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Eigenvalue problems for nonlinear conformable fractional differential equations with multi-point boundary conditions

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ABSTRACT: In this paper, we investigate the eigenvalue problems for a class of nonlinear conformable fractional differential equations with multi-point boundary conditions by using the fixed point index theory and a variant of Krein-Rutman theorem. Finally, two examples are presented to show the effectiveness of our main result.

KEYWORDS: positive solution, multi-point boundary value problem, fixed point index theory

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INTRODUCTION

Since fractional differential equations can describe natural phenomena better than integer order differential equations, they have been extensively applied in various fields such as physics, chemistry, and engineering[1–3]. Fractional differential equations have attracted extensive attention, there exist a large number of published papers on fractional differential equations, we refer the readers to Refs. 4–6 and the references quoted therein. By using the properties of Green's functions and the Krasnoselskii-Zabreiko fixed point theorem, Yang and Qin[7] investigated the existence of positive solutions for a class of nonlinear Hadamard fractional differential equations with integral boundary conditions. Zhou[8,9] established sufficient conditions for the existence of globally attractive solutions for the Cauchy problems of fractional differential (evolution) equations in abstract space and in the cases that the semigroup is compact as well as non-compact, respectively. By Avery-Peterson's fixed point theorem, Guo et al[10] considered the existence of multiple positive solutions to a singular Caputo's fractional differential equations with infinite-point boundary conditions. Based on the direct calculation of quantum derivatives of the Lyapunov-Kravsovskii functionals and linear matrix inequality method, Yang et al[11] obtained some sufficient conditions on delay-dependent and delay-independent asymptotical stability of Riemann-Liouville q-fractional neutral systems with mixed delays. Jiang[12] obtained the eigenvalue interval for multi-point boundary value problems of fractional differential equations. By using some properties of the classical Mittag-Leffler functions, Nieto[13] presented two new maximum principles for a linear fractional differential equation with initial or periodic boundary conditions. Based on Krasnosel'skii and Schauder fixed point theorems and monotone iterative technique, the authors[14] studied existence and uniqueness of periodic solutions for a particular class of nonlinear fractional differential equations admitting its right-hand side with certain singularities. By applying the Green's function and Guo-Krasnoselskii fixed point theorems, Yang[15,16] investigated the positive solutions for the coupled integral and four-point coupled boundary value problem of nonlinear semipositone Hadamard fractional differential equations, respectively. Zhou et al[17] considered the time-fractional reaction-diffusion equation with nonlocal boundary condition and established the existence and uniqueness of a weak solution of the proposed model using the Faedo-Galerkin method and compactness arguments.

In the past years, the concepts of fractional derivatives and integrals are given by the following considerable operators: the Riemann-Liouville fractional derivative and integral, Caputo fractional derivative and integral, Hadamard fractional derivative and integral as well as other fractional derivative and integral. Recently, based on the basic limit definition of the derivative, Khalil et al[18] gave a new well-behaved definition called the conformable fractional derivative. Later, Abdeljawad[19] obtained chain rule, exponential functions, Gronwall type inequality, integration by parts, Taylor power series expansions as well as Laplace transforms of the conformable fractional derivative and integral. Conformable fractional differential equations are getting an increasing interest[20–22]. According to the Banach, Schaefer, and Rothe fixed point theorems and degree theory, Asawasamrit et al[23] studied the existence and uniqueness of solutions for impulsive multi-orders Caputo-Hadamard fractional differential equations equipped with boundary and integral conditions. By using the Picone identity of conformable fractional differential equations on arbitrary time scales, Zhang and Sun[24] obtained the Sturm-Picone comparison theorem of conformable fractional differential equations on time scales. Karayer et al[25] introduced conformable fractional Nikiforov-Uvarov method by means of conformable fractional derivative and gave exact eigenstate solutions of Schrödinger equation for certain potentials in quantum mechanics. By using the equivalence transformation and the associated Riccati techniques, Tariboon and Ntouyas[26] investigated oscillation for the solutions of impulsive conformable fractional differential equations. By using the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions, Liu et al[27] considered a coupled system of nonlinear conformable fractional differential equations. Yang et al[28] obtained the exact solutions for nonlinear local fractional FitzHugh-Nagumo and Newell-Whitehead equations by applying the traveling-wave transformation.

Motivated by the works mentioned above, we consider the following eigenvalue problem for nonlinear conformable fractional differential equation with multi-point boundary condition:

$$\begin{aligned} &x^{(\alpha)}(t)-\Theta x(t)=\lambda f(t,x(t)),\\ &x(a)=\sum_{j=1}^{n}\beta_j x(\xi_j),\ t\in[a,b],\ \alpha\in(0,1], \end{aligned} \tag{1}$$

where $x^{(\alpha)}$ is a conformable fractional derivative with $0<\alpha\leqslant 1$, $\Theta\geqslant 0$, $a<\xi_1<\xi_2<\cdots<\xi_n\leqslant b$, and the nonlinear term f is a continuous function. The main purpose of this paper is to obtain the eigenvalue interval of a positive solution for the multi-point boundary value problem by using the fixed point index theory and a variant of Krein-Rutman theorem.

PRELIMINARIES

For the convenience of the readers, we present some necessary definitions and lemmas from conformable fractional calculus theory in this section.

Definition 1 [Refs. 18, 19] The conformable fractional derivative starting from a point a of a function $f:[a,\infty)\to\mathbb{R}$ of order $\alpha\in(0,1]$ is defined by

$$T_a^{(\alpha)}f(t)=\lim_{\varepsilon\to 0}\frac{h(t+\varepsilon(t-a)^{1-\alpha})-h(t)}{\varepsilon},$$

for all $t>a$, $\alpha\in(0,1]$, provided that the limit exists. If $\lim_{t\to a^+}T_a^{(\alpha)}f(t)$ exists, then by definition, one has $T_a^{(\alpha)}f(a)=\lim_{t\to a^+}T_a^{(\alpha)}f(t)$.

Definition 2 [Refs. 18, 19] The conformable fractional integral starting from a point a of a function $f:[a,\infty)\to\mathbb{R}$ of order $\alpha\in(0,1]$ is defined by

$$I_a^{\alpha}f(t)=\int_a^t (s-a)^{\alpha}f(s)\,\mathrm{d}s.$$

Next, we write $f^{(\alpha)}$ for $T_a^{(\alpha)}f(t)$ to denote α-order conformable fractional derivative of $f(t)$.

Lemma 1 (Ref. 27) *Let $\sigma\in C([a,b],\mathbb{R})$ and $\Theta\in\mathbb{R}$. The linear initial value problem*

$$\begin{aligned} &x^{(\alpha)}(t)-\Theta x(t)=\sigma(t),\\ &x(a)=x_0^*,\ t\in[a,b],\ \alpha\in(0,1], \end{aligned}$$

has a unique solution

$$\begin{aligned} x(t)=&x_0^*\exp(\omega(t^{\alpha}-a^{\alpha}))\\ &+\int_a^t s^{\alpha-1}\exp(\omega(t^{\alpha}-s^{\alpha}))\sigma(s)\,\mathrm{d}s,\ \omega=\frac{\Theta}{\alpha}. \end{aligned}$$

Now we present a variant of Krein-Rutman theorem, which plays an important role in proving the main results.

Lemma 2 (Ref. 29) *Let K be a reproducing cone in a real Banach space X, $L:X\to X$ be a compact linear operator with $L(K)\subseteq K$, and $r(L)$ be the spectral radius of L. If $r(L)>0$, then there exists $\varphi\in K\backslash\{0\}$ such that $L\varphi=r(L)\varphi$.*

Lemma 3 (Ref. 30) *Let X be a Banach space, P be a cone in X and $\Omega(P)$ be a bounded open subset in P. Suppose that $A\colon\overline{\Omega(P)}\to P$ is a completely continuous operator. Then the following results hold:*

(I) *If there exists* $u_0 \in P\backslash\{0\}$ *such that* $u \neq Au + \lambda u_0$ *for all* $u \in \partial\Omega(P)$, $\lambda \geqslant 0$, *then the fixed point index* $i(A,\Omega(P),P) = 0$.

(II) *If* $0 \in \partial\Omega(P)$ *and* $Au \neq \lambda u$ *for all* $u \in \partial\Omega(P)$, $\lambda \geqslant 1$, *then the fixed point index* $i(A,\Omega(P),P) = 1$.

For convenience, we assume that the following conditions hold.

(H_1) $\Theta \geqslant 0$, $f \in C([a,b]\times\mathbb{R},\mathbb{R}^+)$, $a < \xi_1 < \xi_2 < \cdots < \xi_n \leqslant b$.

(H_2) $\beta_j > 0$, $j = 1,2,\dots,n$, $\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha)) < 1$, where $\omega = \Theta/\alpha$.

At the same time, we make the following definitions

$$f_0 = \liminf_{x\to 0^+} \inf_{t\in[a,b]} \frac{f(t,x)}{x},$$

$$f^\infty = \limsup_{x\to+\infty} \sup_{t\in[a,b]} \frac{f(t,x)}{x},$$

$$\mu_0 = \frac{1}{r(L)}.$$

MAIN RESULTS

Lemma 4 *Let* $\sigma \in C([a,b],\mathbb{R})$ *and* $\Theta \in \mathbb{R}$. *The following multi-point boundary value problem*

$$x^{(\alpha)}(t) - \Theta x(t) = \sigma(t),$$

$$x(a) = \sum_{j=1}^n \beta_j x(\xi_j),\ t \in [a,b],\ \alpha \in (0,1],$$

has a unique solution

$$x(t) = \frac{\exp(\omega(t^\alpha - a^\alpha))}{1 - \sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \times \sum_{j=1}^n \beta_j \int_a^{\xi_j} s^{\alpha-1} \exp(\omega(\xi_j^\alpha - s^\alpha))\sigma(s)\,\mathrm{d}s + \int_a^t s^{\alpha-1} \exp(\omega(t^\alpha - s^\alpha))\sigma(s)\,\mathrm{d}s.$$

Proof: Following from Lemma 1, we obtain the following integral equation

$$x(t) = \sum_{j=1}^n \beta_j x(\xi_j) \exp(\omega(t^\alpha - a^\alpha)) + \int_a^t s^{\alpha-1} \exp(\omega(t^\alpha - s^\alpha))\sigma(s)\,\mathrm{d}s. \quad (2)$$

Letting $t = \xi_j$ in (2), we get

$$\sum_{i=1}^n \beta_i x(\xi_i) = \sum_{i=1}^n \beta_i \left(\sum_{j=1}^n \beta_j x(\xi_j) \exp(\omega(\xi_i^\alpha - a^\alpha)) + \int_a^{\xi_i} s^{\alpha-1} \exp(\omega(\xi_i^\alpha - s^\alpha))\sigma(s)\,\mathrm{d}s \right). \quad (3)$$

Solving (3) with respect to $\sum_{j=1}^n \beta_j x(\xi_j)$, we have

$$\sum_{j=1}^n \beta_j x(\xi_j) = \frac{1}{1 - \sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \times \sum_{j=1}^n \beta_j \int_a^{\xi_j} s^{\alpha-1} \exp(\omega(\xi_j^\alpha - s^\alpha))\sigma(s)\,\mathrm{d}s. \quad (4)$$

Substituting (4) into (2), we obtain the desired result. □

Let $X = C[a,b]$ with the norm $\|x\| = \max_{t\in[a,b]}|x(t)|$ and $P = \{x \in X : x(t) \geqslant 0, t \in [a,b]\}$. It is easy to see that $P \subset X$ is a reproducing cone. Define operators $A\colon X \to X$ and $L\colon X \to X$ as follows:

$$Ax(t) = \frac{\lambda \exp(\omega(t^\alpha - a^\alpha))}{1 - \sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \sum_{j=1}^n \beta_j \times \int_a^{\xi_j} s^{\alpha-1} \exp(\omega(\xi_j^\alpha - s^\alpha)) f(s,x(s))\,\mathrm{d}s + \lambda \int_a^t s^{\alpha-1} \exp(\omega(t^\alpha - s^\alpha)) f(s,x(s))\,\mathrm{d}s,$$

$$Lx(t) = \frac{\exp(\omega(t^\alpha - a^\alpha))}{1 - \sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \times \sum_{j=1}^n \beta_j \int_a^{\xi_j} s^{\alpha-1} \exp(\omega(\xi_j^\alpha - s^\alpha)) x(s)\,\mathrm{d}s + \int_a^t s^{\alpha-1} \exp(\omega(t^\alpha - s^\alpha)) x(s)\,\mathrm{d}s.$$

Obviously, $x \in P$ is a solution of boundary value problem (1) if and only if it is a fixed point of the operator A. x is said to be a positive solution of boundary value problem (1) if it satisfies boundary value problem (1) and $x \in P\backslash\{0\}$.

Lemma 5 *Let* $\lambda > 0$. *Then the operator* $A\colon P \to P$ *is completely continuous.*

Proof: From the continuity of $f(t,x)$, we know that $A\colon P \to P$ is continuous. Let $G \subset P$ be bounded. There exist constants $\Delta, \Delta_0 > 0$ such that $\|x\| \leqslant \Delta_0$, $x \in G$ and $|f(t,x)| \leqslant \Delta$, $(t,x) \in [a,b]\times[0,\Delta_0]$. For

$x \in G$, we obtain

$$
\begin{aligned}
|Ax(t)| &\leqslant \lambda\Delta \exp(\omega(b^\alpha - a^\alpha)) \\
&\quad \times \left(\frac{\exp(\omega(b^\alpha - a^\alpha))}{1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \right. \\
&\quad \left. \times \sum_{j=1}^n \beta_j \int_a^{\xi_j} s^{\alpha-1}\,\mathrm{d}s + \int_a^t s^{\alpha-1}\,\mathrm{d}s \right) \\
&\leqslant \frac{\lambda\Delta \exp(\omega(b^\alpha - a^\alpha))}{\alpha} \\
&\quad \times \left(\frac{\exp(\omega(b^\alpha - a^\alpha)) \sum_{j=1}^n \beta_j (\xi_j^\alpha - a^\alpha)}{1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \right. \\
&\quad \left. + (b^\alpha - a^\alpha) \right) < \infty,
\end{aligned}
$$

which implies that $A(G) \subset P$ is bounded. For $x \in G$, $t_1, t_2 \in [a,b]$ with $t_1 < t_2$, we get

$$
\begin{aligned}
&|Ax(t_2) - Ax(t_1)| \\
&\leqslant \frac{\lambda\Delta \exp(\omega(b^\alpha - a^\alpha)) \sum_{j=1}^n \beta_j (\xi_j^\alpha - a^\alpha)}{\alpha(1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha)))} \\
&\quad \times \left| \exp(\omega(t_2^\alpha - a^\alpha)) - \exp(\omega(t_1^\alpha - a^\alpha)) \right| \\
&\quad + \lambda \left| \int_a^{t_2} s^{\alpha-1} \exp(\omega(t_2^\alpha - s^\alpha)) f(s, x(s))\,\mathrm{d}s \right. \\
&\qquad \left. - \int_a^{t_1} s^{\alpha-1} \exp(\omega(t_1^\alpha - s^\alpha)) f(s, x(s))\,\mathrm{d}s \right| \\
&\leqslant \frac{\lambda\Delta \exp(\omega(b^\alpha - a^\alpha)) \sum_{j=1}^n \beta_j (\xi_j^\alpha - a^\alpha)}{\alpha(1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha)))} \\
&\quad \times \left| \exp(\omega(t_2^\alpha - a^\alpha)) - \exp(\omega(t_1^\alpha - a^\alpha)) \right| \\
&\quad + \lambda\Delta \int_a^{t_1} s^{\alpha-1} \left| \exp(\omega(t_2^\alpha - s^\alpha)) - \exp(\omega(t_1^\alpha - s^\alpha)) \right| \mathrm{d}s \\
&\quad + \lambda\Delta \exp(\omega(b^\alpha - a^\alpha)) \int_{t_1}^{t_2} s^{\alpha-1}\,\mathrm{d}s \\
&= \frac{\lambda\Delta \exp(\omega(b^\alpha - a^\alpha)) \sum_{j=1}^n \beta_j (\xi_j^\alpha - a^\alpha)}{\alpha(1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha)))} \\
&\quad \times \left| \exp(\omega(t_2^\alpha - a^\alpha)) - \exp(\omega(t_1^\alpha - a^\alpha)) \right| \\
&\quad + \lambda\Delta \int_a^{t_1} s^{\alpha-1} \left| \exp(\omega(t_2^\alpha - s^\alpha)) - \exp(\omega(t_1^\alpha - s^\alpha)) \right| \mathrm{d}s \\
&\quad + \frac{\lambda\Delta \exp(\omega(b^\alpha - a^\alpha))}{\alpha} \left(t_2^\alpha - t_1^\alpha \right).
\end{aligned}
$$

From the fact that $\exp(\cdot)$, and $\exp(\omega(t^\alpha - a^\alpha))$ and $\exp(\omega(t^\alpha - s^\alpha))$ are uniformly continuous on $[a,b]$ and $[a,b] \times [a,t]$, respectively. Therefore, $A(G)$ is equicontinuous on $[a,b]$. By Arzela-Ascoli theorem, we get that A is compact. □

Lemma 6 *The operator $L: P \to P$ and $r(L) > 0$.*

Proof: It is easy to see that $L: P \to P$. Taking $x \equiv 1$, we have

$$
\begin{aligned}
Lx(t) &= \frac{\exp(\omega(t^\alpha - a^\alpha))}{1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \\
&\quad \times \sum_{j=1}^n \beta_j \int_a^{\xi_j} s^{\alpha-1} \exp(\omega(\xi_j^\alpha - s^\alpha))\,\mathrm{d}s \\
&\quad + \int_a^t s^{\alpha-1} \exp(\omega(t^\alpha - s^\alpha))\,\mathrm{d}s \\
&\geqslant \frac{1}{1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \\
&\quad \times \sum_{j=1}^n \beta_j \int_a^{\xi_j} s^{\alpha-1} \exp(\omega(\xi_j^\alpha - s^\alpha))\,\mathrm{d}s \\
&=: \mathscr{L} > 0,
\end{aligned}
$$

$$
\begin{aligned}
L^2 x(t) &= \frac{\exp(\omega(t^\alpha - a^\alpha))}{1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha))} \sum_{j=1}^n \beta_j \\
&\quad \times \int_a^{\xi_j} s^{\alpha-1} \exp(\omega(\xi_j^\alpha - s^\alpha)) Lx(s)\,\mathrm{d}s \\
&\quad + \int_a^t s^{\alpha-1} \exp(\omega(t^\alpha - s^\alpha)) Lx(s)\,\mathrm{d}s \\
&\geqslant \frac{1}{1-\sum_{j=1}^n \beta_j \exp(\omega(\xi_j^\alpha - a^\alpha)) \sum_{j=1}^n \beta_j} \\
&\quad \times \int_a^{\xi_j} s^{\alpha-1} \exp(\omega(\xi_j^\alpha - s^\alpha)) L(s)\,\mathrm{d}s \\
&= \mathscr{L}^2.
\end{aligned}
$$

Repeating the above progress, we observe $L^n x(t) \geqslant \mathscr{L}^n$. So, $r(L) = \lim_{n\to\infty} \sqrt[n]{\|L^n\|} \geqslant \mathscr{L} > 0$. □

From Lemma 2, we can see that there exist $\varphi \in P \backslash \{0\}$ such that

$$
L\varphi = r(L)\varphi. \tag{5}
$$

Lemma 7 *Suppose that $0 < f_0 \leqslant \infty$. For $\mu_0/f_0 < \lambda < \infty$, there exists $r_0 > 0$ such that either A has at least one fixed point in $\partial\Omega_r(P)$ or $i(A, \Omega_r(P), P) = 0$, for all $0 < r < r_0$, where $\Omega_r(P) = \{x \in P : \|x\| < r\}$.*

Proof: From $\lambda > \mu_0/f_0$, there exists $r_0 > 0$ such that

$$
f(t,x) \geqslant \frac{\mu_0}{\lambda} x, \quad t \in [a,b], \quad x \in [0, r_0]. \tag{6}
$$

Assume that A does not have a fixed point in $\partial\Omega_r(P)$, $0 < r < r_0$. By using the condition (I) of Lemma 3, we only need to confirm that

$$x \neq Ax + k\varphi, \quad \forall x \in \partial\Omega_r(P), \quad k > 0,$$

where φ is similarly defined as in (5). If not, there exists $\forall x_0 \in \partial\Omega_r(P)$, $k_0 > 0$ such that $x_0 = Ax_0 + k_0\varphi$. Combining with (6), we have $Ax_0 \geqslant \mu_0 Lx_0$ and $x_0 \geqslant k_0\varphi$. From $L(P) \subset P$, we obtain $Lx_0 \geqslant k_0 r(L)\varphi = k_0\varphi/\mu_0$. So, we get $x_0 = Ax_0 + k_0\varphi \geqslant \mu_0 Lx_0 + k_0\varphi \geqslant 2k_0\varphi$. Repeating the above process, we can get that $x_0 \geqslant nk_0\varphi$. This is a contradiction with $x_0 \in \partial\Omega_r(P)$. □

Lemma 8 *Suppose that* $0 \leqslant f^{\infty} < \infty$. *For* $0 \leqslant \lambda < \mu_0/f^{\infty}$, *there exists* $\rho_0 > 0$ *such that either A has at least one fixed point in* $\partial\Omega_\rho(P)$ *or* $i(A, \Omega_\rho(P), P) = 1$, *for all* $\rho > \rho_0$, *where* $\Omega_r(P) = \{x \in P : \|x\| < \rho\}$.

Proof: From $\lambda < \mu_0/f^{\infty}$, we see that there exists $\rho_1 > 0$ such that

$$f(t,x) \leqslant \frac{\mu_0 - \varepsilon}{\lambda} x, \quad t \in [a,b], \quad x > \rho_1,$$

where $\varepsilon > 0$ is small enough. According to the continuity of $f(t,x)$ on $[a,b] \times [0,\rho_1]$, there exists a constant $\Lambda_0 > 0$ such that $f(t,x) \leqslant \Lambda_0/\lambda$, $(t,x) \in [a,b] \times [0,\rho_1]$. Therefore, we have

$$f(t,x) \leqslant \frac{\mu_0 - \varepsilon}{\lambda} x + \frac{\Lambda_0}{\lambda}, \ (t,x) \in [a,b] \times \mathbb{R}^+. \quad (7)$$

Let $\rho_0 = \|(I/(\mu_0-\varepsilon) - L)^{-1} L(\Lambda_0/(\mu_0-\varepsilon))\|$. Assume that A does not have a fixed point in $\partial\Omega_\rho(P)$, $\rho > \rho_0$. Using the condition (II) of Lemma 3, we need only to prove that $Ax \neq kx$, $x \in \partial\Omega_\rho(P)$, $k > 1$. Otherwise, there exist $x_0 \in \partial\Omega_\rho(P)$, $k_0 > 1$ such that $Ax_0 = k_0 x_0$. Combining with (7), we obtain $x_0 \leqslant Ax_0 \leqslant (\mu_0 - \varepsilon)Lx_0 + L\Lambda_0$. Furthermore, we have

$$\left(\frac{1}{\mu_0 - \varepsilon} I - L\right) x_0 \leqslant L\left(\frac{\Lambda_0}{\mu_0 - \varepsilon}\right).$$

According to the fact that $(I/(\mu_0 - \varepsilon) - L)^{-1} = \sum_{n=0}^{\infty} (\mu_0 - \varepsilon)^{n+1} L^n$ and $L(P) \subseteq P$, we have

$$\begin{aligned} x_0 &\leqslant \left(\frac{1}{\mu_0 - \varepsilon} I - L\right)^{-1} L\left(\frac{\Lambda_0}{\mu_0 - \varepsilon}\right) \\ &\leqslant \left\| \left(\frac{1}{\mu_0 - \varepsilon} I - L\right)^{-1} L\left(\frac{\Lambda_0}{\mu_0 - \varepsilon}\right) \right\| = \rho_0. \end{aligned}$$

This is a contradiction with $x_0 \in \partial\Omega_\rho(P)$. □

Next, we will give our main result in this paper.

Theorem 1 *Let* $0 \leqslant f^{\infty} < f_0 \leqslant \infty$. *Then the boundary value problem* (1) *has at least one positive solution for* $\mu_0/f_0 < \lambda < \mu_0/f^{\infty}$.

Proof: Let $0 < r < \min\{r_0, \rho_0\}$ and $\rho > \max\{r_0, \rho_0\}$. Assume that A does not have a fixed point in $\partial\Omega_r(P)$ and $\partial\Omega_\rho(P)$. By Lemma 7 and Lemma 8, we get that $i(A, \Omega_r(P), P) = 0$ and $i(A, \Omega_\rho(P), P) = 1$. By using the fixed point index theory, we can obtained that A has at least one fixed point in $\Omega_\rho(P) \backslash \overline{\Omega_r(P)}$. □

EXAMPLES

Example 1 Consider the following boundary value problem

$$\begin{aligned} &x^{(\alpha)}(t) = \lambda \exp(t) \sqrt[3]{|x(t)|}, \\ &x(0) = \sum_{j=1}^{n} \beta_j x(\xi_j), \ t \in [0,1], \ \alpha \in (0,1]. \end{aligned} \quad (8)$$

Corresponding to the boundary value problem (1), we see that $M = 0$, $f^{\infty} = 0$ and $f_0 = \infty$. From Theorem 1, we obtain that the boundary value problem (8) has at least one positive solution for $0 < \lambda < \infty$, if $\beta_j > 0$, $j = 1,2,\dots,n$, $\sum_{j=1}^{n} \beta_j < 1$.

Example 2 Consider the following boundary value problem

$$\begin{aligned} &x^{(\alpha)}(t) = \lambda |x(t)| \left(3 + \exp\left(\frac{1+t}{1+x^2(t)}\right)\right), \\ &x(0) = \sum_{j=1}^{n} \beta_j x(\xi_j), \ t \in [0,1], \ \alpha \in (0,1]. \end{aligned} \quad (9)$$

Corresponding to the boundary value problem (1), we see that $M = 0$, $f^{\infty} = 3$ and $f_0 = 3 + \mathrm{e}$. From Theorem 1, we obtain that the boundary value problem (9) has at least one positive solution for $\frac{\mu_0}{3+\mathrm{e}} < \lambda < \frac{\mu_0}{3}$, if $\beta_j > 0$, $j = 1,2,\dots,n$, $\sum_{j=1}^{n} \beta_j < 1$.

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