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Finite-time stabilization of fractional-order delayed bidirectional associative memory neural networks

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ABSTRACT: This paper investigates the finite-time stabilization problem of fractional-order delayed bidirectional associative memory neural networks with the fractional-order $\alpha \in (1,2)$. Based on feedback control, a sufficient condition is derived to realize the finite-time stabilization of systems by using the Cauchy-Schwartz inequality and the generalized Gronwall inequality. Furthermore, two sufficient conditions are directly given to realize the finite-time stabilization of systems via partial feedback control. In particular, these conditions can be expressed as some algebraic inequalities, so the settling time can be easily calculated in practical applications. Finally, some numerical examples are provided to present the feasibility and effectiveness of our main results.

KEYWORDS: fractional-order, bidirectional associative memory neural networks, time delays, feedback control

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INTRODUCTION

In practical applications, the behaviour of many interacting units is always required to be regulated. It is desired that the unpredicted ultimate states of systems can be controlled to the required objective ones[1]. This kind of stabilization problems has attracted increasing attention from many researchers starting from the pioneering work[2]. To meet the practical requirements, some researchers have proposed various types of stabilization, such as exponential stabilization[3], guaranteed cost stabilization[4], Mittag-Leffler stabilization[5], and finite-time stabilization[6]. Meanwhile, many suitable stabilization control schemes have been put forward to regulate the system behaviour.

Nowadays, fractional-order bidirectional associative memory (BAM) neural networks have been paid great attention due to their potential applications in many fields. These applications heavily depend on the dynamical behaviour of networks, such as stability and synchronization. In the last decade, there have been a lot of important works on fractional-order BAM neural networks[7–10]. For example, Yang et al[10] discussed the uniform stability of fractional-order BAM neural networks with constant delays in the leakage terms. Ke[7] reported the finite-time stability of fractional-order BAM delayed neural networks. Wang et al[9] investigated the global asymptotic stability of Riemann-Liouville fractional-order delayed BAM neural networks with impulsive effects. Rajivganthi et al[8] considered the finite-time stability of a class of fractional-order Cohen-Grossberg BAM neural networks with time delays. Recently, Wu et al[5] discussed the Mittag-Leffler stabilization of fractional-order BAM neural networks without time delays based on linear feedback control and partial feedback control.

In the abovementioned works, notice that the fractional-order of systems lies in the interval $(0,1)$. However, it is also very significant to carry out the study on fractional-order systems with the fractional-order $\alpha \in (1,2)$. For example, for the second order multi-agent dynamics, a fractional-order observer with the fractional-order $\alpha \in (1,2)$ can be used to obtain the velocity information which is not always available[11]. In addition, the fractional-order systems with $\alpha \in (1,2)$ have been extensively studied in mechanics, physics, and information science[12–14]. To the best of the authors' knowledge, the results for the fractional-order $\alpha \in (1,2)$ would not be easily obtained by generalizing those for the case $\alpha \in (0,1)$ owing to the more complicated mathematical theory. Thus it is very interesting to investigate the problems on fractional-order BAM neural networks with the fractional-order $\alpha \in (1,2)$. For example, Cao and Bai[15] studied the finite-time stability for a class of fractional-order BAM neural networks with distributed delays. Xu et al[16] considered the finite-time stability for fractional-order BAM neural networks with time delays. In these two references, the proofs mainly rely on the Laplace transform, the generalized Gronwall-Bellman inequality and some properties of Mittag-Leffler functions. It is noted that the obtained sufficient conditions are some inequalities related to the Mittag-Leffler functions.

In this paper, we consider the finite-time stabilization problem of fractional-order delayed BAM neural networks with the fractional order $\alpha \in (1,2)$. Based on linear feedback control, we derive a sufficient condition to realize the finite-time stabilization of systems. Different from those in Refs. 5, 15–17, our method mainly relies on the Cauchy-Schwartz inequality, the generalized Gronwall inequality and some elementary inequalities. In particular, our condition can be expressed as an algebraic inequality, so the settling time can be easily calculated in practical applications. Based on this result, we directly give two sufficient conditions to realize the finite-time stabilization of systems via partial feedback control.

PRELIMINARIES AND MODEL DESCRIPTION

In this section, we first recall some definitions and properties associated with the Caputo fractional-order derivative. Next we list some inequalities and give the description of the network model.

Definition 1 [Ref. 18] The fractional integral with non-integer order $\alpha > 0$ of a function $f(t)$ is defined by

$$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \geqslant 0,$$

where $\Gamma(\cdot)$ is the Gamma function, i.e., $\Gamma(s) = \int_0^\infty t^{s-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$.

Definition 2 [Ref. 18] The Caputo derivative of fractional order α of a function $f(t) \in C^n([0,\infty),\mathbb{R})$ is defined by

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \quad t \geqslant 0,$$

where $\alpha > 0$, n is a positive integer satisfying $n-1 < \alpha < n$ and $\Gamma(\cdot)$ is the Gamma function.

We now present some properties and some inequalities which are crucial to the proof of the main results.

Proposition 1 (Ref. 19) *Let $\alpha > 0$ and let n be a positive integer satisfying $n-1 < \alpha < n$. If $f(t) \in C^n([0,\infty),\mathbb{R})$, then*

$$D_t^{-\alpha}\,{}_0^C D_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

Proposition 2 (Ref. 20) *Let $x < 1$ and $x \neq 0$. For $0 < n < 1$, we have $(1-x)^n < 1-nx$. Furthermore, $(1-(1-x)^n)^{-1} < (nx)^{-1}$.*

Proposition 3 (Generalized Gronwall[21]) *Suppose that $h(t)$, $v(t)$ and $w(t)$ are nonnegative L_p functions on the interval $[0,T]$. For $1 \leqslant p < \infty$, if*

$$h(t) \leqslant v(t) + w(t)\left[\int_0^t h^p(s)\,\mathrm{d}s\right]^{1/p}, \quad t \in [0,T],$$

then

$$\int_0^t h^p(s)\,\mathrm{d}s \leqslant \left[1-(1-W(t))^{1/p}\right]^{-p} \int_0^t v^p(s) W(s)\,\mathrm{d}s,$$

where $W(t) = \exp(-\int_0^t w^p(s)\,\mathrm{d}s)$.

The network model is described as follows:

$$\left.\begin{aligned}
{}_0^C D_t^\alpha x_i(t) &= -c_i x_i(t) + \sum_{j=1}^m a_{ij}(t) f_{1j}(y_j(t)) \\
&\quad + \sum_{j=1}^m b_{ij}(t) f_{2j}(y_j(t-\tau)) + u_i(t), \\
{}_0^C D_t^\alpha y_j(t) &= -d_j y_j(t) + \sum_{i=1}^n p_{ji}(t) g_{1i}(x_i(t)) \\
&\quad + \sum_{i=1}^n q_{ji}(t) g_{2i}(x_i(t-\tau)) + v_j(t),
\end{aligned}\right\} \tag{1}$$

for $i = 1,2,\dots,n$, and $j = 1,2,\dots,m$, where $1 < \alpha < 2$, $x_i(t)$ and $y_j(t)$ denote the states of the ith unit in the X-layer and the jth unit in the Y-layer, respectively. The constants $c_i > 0$ and $d_j > 0$ are the self-regulating parameters of the neurons. The constant $\tau > 0$ is the transmission delay. $a_{ij}(t)$ and $b_{ij}(t)$ are the connections of the jth neuron to the ith neuron at times t and $t-\tau$, respectively. $p_{ji}(t)$ and $q_{ji}(t)$ have the same meanings as $a_{ij}(t)$ and $b_{ij}(t)$, respectively. $u_i(t)$ and $v_j(t)$ represent the time-varying external controls. f_{1j}, f_{2j}, g_{1i}, and g_{2i} stand for the activation functions satisfying $f_{1j}(0) = 0$, $f_{2j}(0) = 0$, $g_{1i}(0) = 0$, and $g_{2i}(0) = 0$.

The initial conditions of system (1) are given as follows:

$$x_i^{(k)}(t) = \psi_i^{(k)}(t),\ y_j^{(k)}(t) = \phi_j^{(k)}(t), \quad t \in [-\tau, 0],$$

or

$$x^{(k)}(t)=\psi^{(k)}(t),\ y^{(k)}(t)=\phi^{(k)}(t), \quad t\in[-\tau,0],$$

where $k=0,1$. $\psi^{(k)}(t)$ and $\phi^{(k)}(t)$ are two real vector-valued continuous functions on $[-\tau,0]$, whose norms are defined as

$$\|\psi^{(k)}\|=\sup_{s\in[-\tau,0]}(\sum_{i=1}^{n}|\psi_i^{(k)}(s)|),$$
$$\|\phi^{(k)}\|=\sup_{s\in[-\tau,0]}(\sum_{j=1}^{m}|\phi_j^{(k)}(s)|).$$

To obtain our results, we make some necessary assumptions[22].

Assumption 1 *The connection functions $a_{ij}(t)$, $b_{ij}(t)$, $p_{ji}(t)$, and $q_{ji}(t)$ ($i=1,2,\dots,n$, $j=1,2,\dots,m$) are continuous and bounded on $[0,\infty)$.*

Assumption 2 *The activation functions $f_{1j}(x)$, $f_{2j}(x)$, $g_{1i}(x)$, and $g_{2i}(x)$ ($i=1,2,\dots,n$, $j=1,2,\dots,m$) satisfy the Lipschitz conditions, that is, there exist positive constants ζ_1, ζ_2, θ_1, and θ_2 such that*

$$|f_{1j}(x)-f_{1j}(y)|\leqslant\zeta_1|x-y|,$$
$$|f_{2j}(x)-f_{2j}(y)|\leqslant\zeta_2|x-y|,$$
$$|g_{1i}(x)-g_{1i}(y)|\leqslant\theta_1|x-y|,$$
$$|g_{2i}(x)-g_{2i}(y)|\leqslant\theta_2|x-y|,$$

for any $x,y\in\mathbb{R}$.

Based on Refs. 5, 22, we introduce the following definitions.

Definition 3 Let $u_i(t)=0$ ($i=1,2,\dots,n$) and $v_j(t)=0$ ($j=1,2,\dots,m$). Suppose that δ and ε are any positive constants such that $\delta<\varepsilon$. Let $(x(t),y(t))$ be the solution of system (1) with $\|\psi^{(k)}\|+\|\phi^{(k)}\|<\delta$, $k=0,1$. System (1) is said to achieve the finite-time stability with respect to $\{\delta,\varepsilon,T\}$, if

$$\|x(t)\|+\|y(t)\|<\varepsilon, \quad \forall t\in[0,T],$$

where $\|x(t)\|=\sum_{i=1}^{n}|x_i(t)|$ and $\|y(t)\|=\sum_{j=1}^{m}|y_j(t)|$.

Definition 4 Suppose that δ and ε are any positive constants such that $\delta<\varepsilon$. System (1) is said to achieve the finite-time stabilization with respect to $\{\delta,\varepsilon,T\}$ if there exist suitable feedback controls $u(t)$ and $v(t)$ such that system (1) is finite-time stable with respect to $\{\delta,\varepsilon,T\}$.

MAIN RESULTS

In this section, we will investigate the finite-time stabilization problem of fractional-order BAM neural networks. Based on linear feedback control or partial feedback control, we obtain some sufficient conditions to guarantee the finite-time stabilization of system (1).

For $i=1,2,\dots,n$ and $j=1,2,\dots,m$, the external controls $u_i(t)$ and $v_j(t)$ are designed as follows:

$$u_i(t)=-k_ix_i(t), \quad v_j(t)=-l_jy_j(t), \tag{2}$$

where k_i and l_j are any positive constants.

We now introduce our main results. For simplicity, we give the following notation. Let

$$a^*=\max_{1\leqslant j\leqslant m}\sum_{i=1}^{n}a_{ij}^*, \quad b^*=\max_{1\leqslant j\leqslant m}\sum_{i=1}^{n}b_{ij}^*,$$
$$p^*=\max_{1\leqslant i\leqslant n}\sum_{j=1}^{m}p_{ji}^*, \quad q^*=\max_{1\leqslant i\leqslant n}\sum_{j=1}^{m}q_{ji}^*,$$

where $a_{ij}^*=\sup_{t\geqslant0}|a_{ij}(t)|$, $b_{ij}^*=\sup_{t\geqslant0}|b_{ij}(t)|$, $p_{ji}^*=\sup_{t\geqslant0}|p_{ji}(t)|$, and $q_{ji}^*=\sup_{t\geqslant0}|q_{ji}(t)|$. Moreover, let

$$\xi=\max\{a^*\zeta_1,p^*\theta_1\} \text{ and } \eta=\max\{b^*\zeta_2,q^*\theta_2\}.$$

Theorem 3 *Suppose that Assumptions 1 and 2 hold. Let δ and ε be any positive constants such that $\delta<\varepsilon$, and let $\max\{\|\psi^{(0)}\|+\|\phi^{(0)}\|,\|\psi^{(1)}\|+\|\phi^{(1)}\|\}<\delta$. With control (2), system (1) can achieve the finite-time stabilization with respect to $\{\delta,\varepsilon,T\}$ if*

$$(1+t)\Big[1+2\,\mathrm{e}^{(\beta^2+1)t}(1-\mathrm{e}^{-\beta^2t})^{1/2}\Big]<\frac{\varepsilon}{\delta}, \tag{3}$$

for all $t\in[0,T]$, where $\beta=\frac{(\rho+\xi+\eta\,\mathrm{e}^{-\tau})\sqrt{2\Gamma(2\alpha-1)}}{2^\alpha\Gamma(\alpha)}$ with $\rho=\max_{1\leqslant i\leqslant n,1\leqslant j\leqslant m}\{c_i+k_i,d_j+l_j\}$.

Proof: According to Proposition 1, we have

$$x_i(t)=\psi_i^{(0)}(0)+\psi_i^{(1)}(0)t$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Big[-(c_i+k_i)x_i(s)$$
$$+\sum_{j=1}^{m}a_{ij}(s)f_{1j}(y_j(s))+\sum_{j=1}^{m}b_{ij}(s)f_{2j}(y_j(s-\tau))\Big]\mathrm{d}s.$$

Furthermore, we obtain

$$\begin{aligned}|x_i(t)| &\leqslant |\psi_i^{(0)}(0)|+|\psi_i^{(1)}(0)|t \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[(c_i+k_i)|x_i(s)| \\ &\quad+\sum_{j=1}^m |a_{ij}(s)||f_{1j}(y_j(s))|+\sum_{j=1}^m |b_{ij}(s)||f_{2j}(y_j(s-\tau))|\Big]\mathrm{d}s \\ &\leqslant |\psi_i^{(0)}(0)|+|\psi_i^{(1)}(0)|t \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[(c_i+k_i)|x_i(s)| \\ &\quad+\sum_{j=1}^m a_{ij}^*\zeta_1|y_j(s)|+\sum_{j=1}^m b_{ij}^*\zeta_2|y_j(s-\tau)|\Big]\mathrm{d}s.\end{aligned}$$

In the same way, it follows that

$$\begin{aligned}|y_j(t)| &\leqslant |\phi_j^{(0)}(0)|+|\phi_j^{(1)}(0)|t \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[(d_j+l_j)|y_j(s)| \\ &\quad+\sum_{i=1}^n p_{ji}^*\theta_1|x_i(s)|+\sum_{i=1}^n q_{ji}^*\theta_2|x_i(s-\tau)|\Big]\mathrm{d}s.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\|x(t)\|+\|y(t)\|=\sum_{i=1}^n |x_i(t)|+\sum_{j=1}^m |y_j(t)| \\ &\leqslant \|\psi^{(0)}(0)\|+\|\psi^{(1)}(0)\|t+\|\phi^{(0)}(0)\|+\|\phi^{(1)}(0)\|t \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\rho\|x(s)\|+a^*\zeta_1\|y(s)\|+b^*\zeta_2\|y(s-\tau)\|)\,\mathrm{d}s \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\rho\|y(s)\|+p^*\theta_1\|x(s)\|+q^*\theta_2\|x(s-\tau)\|)\,\mathrm{d}s \\ &\quad=\|\psi^{(0)}(0)\|+\|\phi^{(0)}(0)\|+(\|\psi^{(1)}(0)\|+\|\phi^{(1)}(0)\|)t \\ &\quad+\frac{\rho+\xi}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\|x(s)\|+\|y(s)\|)\,\mathrm{d}s \\ &\quad+\frac{\eta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\|x(s-\tau)\|+\|y(s-\tau)\|)\,\mathrm{d}s.\end{aligned}$$

Making use of the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&\|x(t)\|+\|y(t)\|\leqslant \|\psi^{(0)}\|+\|\phi^{(0)}\|+(\|\psi^{(1)}\|+\|\phi^{(1)}\|)t \\ &+\Bigg\{\frac{\rho+\xi}{\Gamma(\alpha)}\left[\int_0^t \mathrm{e}^{-2s}(\|x(s)\|+\|y(s)\|)^2\,\mathrm{d}s\right]^{1/2} \\ &\quad+\frac{\eta}{\Gamma(\alpha)}\left[\int_0^t \mathrm{e}^{-2s}(\|x(s-\tau)\|+\|y(s-\tau)\|)^2\,\mathrm{d}s\right]^{1/2}\Bigg\} \\ &\quad\times\left[\int_0^t (t-s)^{2(\alpha-1)}\,\mathrm{e}^{2s}\,\mathrm{d}s\right]^{1/2}.\end{aligned}$$

Together with the following inequality

$$\begin{aligned}\int_0^t (t-s)^{2(\alpha-1)}\,\mathrm{e}^{2s}\,\mathrm{d}s &= \mathrm{e}^{2t}\int_0^t s^{2(\alpha-1)}\,\mathrm{e}^{-2s}\,\mathrm{d}s \\ &=\frac{2\,\mathrm{e}^{2t}}{4^\alpha}\int_0^{2t} s^{2\alpha-2}\,\mathrm{e}^{-s}\,\mathrm{d}s<\frac{2\,\mathrm{e}^{2t}}{4^\alpha}\Gamma(2\alpha-1),\end{aligned}$$

we obtain

$$\begin{aligned}&\|x(t)\|+\|y(t)\|\leqslant\|\varphi\|+\|\varphi\|t \\ &+\frac{\mathrm{e}^t\sqrt{2\Gamma(2\alpha-1)}}{2^\alpha\Gamma(\alpha)}\Bigg\{(\rho+\xi)\left(\int_0^t \mathrm{e}^{-2s}(\|x(s)\|+\|y(s)\|)^2\mathrm{d}s\right)^{\frac12} \\ &\quad+\eta\left(\int_0^t \mathrm{e}^{-2s}(\|x(s-\tau)\|+\|y(s-\tau)\|)^2\,\mathrm{d}s\right)^{\frac12}\Bigg\},\end{aligned}$$

where $\|\varphi\|=\max\{\|\psi^{(0)}\|+\|\phi^{(0)}\|,\|\psi^{(1)}\|+\|\phi^{(1)}\|\}$. Furthermore,

$$\begin{aligned}&(\|x(t)\|+\|y(t)\|)\,\mathrm{e}^{-t}\leqslant\|\varphi\|\,\mathrm{e}^{-t}+\|\varphi\|t\,\mathrm{e}^{-t} \\ &+\frac{\sqrt{2\Gamma(2\alpha-1)}}{2^\alpha\Gamma(\alpha)}\Bigg\{(\rho+\xi)\left(\int_0^t \mathrm{e}^{-2s}(\|x(s)\|+\|y(s)\|)^2\,\mathrm{d}s\right)^{\frac12} \\ &+\eta\left(\int_0^t \mathrm{e}^{-2s}(\|x(s-\tau)\|+\|y(s-\tau)\|)^2\,\mathrm{d}s\right)^{\frac12}\Bigg\}. \qquad (4)\end{aligned}$$

Let $\omega(t)=\sup_{t-\tau\leqslant\tilde{t}\leqslant t}(\|x(\tilde{t})\|+\|y(\tilde{t})\|)\,\mathrm{e}^{-\tilde{t}}$. Then

$$\begin{aligned}(\|x(s)\|+\|y(s)\|)\,\mathrm{e}^{-s}&\leqslant\omega(s), \\ (\|x(s-\tau)\|+\|y(s-\tau)\|)\,\mathrm{e}^{-(s-\tau)}&\leqslant\omega(s).\end{aligned}$$

For (4), we obtain

$$\begin{aligned}\omega(t)&\leqslant\|\varphi\|\,\mathrm{e}^{-t}+\|\varphi\|t\,\mathrm{e}^{-t}+\frac{\sqrt{2\Gamma(2\alpha-1)}}{2^\alpha\Gamma(\alpha)} \\ &\quad\times\left\{(\rho+\xi)\left(\int_0^t\omega^2(s)\,\mathrm{d}s\right)^{\frac12}+\eta\,\mathrm{e}^{-\tau}\left(\int_0^t\omega^2(s)\,\mathrm{d}s\right)^{\frac12}\right\} \\ &=\|\varphi\|\,\mathrm{e}^{-t}+\|\varphi\|t\,\mathrm{e}^{-t} \\ &\quad+\frac{(\rho+\xi+\eta\,\mathrm{e}^{-\tau})\sqrt{2\Gamma(2\alpha-1)}}{2^\alpha\Gamma(\alpha)}\left(\int_0^t\omega^2(s)\,\mathrm{d}s\right)^{\frac12} \\ &=\|\varphi\|(1+t)\,\mathrm{e}^{-t}+\beta\left(\int_0^t\omega^2(s)\,\mathrm{d}s\right)^{\frac12}. \qquad (5)\end{aligned}$$

According to Proposition 3, we obtain

$$\left(\int_0^t\omega^2(s)\,\mathrm{d}s\right)^{1/2}\leqslant\frac{\left(\int_0^t\left(\|\varphi\|(1+s)\,\mathrm{e}^{-s}\right)^2\mathrm{e}^{-\beta^2 s}\,\mathrm{d}s\right)^{1/2}}{1-\left(1-\mathrm{e}^{-\beta^2 t}\right)^{1/2}}.$$

Combining this with (5), we have

$$\omega(t) \leqslant \|\varphi\|(1+t)\,\mathrm{e}^{-t} + \beta\,\frac{\left[\int_0^t \left(\|\varphi\|(1+s)\,\mathrm{e}^{-s}\right)^2 \mathrm{e}^{-\beta^2 s}\mathrm{d}s\right]^{\frac{1}{2}}}{1-\left(1-\mathrm{e}^{-\beta^2 t}\right)^{1/2}}.$$

Applying Proposition 2, we have

$$\begin{aligned}\omega(t) &\leqslant \|\varphi\|(1+t)\,\mathrm{e}^{-t}\\ &\quad + 2\beta\,\mathrm{e}^{\beta^2 t}\left[\int_0^t \left(\|\varphi\|(1+s)\,\mathrm{e}^{-s}\right)^2 \mathrm{e}^{-\beta^2 s}\,\mathrm{d}s\right]^{\frac{1}{2}}\\ &\leqslant \|\varphi\|(1+t)\,\mathrm{e}^{-t} + 2\beta\|\varphi\|\,\mathrm{e}^{\beta^2 t}(1+t)\left(\int_0^t \mathrm{e}^{-\beta^2 s}\mathrm{d}s\right)^{\frac{1}{2}}\\ &= \|\varphi\|(1+t)\,\mathrm{e}^{-t} + 2\|\varphi\|\,\mathrm{e}^{\beta^2 t}(1+t)(1-\mathrm{e}^{-\beta^2 t})^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\|x(t)\| + \|y(t)\| \leqslant \|\varphi\|(1+t)\left(1+2\,\mathrm{e}^{(\beta^2+1)t}(1-\mathrm{e}^{-\beta^2 t})^{\frac{1}{2}}\right).$$

In view of the assumptions of Theorem 3, it follows that $\|x(t)\| + \|y(t)\| < \varepsilon$ for any $t \in [0, T]$. This indicates that system (1) can achieve the finite-time stabilization with respect to $\{\delta, \varepsilon, T\}$ under control (2). □

Remark 1 Notice that the condition in Theorem 3 is independent of the Mittag-Leffler function. In addition, this condition can be expressed as an algebraic inequality, so the settling time T can be easily calculated in practical applications.

Remark 2 In the existing literature, there have been a few works [15, 16] involved in the finite-time stability of fractional-order neural networks with the order satisfying $1 < \alpha < 2$. The proofs mainly rely on the Laplace transform, the generalized Gronwall-Bellman inequality and some properties of Mittag-Leffler functions. The obtained conditions are related to the Mittag-Leffler functions. Here, we consider the finite-time stabilization of fractional-order BAM neural networks with $1 < \alpha < 2$ based on linear feedback control. Different from those in some earlier works [15–17, 22], our proof mainly relies on the Cauchy-Schwartz inequality and the Gronwall inequality.

Remark 3 Following methods in Refs. 15–17, 22, we can also derive a sufficient condition to ensure the finite-time stabilization of system (1). This condition is related to the Mittag-Leffler function, which is given as follows:

$$\mathrm{e}^{-\gamma t}(1+t)E_\alpha\left((\xi+\eta\,\mathrm{e}^{\gamma\tau})\Gamma(\alpha)t^\alpha\right) < \frac{\varepsilon}{\delta}, \tag{6}$$

where $\gamma = \min_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{c_i + k_i, d_j + l_j\}$.

Let all parameters be given. From (3), the estimated settling time T_1 can be easily obtained. For (6), it is not easy to estimate the settling time T_2. Furthermore, even if we can obtain T_2, it is very difficult to derive the specific relationship between T_1 and T_2 from a mathematical point of view.

Remark 4 For the case without time delays, i.e., $b_{ij}(t) = q_{ji}(t) = 0$, the finite-time stabilization with respect to $\{\delta, \varepsilon, T\}$ can be ensured under control (2) if $\max\{\|\psi^{(0)}\| + \|\phi^{(0)}\|, \|\psi^{(1)}\| + \|\phi^{(1)}\|\} < \delta$ and

$$(1+t)\{1+2\,\mathrm{e}^{(\beta^2+1)t}(1-\mathrm{e}^{-\beta^2 t})^{1/2}\} < \frac{\varepsilon}{\delta}, \quad \forall t \in [0, T],$$

where $\beta = (\rho+\xi)\sqrt{2\Gamma(2\alpha-1)}/2^\alpha\Gamma(\alpha)$ and $\rho = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{c_i + k_i, d_j + l_j\}$.

In practical applications, the scheme with partial feedback control is always desirable due to its lower complexity [5]. Assume that the external control $u_i(t)$ $(i = 1, 2, \ldots, n)$ and $v_j(t)$ $(j = 1, 2, \ldots, m)$ are designed as follows:

$$u_i(t) = -k_i x_i(t), \quad v_j(t) = 0, \tag{7}$$

$$\text{or} \qquad u_i(t) = 0, \quad v_j(t) = -l_j y_j(t), \tag{8}$$

where k_i and l_j are positive constants.

Corollary 1 *Suppose that Assumptions 1 and 2 hold. Let δ and ε be any positive constants such that $\delta < \varepsilon$, and let $\max\{\|\psi^{(0)}\| + \|\phi^{(0)}\|, \|\psi^{(1)}\| + \|\phi^{(1)}\|\} < \delta$. System (1) can achieve the finite-time stabilization with respect to $\{\delta, \varepsilon, T\}$ under control (7), if $\max\{\|\psi^{(0)}\| + \|\phi^{(0)}\|, \|\psi^{(1)}\| + \|\phi^{(1)}\|\} < \delta$ and*

$$(1+t)\{1+2\,\mathrm{e}^{(\beta^2+1)t}(1-\mathrm{e}^{-\beta^2 t})^{1/2}\} < \frac{\varepsilon}{\delta}, \quad \forall t \in [0, T],$$

where $\beta = (\rho+\xi+\eta\,\mathrm{e}^{-\tau})\sqrt{2\Gamma(2\alpha-1)}/2^\alpha\Gamma(\alpha)$ and $\rho = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{c_i + k_i, d_j\}$.

Corollary 2 *Suppose that Assumptions 1 and 2 hold. Let δ and ε be any positive constants such that $\delta < \varepsilon$, and let $\max\{\|\psi^{(0)}\| + \|\phi^{(0)}\|, \|\psi^{(1)}\| + \|\phi^{(1)}\|\} < \delta$. System (1) can achieve the finite-time stabilization with respect to $\{\delta, \varepsilon, T\}$ under control (8), if $\max\{\|\psi^{(0)}\| + \|\phi^{(0)}\|, \|\psi^{(1)}\| + \|\phi^{(1)}\|\} < \delta$ and*

$$(1+t)\{1+2\,\mathrm{e}^{(\beta^2+1)t}(1-\mathrm{e}^{-\beta^2 t})^{1/2}\} < \frac{\varepsilon}{\delta}, \quad \forall t \in [0, T],$$

where $\beta = (\rho+\xi+\eta\,\mathrm{e}^{-\tau})\sqrt{2\Gamma(2\alpha-1)}/2^\alpha\Gamma(\alpha)$ and $\rho = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{c_i, d_j + l_j\}$.

Remark 5 For the fractional-order $0 < \alpha < 1$, Wu et al[5] considered the global Mittag-Leffler stabilization of fractional-order BAM neural networks without time delays. Based on feedback control or partial feedback control, they obtained three sufficient conditions to realize the global Mittag-Leffler stabilization of systems by using the Lyapunov method.

NUMERICAL SIMULATIONS

In this section, we will give three examples to illustrate the effectiveness of our results.

Example 1 The network model is as follows:

$$\left.\begin{aligned} {}_0^C D_t^{\alpha} x_i(t) &= -c_i x_i(t) + \sum_{j=1}^{2} a_{ij}(t) f_{1j}(y_j(t)) \\ &\quad + \sum_{j=1}^{2} b_{ij}(t) f_{2j}(y_j(t-\tau)) + u_i(t), \\ {}_0^C D_t^{\alpha} y_j(t) &= -d_j y_j(t) + \sum_{i=1}^{3} p_{ji}(t) g_{1i}(x_i(t)) \\ &\quad + \sum_{i=1}^{3} q_{ji}(t) g_{2i}(x_i(t-\tau)) + v_j(t), \end{aligned}\right\} \tag{9}$$

or

$$\left.\begin{aligned} {}_0^C D_t^{\alpha} x(t) &= -Cx(t) + A(t) f_1(y(t)) \\ &\quad + B(t) f_2(y(t-\tau)) + u(t), \\ {}_0^C D_t^{\alpha} y(t) &= -Dy(t) + P(t) g_1(x(t)) \\ &\quad + Q(t) g_2(x(t-\tau)) + v(t), \end{aligned}\right\} \tag{10}$$

where $\alpha = 1.4$, $\tau = 0.1$,

$$C = \mathrm{diag}[0.003, 0.002, 0.001],$$
$$D = \mathrm{diag}[0.002, 0.001]$$
$$f_1(y) = [0.1 \tanh y_1, 0.1 \tanh y_2]^{\mathrm{T}},$$
$$f_2(y) = [0.1 \sin y_1, 0.1 \sin y_2]^{\mathrm{T}},$$
$$g_1(x) = [0.1 \tanh x_1, 0.1 \tanh x_2, 0.1 \tanh x_3]^{\mathrm{T}},$$
$$g_2(x) = [0.1 \sin x_1, 0.1 \sin x_2, 0.1 \sin x_3]^{\mathrm{T}},$$
$$u_i(t) = -k_i x_i(t), \quad v_j(t) = -l_j y_j(t), \text{ and}$$

$$A(t) = \begin{bmatrix} 0.06 \cos t & 0.01\, e^{-t} \\ 0.02 & 0.03 \sin t \\ 0.04 \sin t & 0.025 \cos t \end{bmatrix},$$

$$B(t) = \begin{bmatrix} -0.03\, e^{-t} & 0.05 \sin t \\ 0.02 \cos t & 0.025\, e^{-t} \\ -0.01 & 0.04 \cos t \end{bmatrix},$$

$$P(t) = \begin{bmatrix} 1.15\, e^{-t} & 1.6 & 1.2 \cos t \\ 1.45 \cos t & -1.1 \sin t & 1.4\, e^{-t} \end{bmatrix},$$

$$Q(t) = \begin{bmatrix} -1.3 & 1.9\, e^{-t} & -1.7 \sin t \\ 1.2\, e^{-t} & 1.4 \sin t & 1.5 \cos t \end{bmatrix}.$$

By calculating, we obtain $a^* = 0.12$, $b^* = 0.115$, $p^* = 2.7$, $q^* = 3.3$. The activation functions $f_{1j}(x)$, $f_{2j}(x)$, $g_{1i}(x)$, $g_{2i}(x)$ satisfy Assumption 2 with $\zeta_1 = \zeta_2 = \theta_1 = \theta_2 = 0.1$. The initial conditions of system (9) are given as $x(t) = [0.002\, e^{-t}, 0.003, 0.001 \sin t]^{\mathrm{T}}$, $x'(t) = [-0.002\, e^{-t}, 0, 0.001 \cos t]^{\mathrm{T}}$, $y(t) = [0.003t + 0.004, 0.002 \sin t - 0.001]^{\mathrm{T}}$ and $y'(t) = (0.003, 0.002 \cos t)^{T}$ for any $t \in [-0.1, 0]$. Fig. 1a shows the time evolution of system (9) without external control. Let $k_i = 0.9$ ($i = 1, 2, 3$) and $l_j = 0.9$ ($j = 1, 2$). The feedback control is written as

$$\begin{aligned} u(t) &= [-0.9x_1(t), -0.9x_2(t), -0.9x_3(t)]^{\mathrm{T}}, \\ v(t) &= [-0.9y_1(t), -0.9y_2(t)]^{\mathrm{T}}. \end{aligned} \tag{11}$$

By a simple calculation, we obtain $\beta = 0.8577$. According to the initial conditions, we can take $\delta = 0.01 > \max\{\|\psi^{(0)}\| + \|\phi^{(0)}\|, \|\psi^{(1)}\| + \|\phi^{(1)}\|\}$. Let $\varepsilon = 1$, from Theorem 3, we obtain the estimated settling time $T_1 = 1.7485$. Fig. 1b shows the time evolution of system (9) with control (11).

Example 2 Let initial conditions of system (9) be

$$\begin{aligned} x(t) &= [0.003 \cos t, -0.001, 0.012t^2]^{\mathrm{T}}, \\ x'(t) &= [-0.003 \sin t, 0, 0.024t]^{\mathrm{T}}, \\ y(t) &= [0.002 + 0.001 \sin t, 0.002t - 0.004]^{\mathrm{T}}, \\ y'(t) &= [0.001 \cos t, 0.002]^{\mathrm{T}} \end{aligned}$$

for any $t \in [-0.1, 0]$. Fig. 2a shows the time evolution of system without external control.

Let $k_i = 0$ ($i = 1, 2, 3$) and $l_j = 0.9$ ($j = 1, 2$). The feedback control is written as

$$\begin{aligned} u(t) &= [0, 0, 0]^{\mathrm{T}}, \\ v(t) &= [-0.9y_1(t), -0.9y_2(t)]^{\mathrm{T}}. \end{aligned} \tag{12}$$

By calculation, we obtain $\beta = 0.8572$. In view of the initial conditions, we can choose $\delta = 0.01 > \max\{\|\psi^{(0)}\| + \|\phi^{(0)}\|, \|\psi^{(1)}\| + \|\phi^{(1)}\|\}$. Let $\varepsilon = 1$, from Corollary 2, we obtain the estimated settling time $T_2 = 1.7492$. Fig. 2b presents the time evolution of system with control (12).

Example 3 Suppose that some parameters of system (9) are given as follows: $C =$

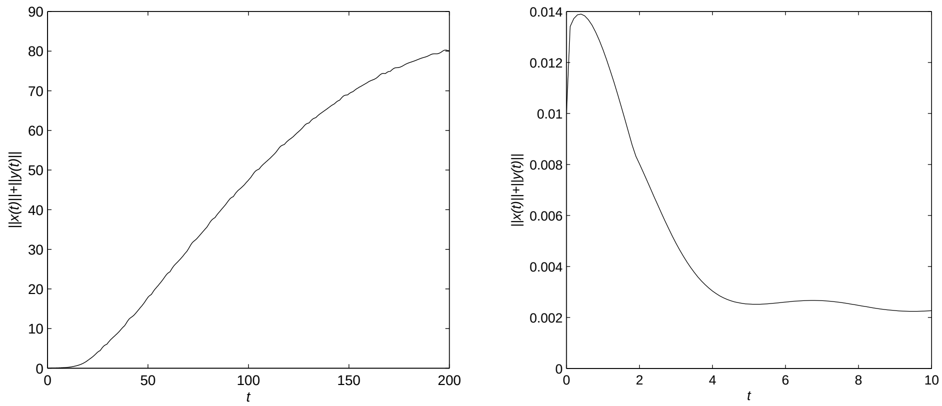

Fig. 1 The time evolution of system in Example 1, (a) without external control and (b) under control (11).

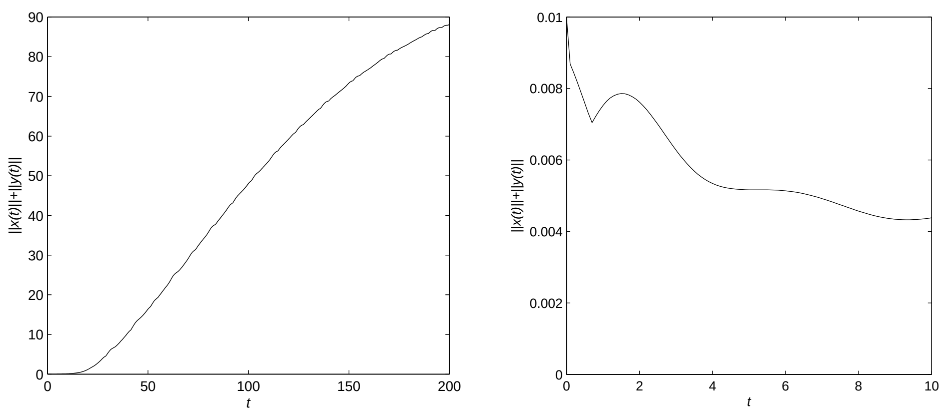

Fig. 2 The time evolution of system in Example 2, (a) without external control and (b) under control (12).

$\mathrm{diag}(0.002, 0.003, 0.001)$, $D = \mathrm{diag}(0.002, 0.005)$,

$$A(t) = \begin{bmatrix} 0.8\cos t & 0.55\,\mathrm{e}^{-t} \\ 1 & 0.15 + 0.5\sin t \\ 0.7\sin t & 1.25\cos t \end{bmatrix},$$

$$B(t) = \begin{bmatrix} -0.65\,\mathrm{e}^{-t} & 0.75\sin t \\ 0.25 + 0.6\cos t & 0.625\,\mathrm{e}^{-t} \\ -0.9 & 0.7\cos t \end{bmatrix},$$

$$P(t) = \begin{bmatrix} 0.05\,\mathrm{e}^{-t} & -0.06 & -0.02\cos t \\ 0.05\cos t & -0.01\sin t & 0.04\,\mathrm{e}^{-t} \end{bmatrix},$$

$$Q(t) = \begin{bmatrix} -0.012 & 0.01\,\mathrm{e}^{-t} & -0.03\sin t \\ -0.02\,\mathrm{e}^{-t} & -0.04\sin t & 0.015\cos t \end{bmatrix}.$$

The other parameters of system (9) are the same as those in Example 1. The initial conditions of system (9) are taken as follows: $x(t) = [0.002\,\mathrm{e}^{-t}, 0.003, -0.001\sin t]^{\mathrm{T}}$, $x'(t) = [-0.002\,\mathrm{e}^{-t}, 0, -0.001\cos t]^{\mathrm{T}}$ and $y(t) = [0.002\cos t, -0.001]^{\mathrm{T}}$ and $y'(t) = [-0.002\sin t, 0]^{\mathrm{T}}$ for any $t \in [-0.1, 0]$.

Fig. 3a shows the time evolution of system without external control. By calculation, we can obtain $a^* = 2.5$, $b^* = 2.4$, $p^* = 0.1$, and $q^* = 0.05$. Let $k_i = 0.9$ ($i = 1, 2, 3$) and $l_j = 0$ ($j = 1, 2$). The feedback control is written as

$$\begin{aligned} u(t) &= [-0.9x_1(t), -0.9x_2(t), -0.9x_3(t)]^{\mathrm{T}}, \\ v(t) &= [0, 0]^{\mathrm{T}}. \end{aligned} \tag{13}$$

By calculation, it follows that $\beta = 0.7986$. Fig. 3b shows the time evolution of system with control (13). We choose $\delta = 0.01 > \max\{\|\psi^{(0)}\| + \|\phi^{(0)}\|, \|\psi^{(1)}\| + \|\phi^{(1)}\|\}$ and $\varepsilon = 1$. According to Corollary 1, we obtain the estimated settling time $T_3 = 1.8450$.

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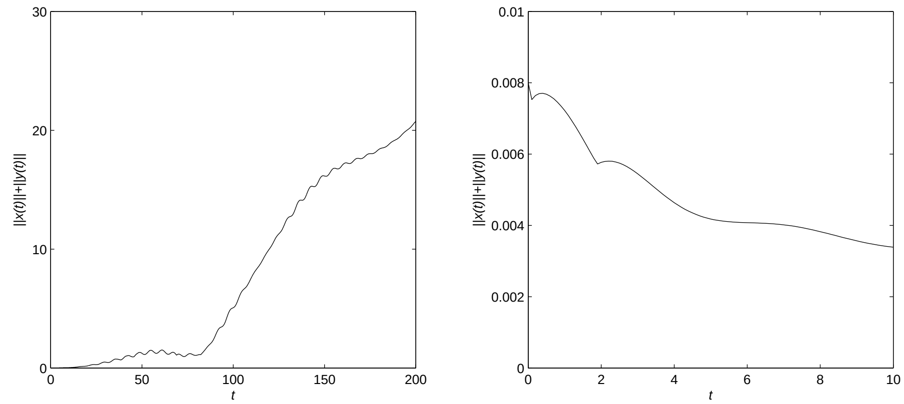

Fig. 3 The time evolution of system in Example 3, (a) without external control and (b) with control (13).

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