite-Hadamard inequalities involving fractional integrals for different classes of functions have been established. Sarikaya et al [4] presented Hermite-Hadamard's inequalities for fractional integrals as follows.

Theorem 1 *Let $g : [x_1, x_2] \to \mathbb{R}$ with $0 \leqslant x_1$ and $g \in L[x_1, x_2]$, i.e., an integrable function defined on $[x_1, x_2]$. If g is a positive and convex function, then the following inequalities for fractional integrals hold:*

$$g\left(\frac{x_1+x_2}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_2^-} g(x_1) + J^\alpha_{x_1^+} g(x_2)\right] \leqslant \frac{g(x_1)+g(x_2)}{2},$$

where $J^\alpha_{x_1^+}$ and $J^\alpha_{x_2^-}$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{R}^+$

defined as

$$J^{\alpha}_{x_1^+} g(x) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} (x-t)^{\alpha-1} g(t)\,\mathrm{d}t, \quad x_1 < x \leqslant x_2,$$

$$J^{\alpha}_{x_2^-} g(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{x_2} (t-x)^{\alpha-1} g(t)\,\mathrm{d}t, \quad x_1 \leqslant x < x_2.$$

In the case $\alpha = 1$, the fractional integral reduces to the classical integral. Akkurt [5] proposed Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals. A function $g : H \to \mathbb{R}$ is said to be convex with respect to η for every $x_1, x_2 \in H$ and $t \in [0,1]$, where $H \subseteq \mathbb{R}^n$ is a convex set with respect to η, if

$$g(tx_1 + (1-t)x_2) \leqslant g(x_2) + t\eta(g(x_1), g(x_2)).$$

Gordji et al [6] presented η-convex function as an extension of convex function. Delavar et al [7] generalized the definition of η-convex to (η_1, η_2)-convex function defined as follows. Let $H \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : H \times H \to \mathbb{R}$. Consider $g : H \to \mathbb{R}$ and $\eta_2 : g(H) \times g(H) \to \mathbb{R}$. The function g is said to be (η_1, η_2)-convex if for every $x_1, x_2 \in H$ and $t \in [0,1]$,

$$g(x_1 + t\eta_1(x_2, x_1)) \leqslant g(x_1) + t\eta_2(g(x_2), g(x_1)).$$

An (η_1, η_2)-convex function reduces to:
(i) an η-convex function if we consider $\eta_1(x_1, x_2) = x_1 - x_2$ for every $x_1, x_2 \in H$;
(ii) a preinvex function if we consider $\eta_2(x_1, x_2) = x_1 - x_2$ for every $x_1, x_2 \in g(H)$;
(iii) a convex function for choices of (i) and (ii).

The estimates of Hermite-Hadamard-Fejér type inequalities for (η_1, η_2)-convex functions for fractional domains are not addressed up to the present. In this paper, we develop some new estimates for the left and right hand side of Hermite-Hadamard-Fejér inequalities for fractional integral of (η_1, η_2)-convex functions.

MAIN RESULTS

Throughout this section, let $\|h\|_\infty = \sup_{t\in[x_1,x_2]} |h|$, where $h : [x_1, x_2] \to \mathbb{R}$ is a continuous function.

Lemma 1 *Let $H \subseteq \mathbb{R}$ be an open invex set and $\eta_1 : H \times H \to \mathbb{R}$. Suppose there is a differentiable mapping $g : H \to \mathbb{R}$ such that $g' \in L[x_1, x_1 + \eta_1(x_2, x_1)]$ where $x_1, x_2 \in H$ with $x_1 < x_1 + \eta_1(x_2, x_1)$. If $h : [x_1, x_1 + \eta_1(x_2, x_1)] \to [0, \infty)$ is an integrable mapping, then for all $x_1, x_2 \in H$ with $\eta_1(x_2, x_1) \neq 0$ the following equality holds:*

$$\begin{aligned} &\frac{g(x_1 + \frac{1}{2}\eta_1(x_2,x_1))\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}} \Big[J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-} h(x_1) \\ &\quad + J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+} h(x_1 + \eta_1(x_2,x_1)) \Big] \\ &\quad - \frac{\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}} \Big[J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-} (gh)(x_1) \\ &\quad + J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+} (gh)(x_1 + \eta_1(x_2,x_1)) \Big] \\ &\qquad = \int_0^1 w(t) g'(x_1 + t\eta_1(x_2,x_1))\,\mathrm{d}t, \end{aligned} \tag{3}$$

where

$$w(t) = \begin{cases} \int_0^t u^{\alpha-1} h(x_1 + u\eta_1(x_2,x_1))\,\mathrm{d}u, & t \in [0, \frac{1}{2}), \\ \int_1^t (1-u)^{\alpha-1} h(x_1 + u\eta_1(x_2,x_1))\,\mathrm{d}u, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof:

$$\begin{aligned} &\int_0^1 w(t) g'(x_1 + u\eta_1(x_2,x_1))\,\mathrm{d}t \\ &= \int_0^{\frac{1}{2}} \left[\int_0^t u^{\alpha-1} h(x_1 + u\eta_1(x_2,x_1))\,\mathrm{d}u \right] \\ &\quad \times g'(x_1 + t\eta_1(x_2,x_1))\,\mathrm{d}t \\ &\quad + \int_{\frac{1}{2}}^1 \left[\int_1^t (1-u)^{\alpha-1} h(x_1 + u\eta_1(x_2,x_1))\,\mathrm{d}u \right] \\ &\quad \times g'(x_1 + t\eta_1(x_2,x_1))\,\mathrm{d}t \\ &= I_1 + I_2 \end{aligned}$$

From the first integral,

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left[\int_0^t u^{\alpha-1} h(x_1 + t\eta_1(x_2,x_1))\,\mathrm{d}u \right] g'(x_1 + t\eta_1(x_2,x_1))\,\mathrm{d}t \\ &= \frac{1}{\eta_1(x_2,x_1)} \left[\int_0^t u^{\alpha-1} h(x_1 + u\eta_1(x_2,x_1))\,\mathrm{d}u \right] g(x_1 + t\eta_1(x_2,x_1)) \Big|_0^{\frac{1}{2}} \\ &\quad - \frac{1}{\eta_1(x_2,x_1)} \int_0^{\frac{1}{2}} t^{\alpha-1} h(x_1 + t\eta_1(x_2,x_1)) g(x_1 + t\eta_1(x_2,x_1))\,\mathrm{d}t \\ &= \frac{g(x_1 + \frac{1}{2}\eta_1(x_2,x_1))}{\eta_1(x_2,x_1)} \int_0^{\frac{1}{2}} t^{\alpha-1} h(x_1 + t\eta_1(x_2,x_1))\,\mathrm{d}t \\ &\quad - \frac{1}{\eta_1(x_2,x_1)} \int_0^{\frac{1}{2}} t^{\alpha-1} g(x_1 + t\eta_1(x_2,x_1)) h(x_1 + t\eta_1(x_2,x_1))\,\mathrm{d}t. \end{aligned} \tag{4}$$

Substituting $x = x_1 + t\eta_1(x_2, x_1)$ in (4)

$$\begin{aligned}
I_1 &= \frac{g(x_1 + \frac{1}{2}\eta_1(x_2, x_1))}{(\eta_1(x_2, x_1))^{\alpha+1}} \int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)} (x - x_1)^{\alpha-1} h(x)\,\mathrm{d}x \\
&\quad - \frac{1}{(\eta_1(x_2, x_1))^{\alpha+1}} \int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)} (x - x_1)^{\alpha-1} g(x)h(x)\,\mathrm{d}x \\
&= \frac{g(x_1 + \frac{1}{2}\eta_1(x_2, x_1))\Gamma(\alpha)}{(\eta_1(x_2, x_1))^{\alpha+1}} J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-} h(x_1) \\
&\quad - \frac{\Gamma(\alpha)}{(\eta_1(x_2, x_1))^{\alpha+1}} J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-} (gh)(x_1). \qquad (5)
\end{aligned}$$

From the second integral

$$\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^{1} \left[\int_{1}^{t} (1-u)^{\alpha-1} h(x_1 + u\eta_1(x_2, x_1))\,\mathrm{d}u \right] \\
&\qquad \times g^{'}(x_1 + t\eta_1(x_2, x_1))\,\mathrm{d}t \\
&= \frac{1}{\eta_1(x_2, x_1)} \left[\int_{1}^{t} (1-u)^{\alpha-1} h(x_1 + u\eta_1(x_2, x_1))\,\mathrm{d}u \right] \\
&\quad \times g(x_1 + t\eta_1(x_2, x_1)) \Big|_{\frac{1}{2}}^{1} \\
&\quad - \frac{1}{\eta_1(x_2, x_1)} \int_{\frac{1}{2}}^{1} (1-t)^{\alpha-1} g(x_1 + t\eta_1(x_2, x_1)) \\
&\quad \times h(x_1 + t\eta_1(x_2, x_1))\,\mathrm{d}t \\
&= - \frac{g(x_1 + \frac{1}{2}\eta_1(x_2, x_1))}{\eta_1(x_2, x_1)} \\
&\quad \times \int_{1}^{\frac{1}{2}} (1-t)^{\alpha-1} h(x_1 + t\eta_1(x_2, x_1))\,\mathrm{d}t \\
&\quad - \frac{1}{\eta_1(x_2, x_1)} \int_{\frac{1}{2}}^{1} (1-t)^{\alpha-1} g(x_1 + t\eta_1(x_2, x_1)) \\
&\qquad \times h(x_1 + t\eta_1(x_2, x_1)\,\mathrm{d}t. \qquad (6)
\end{aligned}$$

Substituting $x = x_1 + t\eta_1(x_2, x_1)$ in (6)

$$\begin{aligned}
I_2 &= - \frac{g(x_1 + \frac{1}{2}\eta_1(x_2, x_1))}{(\eta_1(x_2, x_1))^{\alpha+1}} \int_{x_1+\eta_1(x_2,x_1)}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)} (x_1 + \eta_1(x_2, x_1) - x)^{\alpha-1} h(x)\,\mathrm{d}x \\
&\quad - \frac{1}{(\eta_1(x_2, x_1))^{\alpha+1}} \int_{x_1+\frac{1}{2}\eta_1(x_2,x_1)}^{x_1+\eta_1(x_2,x_1)} (x_1 + \eta_1(x_2, x_1) - x)^{\alpha-1} g(x)h(x)\,\mathrm{d}x \\
&= \frac{g(x_1 + \frac{1}{2}\eta_1(x_2, x_1))\Gamma(\alpha)}{(\eta_1(x_2, x_1))^{\alpha+1}} J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+} h(x_1 + \eta_1(x_2, x_1)) \\
&\quad - \frac{\Gamma(\alpha)}{(\eta_1(x_2, x_1))^{2}} J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+} (gh)(x_1 + \eta_1(x_2, x_1)). \qquad (7)
\end{aligned}$$

On combining (5) and (7), we get the result. □

Remark 1 If we take $\eta_1(x_2, x_1) = x_2 - x_1$ then Lemma 1 reduces to Lemma 4 in [8].

Lemma 2 *If $h : [x_1, x_1 + \eta_1(x_2, x_1)] \to \mathbb{R}$ is an integrable function and symmetric about $x_1 + \frac{1}{2}\eta_1(x_2, x_1)$ with $x_1 < x_1 + \eta_1(x_2, x_1)$, then for $\alpha > 0$*

$$\begin{aligned}
J^{\alpha}_{x_1^+} h(x_1 + \eta_1(x_2, x_1)) &= J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} h(x_1) \\
&= \tfrac{1}{2} \left[J^{\alpha}_{x_1^+} h(x_1 + \eta_1(x_2, x_1)) + J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} h(x_1) \right]. \qquad (8)
\end{aligned}$$

Proof: Since h is symmetric about $x_1 + \frac{1}{2}\eta_1(x_2, x_1)$, we have $h(2x_1 + \eta_1(x_2, x_1) - x) = h(x)$ for all $x \in [x_1, x_1 + \eta_1(x_2, x_1)]$. Taking $2x_1 + \eta_1(x_2, x_1) - t = x$,

$$\begin{aligned}
&J^{\alpha}_{x_1^+} h(x_1 + \eta_1(x_2, x_1)) \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_1+\eta_1(x_2,x_1)} [(x_1 + \eta_1(x_2, x_1) - t]^{\alpha-1} h(t)\,\mathrm{d}t \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_1+\eta_1(x_2,x_1)} (x - x_1)^{\alpha-1} h(2x_1 + \eta_1(x_2, x_1) - x) dx \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_1+\eta_1(x_2,x_1)} (x - x_1)^{\alpha-1} h(x)\,\mathrm{d}x \\
&\quad = J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} h(x_1).
\end{aligned}$$

□

Remark 2 If we take $\eta_1(x_2, x_1) = x_2 - x_1$ then Lemma 2 reduces to Lemma 2.1 in [9].

Lemma 3 *Let $H \subseteq \mathbb{R}$ be an open invex set and $\eta_1 : H \times H \to \mathbb{R}$. Suppose there is a differentiable mapping $g : H \to \mathbb{R}$ such that $g' \in L[x_1, x_1 + \eta_1(x_2, x_1)]$ where $x_1, x_2 \in H$ with $x_1 < x_1 + \eta_1(x_2, x_1)$. If $h : [x_1, x_1 + \eta_1(x_2, x_1)] \to [0, \infty)$ is integrable, then for all $x_1, x_2 \in H$ with $\eta_1(x_2, x_1) \neq 0$ the following equality holds:*

$$\begin{aligned}
&\frac{g(x_1) + g(x_1 + \eta_1(x_2, x_1))}{2} \\
&\quad \times \left(J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} h(x_1) + J^{\alpha}_{x_1^+} h(x_1 + \eta_1(x_2, x_1)) \right) \\
&\quad - \left(J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} (gh)(x_1) + J^{\alpha}_{x_1^+} (gh)(x_1 + \eta_1(x_2, x_1)) \right) \\
&\qquad = \frac{(\eta_1(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha)} \int_{0}^{1} w(t) g'(x_1 + t\eta_1(x_2, x_1))\,\mathrm{d}t, \qquad (9)
\end{aligned}$$

where

$$\begin{aligned}
w(t) &= \int_{0}^{t} (1-u)^{\alpha-1} h(x_1 + u\eta_1(x_2, x_1))\,\mathrm{d}u \\
&\quad - \int_{t}^{1} u^{\alpha-1} h(x_1 + u\eta_1(x_2, x_1))\,\mathrm{d}u, \quad \forall t \in [0, 1].
\end{aligned}$$

Proof: Consider

$$\int_0^1 w(t)g'(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t = \int_0^1\left[\int_0^t (1-u)^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u + \int_1^t u^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u\right]g'(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t$$
$$= \int_0^1\left[\int_0^t (1-u)^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u\right] \times g'(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t + \int_0^1\left[\int_1^t u^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u\right] \times g'(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t =: I_1+I_2. \quad (10)$$

From the first integral

$$I_1 = \int_0^1\left[\int_0^t (1-u)^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u\right] \times g'(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t$$
$$= \frac{1}{\eta_1(x_2,x_1)}\left[\int_0^t (1-u)^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u\right] \times g(x_1+t\eta_1(x_2,x_1))\Big|_0^1 - \frac{1}{\eta_1(x_2,x_1)}\int_0^1 (1-t)^{\alpha-1}h(x_1+t\eta_1(x_2,x_1)) \times g(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t$$
$$= \frac{g(x_1+\eta_1(x_2,x_1))}{\eta_1(x_2,x_1)}\int_0^1 (1-t)^{\alpha-1}h(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t - \frac{1}{\eta_1(x_2,x_1)}\int_0^1 (1-t)^{\alpha-1}g(x_1+t\eta_1(x_2,x_1)) \times h(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t. \quad (11)$$

Substituting $x = x_1+t\eta_1(x_2,x_1)$ in (11),

$$I_1 = \frac{g(x_1+\eta_1(x_2,x_1))}{(\eta_1(x_2,x_1))^{\alpha+1}}\int_{x_1}^{x_1+\eta_1(x_2,x_1)} ((x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\,\mathrm{d}x - \frac{1}{(\eta_1(x_2,x_1))^{\alpha+1}}\int_{x_1}^{x_1+\eta_1(x_2,x_1)} (x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}g(x)h(x)\,\mathrm{d}x$$
$$= \frac{g(x_1+\eta_1(x_2,x_1))\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}}J^{\alpha}_{x_1^+}h(x_1+\eta_1(x_2,x_1)) - \frac{\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}}J^{\alpha}_{x_1^+}(gh)(x_1+\eta_1(x_2,x_1)). \quad (12)$$

Now, for the second integral:

$$I_2 = \int_0^1\left[\int_1^t u^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u\right] \times g'(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t$$
$$= \frac{1}{\eta_1(x_2,x_1)}\left[\int_1^t u^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u\right] \times g(x_1+t\eta_1(x_2,x_1))\Big|_0^1 - \frac{1}{\eta_1(x_2,x_1)}\int_0^1 t^{\alpha-1}g(x_1+t\eta_1(x_2,x_1)) \times h(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t$$
$$= \frac{g(x_1)}{\eta_1(x_2,x_1)}\int_0^1 t^{\alpha-1}h(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t - \frac{1}{\eta_1(x_2,x_1)}\int_0^1 t^{\alpha-1}g(x_1+t\eta_1(x_2,x_1)) \times h(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t. \quad (13)$$

Substituting $x = x_1+t\eta_1(x_2,x_1)$ in (13),

$$I_2 = \frac{g(x_1)}{(\eta_1(x_2,x_1))^{\alpha+1}}\int_{x_1}^{x_1+\eta_1(x_2,x_1)} (x-x_1)^{\alpha-1}h(x)\,\mathrm{d}x - \frac{1}{(\eta_1(x_2,x_1))^{\alpha+1}}\int_{x_1}^{x_1+\eta_1(x_2,x_1)} (x-x_1)^{\alpha-1}g(x)h(x)\,\mathrm{d}x$$
$$= \frac{g(x_1)\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}}J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-}h(x_1) - \frac{\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}}J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-}(gh)(x_1). \quad (14)$$

Combining (8), (12), and (14) to get the result. □

Remark 3 If we take $\eta_1(x_2,x_1) = x_2 - x_1$ then Lemma 3 reduces to Lemma 2.4 in [9].

Theorem 2 *Let $H \subseteq \mathbb{R}$ be an open invex set and $\eta_1 : H\times H \to \mathbb{R}$. Suppose there is a differentiable mapping $g : H \to \mathbb{R}$ such that $g' \in L[x_1, x_1+\eta_1(x_2,x_1)]$ where $x_1, x_2 \in H$ with $x_1 < x_1+\eta_1(x_2,x_1)$. If $h : [x_1, x_1+\eta_1(x_2,x_1)] \to [0,\infty)$ is integrable and symmetric to $x_1+\frac{1}{2}\eta_1(x_2,x_1)$ and $|g'|$ is an (η_1,η_2)-convex function on H, then for all $x_1, x_2 \in H$ with*

$\eta_1(x_2,x_1)\neq 0$ *the following inequality holds:*

$$\Big|\big(x_1+\tfrac{1}{2}\eta_1(x_2,x_1)\big)\Big[J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-}h(x_1)+J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+}h(x_1+\eta_1(x_2,x_1))\Big]-\Big[J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-}(gh)(x_1)+J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+}(gh)(x_1+\eta_1(x_2,x_1))\Big]\Big|\leqslant\frac{\|h\|_\infty}{\Gamma(\alpha+2)}\Big[2|g'(x_1)|+\eta_2(|g'(x_2)|,|g'(x_1)|)\Big]\times\frac{(\eta_1(x_2,x_1))^{\alpha+1}}{2^{\alpha+1}}.\quad(15)$$

Proof: Applying modulus on both sides of (3),

$$\Big|\frac{g(x_1+\frac{1}{2}\eta_1(x_2,x_1))\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}}\Big[J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-}h(x_1)+J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+}h(x_1+\eta_1(x_2,x_1))\Big]-\frac{\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}}\Big[J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-}(gh)(x_1)+J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+}(gh)(x_1+\eta_1(x_2,x_1))\Big]\Big|$$
$$=\Big|\int_0^{1/2}\Big[\int_0^t u^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u\Big]\times g'(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t+\int_{1/2}^1\Big[-\int_t^1((1-u)^{\alpha-1}h(x_1+u\eta_1(x_2,x_1))\,\mathrm{d}u)\Big]\times g'(x_1+t\eta_1(x_2,x_1))\,\mathrm{d}t\Big|.\quad(16)$$

From the (η_1,η_2)-convexity of $|g'|$ on H, we have

$$\Big|\frac{g(x_1+\frac{1}{2}\eta_1(x_2,x_1))\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}}\Big[J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-}h(x_1)+J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+}h(x_1+\eta_1(x_2,x_1))\Big]-\frac{\Gamma(\alpha)}{(\eta_1(x_2,x_1))^{\alpha+1}}\Big[J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^-}(gh)(x_1)+J^{\alpha}_{(x_1+\frac{1}{2}\eta_1(x_2,x_1))^+}(gh)(x_1+\eta_1(x_2,x_1))\Big]\Big|$$
$$\leqslant\int_0^{1/2}\Big[\int_0^t u^{\alpha-1}|h(x_1+u\eta_1(x_2,x_1))|\,\mathrm{d}u\Big]\times\Big[|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\Big]\mathrm{d}t+\int_{1/2}^1\Big[\int_t^1((1-u)^{\alpha-1}|h(x_1+u\eta_1(x_2,x_1))|\,\mathrm{d}u\Big]\times\Big[|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\Big]\mathrm{d}t=I_1+I_2.\quad(17)$$

Changing the order of integration in I_1 in (17),

$$I_1=\int_0^{1/2}\Big[\int_0^t u^{\alpha-1}|h(x_1+u\eta_1(x_2,x_1))|\,\mathrm{d}u\Big]\times\big(|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\big)\,\mathrm{d}t$$
$$=\int_0^{1/2}u^{\alpha-1}|h(x_1+u\eta_1(x_2,x_1))|\times\int_u^{1/2}\big(|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\big)\,\mathrm{d}t\,\mathrm{d}u$$
$$=\int_0^{1/2}u^{\alpha-1}|h(x_1+u\eta_1(x_2,x_1))|\Big(|g'(x_1)|(\tfrac{1}{2}-u)+\eta_2(|g'(x_2)|,|g'(x_1)|)(\tfrac{1}{8}-\tfrac{u^2}{2})\Big)\,\mathrm{d}u.$$

Using $x=x_1+u\eta_1(x_2,x_1)$ for $u\in[0,1]$,

$$I_1=\frac{|g'(x_1)|}{\eta_1(x_2,x_1)}\int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)}(\tfrac{1}{2}-\tfrac{x-x_1}{\eta_1(x_2,x_1)})(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^{\alpha-1}|h(x)|\,\mathrm{d}x+\frac{\eta_2(|g'(x_2)|,|g'(x_1)|)}{\eta_1(x_2,x_1)}\times\int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)}(\tfrac{1}{8}-\tfrac{1}{2}(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^2)(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^{\alpha-1}|h(x)|\,\mathrm{d}x.$$

Let $\|h\|_\infty=\sup_{t\in[x_1,x_2]}|h(t)|$,

$$I_1=\frac{|g'(x_1)|}{\eta_1(x_2,x_1)}\|h\|_\infty\int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)}(\tfrac{1}{2}-\tfrac{x-x_1}{\eta_1(x_2,x_1)})(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^{\alpha-1}\,\mathrm{d}x+\frac{\eta_2(|g'(x_2)|,|g'(x_1)|)}{\eta_1(x_2,x_1)}\|h\|_\infty\times\int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)}(\tfrac{1}{8}-\tfrac{1}{2}(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^2)(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^{\alpha-1}\,\mathrm{d}x.\quad(18)$$

Similarly, using that h is symmetric with respect to $x_1+\frac{1}{2}\eta_1(x_2,x_1)$ for I_2, we have

$$I_2=\int_{1/2}^1\Big[\int_t^1(1-u)^{\alpha-1}|h(x_1+(1-u)\eta_1(x_2,x_1))|\,\mathrm{d}u\Big]\times\Big[|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\Big]\mathrm{d}t$$
$$=\int_{1/2}^1(1-u)^{\alpha-1}|h(x_1+(1-u)\eta_1(x_2,x_1)|)\times\int_{1/2}^u\big(|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\big)\,\mathrm{d}t\,\mathrm{d}u$$
$$=\int_{1/2}^1(1-u)^{\alpha-1}h(x_1+|(1-u)\eta_1(x_2,x_1)|)\times\Big[|g'(x_1)|(u-\tfrac{1}{2})+\eta_2(|g'(x_2)|,|g'(x_1)|)(\tfrac{u^2}{2}-\tfrac{1}{8})\Big]\mathrm{d}u.$$

By setting $x = x_1 + (1-u)\eta_1(x_2, x_1)$,

$$I_2 = \frac{|g'(x_1)|}{\eta_1(x_2,x_1)} \int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)} (\tfrac{1}{2} - \tfrac{x-x_1}{\eta_1(x_2,x_1)})(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^{\alpha-1} |h(x)|\,\mathrm{d}x + \frac{\eta_2(|g'(x_2)|, |g'(x_1)|)}{\eta_1(x_2,x_1)} \times \int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)} (\tfrac{1}{2}(1-\tfrac{x-x_1}{\eta_1(x_2,x_1)})^2 - \tfrac{1}{8})(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^{\alpha-1} |h(x)|\,\mathrm{d}x,$$

and letting $\|h\|_\infty = \sup_{t\in[x_1,x_2]} |h(t)|$,

$$I_2 = \frac{|g'(x_1)|}{\eta_1(x_2,x_1)} \|h\|_\infty \int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)} (\tfrac{1}{2} - \tfrac{x-x_1}{\eta_1(x_2,x_1)})(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^{\alpha-1}\,\mathrm{d}x + \frac{\eta_2(|g'(x_2)|, |g'(x_1)|)}{\eta_1(x_2,x_1)} \|h\|_\infty \times \int_{x_1}^{x_1+\frac{1}{2}\eta_1(x_2,x_1)} (\tfrac{1}{2}(1-\tfrac{x-x_1}{\eta_1(x_2,x_1)})^2 - \tfrac{1}{8})(\tfrac{x-x_1}{\eta_1(x_2,x_1)})^{\alpha-1}\,\mathrm{d}x. \quad (19)$$

Using (18) and (19) in (17), we get the result. □

Remark 4 If we take $\eta_1(x_2, x_1) = x_2 - x_1$, we recapture the error bound of the left hand side of Hermite-Hadamard-Fejér inequalities for convex functions in fractional domain given in [8] as a special case.

Theorem 3 *Let $H \subseteq \mathbb{R}$ be an open invex set and $\eta_1 : H\times H \to \mathbb{R}$. Suppose there is a differentiable mapping $g : H \to \mathbb{R}$ such that $g' \in L[x_1, x_1 + \eta_1(x_2, x_1)]$ where $x_1, x_2 \in H$ with $x_1 < x_1 + \eta_1(x_2, x_1)$. If $h : [x_1, x_1 + \eta_1(x_2, x_1)] \to [0,\infty)$ is integrable and symmetric to $x_1 + \frac{1}{2}\eta_1(x_2, x_1)$ and $|g'|$ is an (η_1, η_2)-convex function on H, then for all $x_1, x_2 \in H$ with $\eta_1(x_2, x_1) \neq 0$ the following inequality holds:*

$$\Big| \frac{g(x_1) + g(x_1 + \eta_1(x_2, x_1))}{2} \times \Big[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} h(x_1) + J^{\alpha}_{x_1^+} h(x_1 + \eta_1(x_2, x_1)) \Big] - \Big[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} (gh)(x_1) + J^{\alpha}_{x_1^+} (gh)(x_1 + \eta_1(x_2, x_1)) \Big] \Big| \leqslant \frac{\|h\|_\infty (\eta_1(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha+2)} (1 - \frac{1}{2^\alpha}) \times \Big[2|g'(x_1)| + \eta_2(|g'(x_2)|, |g'(x_1)|) \Big]. \quad (20)$$

Proof: Applying the modulus on both sides of (9),

$$\Big| \frac{g(x_1) + g(x_1 + \eta_1(x_2, x_1))}{2} \times \Big[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} h(x_1) + J^{\alpha}_{x_1^+} h(x_1 + \eta_1(x_2, x_1)) \Big] - \Big[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} (gh)(x_1) + J^{\alpha}_{x_1^+} (gh)(x_1 + \eta_1(x_2, x_1)) \Big] \Big| = \Big| \frac{(\eta_1(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 \Big[\int_0^t (1-u)^{\alpha-1} h(x_1 + u\eta_1(x_2, x_1))\,\mathrm{d}u - \int_t^1 u^{\alpha-1} h(x_1 + u\eta_1(x_2, x_1))\,\mathrm{d}u \Big] \times g'(x_1 + t\eta_1(x_2, x_1))\,\mathrm{d}t \Big| \quad (21)$$

$$\leqslant \frac{(\eta_1(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 \Big| \int_0^t (1-u)^{\alpha-1} h(x_1 + u\eta_1(x_2, x_1))\,\mathrm{d}u - \int_t^1 u^{\alpha-1} h(x_1 + u\eta_1(x_2, x_1))\,\mathrm{d}u \Big| \times |g'(x_1 + t\eta_1(x_2, x_1))|\,\mathrm{d}t. \quad (22)$$

By the change of variable $x = x_1 + u\eta_1(x_2, x_1)$,

$$\Big| \frac{g(x_1) + g(x_1 + \eta_1(x_2, x_1))}{2} \times \Big[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} h(x_1) + J^{\alpha}_{x_1^+} h(x_1 + \eta_1(x_2, x_1)) \Big] - \Big[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-} (gh)(x_1) + J^{\alpha}_{x_1^+} (gh)(x_1 + \eta_1(x_2, x_1)) \Big] \Big| = \frac{(\eta_1(x_2, x_1))}{\Gamma(\alpha)} \int_0^1 \Big| \int_{x_1}^{x_1+t\eta_1(x_2,x_1)} (x_1 + \eta_1(x_2, x_1) - x)^{\alpha-1} h(x)\,\mathrm{d}x - \int_{x_1+t\eta_1(x_2,x_1)}^{x_1+\eta_1(x_2,x_1)} (x - x_1)^{\alpha-1} h(x)\,\mathrm{d}x \Big| |g'(x_1 + t\eta_1(x_2, x_1))|\,\mathrm{d}t. \quad (23)$$

Since $h : [x_1, x_1 + \eta_1(x_2, x_1)] \to \mathbb{R}$ is integrable and symmetric about $x_1 + \frac{1}{2}\eta_1(x_2, x_1)$, then

$$\int_{x_1+t\eta_1(x_2,x_1)}^{x_1+\eta_1(x_2,x_1)} (x - x_1)^{\alpha-1} h(x)\,\mathrm{d}x = \int_{x_1}^{x_1+(1-t)\eta_1(x_2,x_1)} (x_1 + \eta_1(x_2, x_1) - x)^{\alpha-1} h(2x_1 + \eta_1(x_2, x_1) - x)\,\mathrm{d}x = \int_{x_1}^{x_1+(1-t)\eta_1(x_2,x_1)} (x_1 + \eta_1(x_2, x_1) - x)^{\alpha-1} h(x)\,\mathrm{d}x. \quad (24)$$

Using (24), we have

$$\left|\int_{x_1}^{x_1+t\eta_1(x_2,x_1)}(x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\,\mathrm{d}x-\int_{x_1+t\eta_1(x_2,x_1)}^{x_1+\eta_1(x_2,x_1)}(x-x_1)^{\alpha-1}h(x)\,\mathrm{d}x\right|$$
$$=\left|\int_{x_1+(1-t)\eta_1(x_2,x_1)}^{x_1+t\eta_1(x_2,x_1)}(x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\,\mathrm{d}x\right|$$
$$\leqslant\begin{cases}\displaystyle\int_{x_1+t\eta_1(x_2,x_1)}^{x_1+(1-t)\eta_1(x_2,x_1)}\left|(x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\right|\,\mathrm{d}x,\ t\in[0,\tfrac{1}{2}],\\ \displaystyle\int_{x_1+(1-t)\eta_1(x_2,x_1)}^{x_1+t\eta_1(x_2,x_1)}\left|(x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\right|\,\mathrm{d}x,\ t\in[\tfrac{1}{2},1].\end{cases}\tag{25}$$

Using (25) in (23), we get

$$\Bigg|\frac{g(x_1)+g(x_1+\eta_1(x_2,x_1))}{2}\times\left[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-}h(x_1)+J^{\alpha}_{x_1^+}h(x_1+\eta_1(x_2,x_1))\right]-\left[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-}(gh)(x_1)+J^{\alpha}_{x_1^+}(gh)(x_1+\eta_1(x_2,x_1))\right]\Bigg|$$
$$\leqslant\frac{(\eta_1(x_2,x_1))}{\Gamma(\alpha)}\left\{\int_0^{\frac{1}{2}}\left[\int_{x_1+t\eta_1(x_2,x_1)}^{x_1+(1-t)\eta_1(x_2,x_1)}\left|(x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\right|\,\mathrm{d}x\right]\times|g'(x_1+t\eta_1(x_2,x_1))|\,\mathrm{d}t\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\left[\int_{x_1+(1-t)\eta_1(x_2,x_1)}^{x_1+t\eta_1(x_2,x_1)}\left|x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\right|\,\mathrm{d}x\right]\times|g'(x_1+t\eta_1(x_2,x_1))|\,\mathrm{d}t\right\}.$$

From the (η_1,η_2)-convexity of $|g'|$ on H, we have

$$\Bigg|\frac{g(x_1)+g(x_1+\eta_1(x_2,x_1))}{2}\times\left[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-}h(x_1)+J^{\alpha}_{x_1^+}h(x_1+\eta_1(x_2,x_1))\right]-\left[J^{\alpha}_{(x_1+\eta_1(x_2,x_1))^-}(gh)(x_1)+J^{\alpha}_{x_1^+}(gh)(x_1+\eta_1(x_2,x_1))\right]\Bigg|$$
$$\leqslant\frac{(\eta_1(x_2,x_1))}{\Gamma(\alpha)}\left\{\int_0^{\frac{1}{2}}\int_{x_1+t\eta_1(x_2,x_1)}^{x_1+(1-t)\eta_1(x_2,x_1)}\left|(x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\right|\,\mathrm{d}x\times\left[|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\right]\mathrm{d}t\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\int_{x_1+(1-t)\eta_1(x_2,x_1)}^{x_1+t\eta_1(x_2,x_1)}\left|x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}h(x)\right|\,\mathrm{d}x\times\left[|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\right]\mathrm{d}t\right\}$$
$$:=\frac{(\eta_1(x_2,x_1))}{\Gamma(\alpha)}[I_1+I_2].\tag{26}$$

Now

$$I_1=\int_0^{\frac{1}{2}}\int_{x_1+t\eta_1(x_2,x_1)}^{x_1+(1-t)\eta_1(x_2,x_1)}(x_1+\eta_1(x_2,x_1)-x)^{\alpha-1}\,\mathrm{d}x\times\left[|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\right]\mathrm{d}t$$
$$=\int_0^{\frac{1}{2}}\Big[|g'(x_1)|((1-t)^{\alpha}-t^{\alpha})-\eta_2(|g'(x_2)|,|g'(x_1)|)\left(t(1-t)^{\alpha}-t^{\alpha+1}\right)\Big]\mathrm{d}t,\tag{27}$$

$$I_2=\int_{\frac{1}{2}}^{1}\int_{x_1+(1-t)\eta_1(x_2,x_1)}^{x_1+t\eta_1(x_2,x_1)}(x_1+\eta_1(x_2,x_1)-x)^{\alpha1}\,\mathrm{d}x\times\left[|g'(x_1)|+t\eta_2(|g'(x_2)|,|g'(x_1)|)\right]\mathrm{d}t$$
$$=\int_{\frac{1}{2}}^{1}\Big[|g'(x_1)|(t^{\alpha}-(1-t))^{\alpha})-\eta_2(|g'(x_2)|,|g'(x_1)|)\left(t^{\alpha+1}-t(1-t)^{\alpha}\right)\Big]\mathrm{d}t.\tag{28}$$

Utilizing (27) and (28) in (26), and taking $\|h\|_{\infty}=\sup_{t\in[x_1,x_2]}|h(t)|$, we get the result. □

Remark 5 If $\eta_1(x_2,x_1)=x_2-x_1$ we recapture the error bound of the right hand side of Hermite-Hadamard-Fejér inequalities for convex functions in fractional domain given in [9] as a special case.

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