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Multiplicity of almost periodic oscillations in delayed harvesting predator-prey model with modified Leslie-Gower Holling-type II schemes

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ABSTRACT: By means of a fixed point theorem of coincidence degree theory, sufficient conditions for the multiplicity of positive almost periodic solutions to a delayed harvesting predator-prey model with modified Leslie-Gower Holling-type II schemes are established. The method used in this paper provides a possible method to study the multiplicity of positive almost periodic solutions of the models in biological populations. Finally, an example and computer simulations are given to illustrate the feasibility and effectiveness of our main results.

KEYWORDS: multiplicity, almost periodic solution, coincidence degree, predator-prey, Leslie-Gower

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INTRODUCTION

In 2003, Aziz-Alaoui et al[1] proposed a kind of two-dimensional system of autonomous differential equation modelling a predator-prey system[2–4] which incorporates a modified version of Leslie-Gower functional response as well as that of the Holling-type II. They considered the model

$$\dot{x}(t) = x(t)\left[r_1 - bx(t) - \frac{a_1 y(t)}{x(t)+k_1}\right],$$
$$\dot{y}(t) = y(t)\left[r_2 - \frac{a_2 y(t)}{x(t)+k_2}\right],$$

where $x(t)$ and $y(t)$ represent the population densities at time t, r_1 is the growth rate of prey x, b measures the strength of competition among individuals of species x, a_1 is the maximum value of the per capita reduction rate of x due to y, k_1 and k_2 measure the extent to which the environment provides protection to prey x and predator y, respectively, r_2 describes the growth rate of y, and a_2 has a similar meaning to a_1.

In many earlier studies, it has been shown that harvesting has a strong impact on dynamic evolution of a population[5,6]. So the study of the population dynamics with harvesting is becoming a very important subject in mathematical bio-economics. The non-autonomous harvesting predator-prey model with modified Leslie-Gower Holling-type II schemes and delays generally is described as

$$\begin{aligned}\dot{x}(t) &= x(t)\Big[r_1(t) - b(t)x(t-\tau(t)) \\ &\qquad - \frac{a_1(t)y(t-\delta(t))}{x(t-\delta(t))+k_1}\Big] - h_1(t), \\ \dot{y}(t) &= y(t)\left[r_2(t) - \frac{a_2(t)y(t-\sigma(t))}{x(t-\sigma(t))+k_2}\right] - h_2(t),\end{aligned} \tag{1}$$

where h_1 and h_2 represent harvesting terms.

In recent years, there are some scholars concerning with the existence of positive periodic solutions of the predator-prey model with modified Leslie-Gower Holling-type II schemes (PPLGH-II). Song and Li[7] studied the existence of positive periodic solutions of PPLGH-II by using Floquet theory of linear periodic impulsive equation. Further, by utilizing Mawhin's continuation theorem of coincidence degree theory, Zhu and Wang[8] also obtained some sufficient conditions for the existence of positive periodic solutions of PPLGH-II. However, in real world phenomenon, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples, see Example 1) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is more realistic and more general than periodicity. In recent years, the almost periodic solutions of the continuous models in biological populations [9,10] and engineering areas [11] have been studied extensively.

Example 1 Consider the following simple PPLGH-II with harvesting terms:

$$\begin{aligned}\dot{x}(t) &= x(t)\left[1-x(t)-\frac{0.1|\sin(\sqrt{2}t)|y(t)}{x(t)+1}\right]-0.02,\\ \dot{y}(t) &= y(t)\left[1-\frac{(0.5+|\sin(\sqrt{3}t)|)y(t)}{x(t)+1}\right]-0.02.\end{aligned} \tag{2}$$

In system (2), corresponding to system (1), $a_1(t) = 0.1|\sin(\sqrt{2}t)|$ is $\frac{1}{2}\sqrt{2}\pi$-periodic function and $a_2(t) = 0.5+|\sin(\sqrt{3}t)|$ is $\frac{1}{3}\sqrt{3}\pi$-periodic function, which imply that system (2) is with incommensurable periods. Then there is no a priori reason to expect the existence of positive periodic solutions of system (2). Thus it is significant to study the existence of positive almost periodic solutions of system (2).

By means of Mawhin's continuation theorem of coincidence degree theory, many scholars are concerning with the existence of multiple positive periodic solutions for some kinds of nonlinear ecosystems [12–14]. However, owing to the complexity of the almost periodic oscillation, it is difficult to investigate the existence of multiple positive almost periodic solutions of nonlinear ecosystems by using Mawhin's continuation theorem. Hence to the best of the authors' knowledge, so far, there is no result concerning with the multiplicity of positive almost periodic solutions of system (1). Motivated by the above reason, the main purpose of this paper is to establish sufficient conditions for the existence of multiple positive almost periodic solutions to system (1) by applying Mawhin's continuation theorem of coincidence degree theory.

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively, $C(\mathbb{X},\mathbb{Y})$ and $C^1(\mathbb{X},\mathbb{Y})$ be the space of continuous functions and continuously differential functions which map $\mathbb{X}$ into $\mathbb{Y}$, respectively. In particular, $C(\mathbb{X}) := C(\mathbb{X},\mathbb{X})$, $C^1(\mathbb{X}) := C^1(\mathbb{X},\mathbb{X})$. Related to a continuous bounded function f, we use the following notation:

$$f^- = \inf_{s\in\mathbb{R}} f(s),\ f^+ = \sup_{s\in\mathbb{R}} f(s),\ |f|_\infty = \sup_{s\in\mathbb{R}}|f(s)|.$$

Throughout this paper, we always make the following assumption for system (1):

(H_1) All the coefficients in system (1) are nonnegative almost periodic functions with $a_2^- > 0$, $b^- > 0$ and $h_i^- > 0$, $i = 1,2$.

PRELIMINARIES

Definition 1 [Refs. 15,16] $x \in C(\mathbb{R},\mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau)-x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is an arbitrary norm of $\mathbb{R}^n$. τ is called the ϵ-almost period of x, $T(x,\epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called the length of the inclusion interval for $T(x,\epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R},\mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R},\mathbb{R})$.

Lemma 1 (Refs. 15,16) *If $x \in AP(\mathbb{R})$, then x is bounded and uniformly continuous on $\mathbb{R}$.*

Lemma 2 (Refs. 15,16) *If $x \in AP(\mathbb{R})$, then $\int_0^t x(s)\,\mathrm{d}s \in AP(\mathbb{R})$ if and only if $\int_0^t x(s)\,\mathrm{d}s$ is bounded on $\mathbb{R}$.*

Lemma 3 (Ref. 17) *Assume that $x \in AP(\mathbb{R})\cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$, $\forall\epsilon > 0$, we have the following conclusions:*

(1) *there is a point $\xi_\epsilon \in [0,+\infty)$ such that $x(\xi_\epsilon) \in [x^*-\epsilon, x^*]$ and $\dot{x}(\xi_\epsilon) = 0$;*
(2) *there is a point $\eta_\epsilon \in [0,+\infty)$ such that $x(\eta_\epsilon) \in [x_*, x_*+\epsilon]$ and $\dot{x}(\eta_\epsilon) = 0$.*

In the following, we recall the famous Mawhin's coincidence degree theorem.

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L\colon \mathrm{Dom}(L) \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N\colon \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if the following conditions hold:

(i) $\mathrm{Im}(L)$ is closed in $\mathbb{Y}$;
(ii) $\dim\mathrm{Ker}(L) = \mathrm{codim}\,\mathrm{Im}(L) < \infty$.

If L is a Fredholm mapping of index zero and there exist continuous projectors $P\colon \mathbb{X} \to \mathbb{X}$ and $Q\colon \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}(P) = \mathrm{Ker}(L)$, $\mathrm{Ker}(Q) = \mathrm{Im}(L) = \mathrm{Im}(I-Q)$. It follows that $L\,|_{\mathrm{Dom}(L)\cap\mathrm{Ker}(P)}\colon (I-P)\mathbb{X} \to \mathrm{Im}(L)$ is invertible, and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if the following conditions are satisfied:

(i) $QN(\bar{\Omega})$ is bounded;
(ii) $K_P(I-Q)N : \bar{\Omega} \to \mathbb{X}$ is compact.

Since $\mathrm{Im}(Q)$ is isomorphic to $\mathrm{Ker}(L)$, there exists an isomorphism $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$.

Lemma 4 (Ref. 18) *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero, and N be L-compact on $\bar{\Omega}$. If all the following conditions hold:*
(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}(L)$, $\lambda \in (0,1)$;
(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}(L)$;
(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}(L), 0\} \neq 0$, *where* $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$ *is an isomorphism.*
Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}(L)$.

Under the invariant transformation $(x, y)^{\mathrm{T}} = (\mathrm{e}^{u}, \mathrm{e}^{v})^{\mathrm{T}}$, system (1) reduces to

$$\begin{aligned} \dot{u}(t) &= r_1(t) - b(t)\,\mathrm{e}^{u(t-\tau(t))} \\ &\quad - \frac{a_1(t)\,\mathrm{e}^{v(t-\delta(t))}}{\mathrm{e}^{u(t-\delta(t))} + k_1} - \frac{h_1(t)}{\mathrm{e}^{u(t)}} := F_1(t), \\ \dot{v}(t) &= r_2(t) - \frac{a_2(t)\,\mathrm{e}^{v(t-\sigma(t))}}{\mathrm{e}^{u(t-\sigma(t))} + k_2} - \frac{h_2(t)}{\mathrm{e}^{v(t)}} := F_2(t). \end{aligned} \tag{3}$$

For $x \in AP(\mathbb{R})$, we denote by

$$\bar{x} = m(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(s)\,\mathrm{d}s,$$

$$a(x, \varpi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(s)\,\mathrm{e}^{-\mathrm{i}\varpi s}\,\mathrm{d}s,$$

$$\Lambda(x) = \left\{ \varpi \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T} \int_0^T x(s)\,\mathrm{e}^{-\mathrm{i}\varpi s}\,\mathrm{d}s \neq 0 \right\},$$

the mean value and the set of Fourier exponents of x, respectively.

Set $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \oplus \mathbb{V}_2$, where

$$\begin{aligned} \mathbb{V}_1 &= \Big\{ z = (u, v)^{\mathrm{T}} \in AP(\mathbb{R}, \mathbb{R}^2) : \\ &\qquad \forall \varpi \in \Lambda(u) \cup \Lambda(v), |\varpi| \geqslant \gamma \Big\}, \\ \mathbb{V}_2 &= \Big\{ z = (u, v)^{\mathrm{T}} \equiv (k_1, k_2)^{\mathrm{T}}, k_1, k_2 \in \mathbb{R} \Big\}, \end{aligned}$$

where γ is a given positive constant. Define the norm for $z = (u, v)^{\mathrm{T}} \in \mathbb{X} = \mathbb{Y}$

$$\|z\|_{\mathbb{X}} = \max\left\{ \sup_{s\in\mathbb{R}} |u(s)|, \sup_{s\in\mathbb{R}} |v(s)| \right\}.$$

Similar to the proof as that in Ref. 2, it follows that

Lemma 5 *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with $\|\cdot\|_{\mathbb{X}}$.*

Lemma 6 *Let $L : \mathbb{X} \to \mathbb{Y}$, $Lz = L(u, v)^{\mathrm{T}} = (\dot{u}, \dot{v})^{\mathrm{T}}$. Then L is a Fredholm mapping of index zero.*

Lemma 7 *Define $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$, and $Q : \mathbb{Y} \to \mathbb{Y}$ for $z = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{X} = \mathbb{Y}$ by*

$$Nz = N\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix},$$

$$Pz = P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m(u) \\ m(v) \end{pmatrix} = Qz.$$

Then N is L-compact on $\bar{\Omega}$, an open and bounded subset of $\mathbb{X}$.

MAIN RESULTS

Let

$$\rho_1 = \ln \frac{r_1^+}{b^-} + r_1^+ \tau^+, \quad \rho_2 = \ln \frac{r_2^+(\mathrm{e}^{\rho_1} + k_2)}{a_2^-} + r_2^+ \sigma^+,$$

$$f_1^- = \frac{r_1^+ - \sqrt{(r_1^+)^2 - 4b^- h_1^-}}{2b^-}.$$

Theorem 1 *Assume (H_1) holds. Suppose further that:*
(H_2) $\mu > 2\sqrt{b^+ h_1^+} > 0$ *and* $r_2^- > 2\sqrt{\nu h_2^+} > 0$, *where*

$$\mu = r_1^- - \frac{a_1^+\,\mathrm{e}^{\rho_2}}{f_1^- + k_1}, \quad \nu = \frac{a_2^+}{f_1^- + k_2}.$$

Then system (1) admits at least four positive almost periodic solutions.

Proof: It is easy to see that if system (3) has one almost periodic solution $(u, v)^{\mathrm{T}}$, then $(x, y)^{\mathrm{T}} = (\mathrm{e}^{u}, \mathrm{e}^{v})^{\mathrm{T}}$ is a positive almost periodic solution of system (1). Hence to complete the proof, it suffices to show that system (3) has four almost periodic solutions.

To use Lemma 4, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ as those in Lemma 5, and L, N, P, Q the same as those defined in Lemma 6 and Lemma 7, respectively. It remains to search for an appropriate open and bounded subset $\Omega \subseteq \mathbb{X}$.

Corresponding to the operator equation $Lz = \lambda z$, $\lambda \in (0,1)$, we have

$$\begin{aligned} \dot{u}(t) &= \lambda\Big[r_1(t) - b(t)\,\mathrm{e}^{u(t-\tau(t))} \\ &\qquad - \frac{a_1(t)\,\mathrm{e}^{v(t-\delta(t))}}{\mathrm{e}^{u(t-\delta(t))} + k_1} - \frac{h_1(t)}{\mathrm{e}^{u(t)}} \Big], \\ \dot{v}(t) &= \lambda\Big[r_2(t) - \frac{a_2(t)\,\mathrm{e}^{v(t-\sigma(t))}}{\mathrm{e}^{u(t-\sigma(t))} + k_2} - \frac{h_2(t)}{\mathrm{e}^{v(t)}} \Big]. \end{aligned} \tag{4}$$

Suppose that $z = (u, v^{\mathrm{T}}) \in \mathrm{Dom}(L) \subseteq \mathbb{X}$ is a solution of system (4) for some $\lambda \in (0,1)$, where $\mathrm{Dom}(L) = \{z = (u, v)^{\mathrm{T}} \in \mathbb{X} : u, v \in C^1(\mathbb{R}), \dot{u}, \dot{v} \in C(\mathbb{R})\}$.

By Lemma 3, for all $\epsilon \in (0,1)$, there are four points $\xi = \xi(\epsilon)$, $\zeta = \zeta(\epsilon)$, $\eta = \eta(\epsilon)$, and $\varsigma = \varsigma(\epsilon) \in [0,+\infty)$ such that

$$\begin{cases}\dot{u}(\xi) = 0,\\ u(\xi) \in [u^* - \epsilon, u^*],\end{cases} \quad \begin{cases}\dot{v}(\zeta) = 0,\\ v(\zeta) \in [v^* - \epsilon, v^*],\end{cases} \tag{5}$$

$$\begin{cases}\dot{u}(\eta) = 0,\\ u(\eta) \in [u_*, u_* + \epsilon],\end{cases} \quad \begin{cases}\dot{v}(\varsigma) = 0,\\ v(\varsigma) \in [v_*, v_* + \epsilon],\end{cases} \tag{6}$$

where $u^* = \sup_{s\in\mathbb{R}} u(s)$, $v^* = \sup_{s\in\mathbb{R}} v(s)$, $u_* = \inf_{s\in\mathbb{R}} u(s)$, and $v_* = \inf_{s\in\mathbb{R}} v(s)$.

Further, in view of (H_2), we may assume that the above ϵ is small enough so that

$$\mu^2 > 4b^+h_1^+ \mathrm{e}^{\epsilon}, \quad (r_2^-)^2 > 4\nu h_2^+ \mathrm{e}^{\epsilon}.$$

From system (4), it follows from (5) and (6) that

$$\begin{aligned} 0 &= r_1(\xi) - b(\xi)\,\mathrm{e}^{u(\xi-\tau(\xi))} \\ &\quad - \frac{a_1(\xi)\,\mathrm{e}^{v(\xi-\delta(\xi))}}{\mathrm{e}^{u(\xi-\delta(\xi))} + k_1} - \frac{h_1(\xi)}{\mathrm{e}^{u(\xi)}}, \\ 0 &= r_2(\zeta) - \frac{a_2(\zeta)\,\mathrm{e}^{v(\zeta-\sigma(\zeta))}}{\mathrm{e}^{u(\zeta-\sigma(\zeta))} + k_2} - \frac{h_2(\zeta)}{\mathrm{e}^{v(\zeta)}} \end{aligned} \tag{7}$$

and

$$\begin{aligned} 0 &= r_1(\eta) - b(\eta)\,\mathrm{e}^{u(\eta-\tau(\eta))} \\ &\quad - \frac{a_1(\eta)\,\mathrm{e}^{v(\eta-\delta(\eta))}}{\mathrm{e}^{u(\eta-\delta(\eta))} + k_1} - \frac{h_1(\eta)}{\mathrm{e}^{u(\eta)}}, \\ 0 &= r_2(\varsigma) - \frac{a_2(\varsigma)\,\mathrm{e}^{v(\varsigma-\sigma(\varsigma))}}{\mathrm{e}^{u(\varsigma-\sigma(\varsigma))} + k_2} - \frac{h_2(\varsigma)}{\mathrm{e}^{v(\varsigma)}}. \end{aligned} \tag{8}$$

In view of the first equation of system (8), we have from (6) that

$$b^-\,\mathrm{e}^{u_*} + \frac{h_1^-}{\mathrm{e}^{u_*+\epsilon}} \leqslant b(\eta)\,\mathrm{e}^{u(\eta-\tau(\eta))} + \frac{h_1(\eta)}{\mathrm{e}^{u(\eta)}} \leqslant r_1(\eta) \leqslant r_1^+.$$

That is,

$$b^-\,\mathrm{e}^{2u_*} - r_1^+\,\mathrm{e}^{u_*} + h_1^-\,\mathrm{e}^{-\epsilon} \leqslant 0,$$

which implies that

$$\begin{aligned} \ln\left[\frac{r_1^+ - \sqrt{(r_1^+)^2 - 4b^-h_1^-\,\mathrm{e}^{-\epsilon}}}{2b^-}\right] &\leqslant u_* \\ &\leqslant \ln\left[\frac{r_1^+ + \sqrt{(r_1^+)^2 - 4b^-h_1^-\,\mathrm{e}^{-\epsilon}}}{2b^-}\right]. \end{aligned} \tag{9}$$

Letting $\epsilon \to 0$ in (9) leads to

$$\ln f_1^- \leqslant u_* \leqslant \ln f_1^+, \tag{10}$$

where

$$f_1^{\pm} = \frac{r_1^+ \pm \sqrt{(r_1^+)^2 - 4b^-h_1^-}}{2b^-}.$$

By the first equation of system (7), we have

$$b^-\,\mathrm{e}^{u(\xi-\tau(\xi))} \leqslant b(\xi)\,\mathrm{e}^{u(\xi-\tau(\xi))} \leqslant r_1(\xi) \leqslant r_1^+,$$

which implies that

$$u(\xi - \tau(\xi)) \leqslant \ln \frac{r_1^+}{b^-}. \tag{11}$$

Since

$$\begin{aligned} \int_{\xi-\tau(\xi)}^{\xi} \dot{u}(s)\,\mathrm{d}s &= \int_{\xi-\tau(\xi)}^{\xi} \lambda\Big[r_1(s) - b(s)\,\mathrm{e}^{u(s-\tau(s))} \\ &\qquad - \frac{a_1(s)\,\mathrm{e}^{v(s-\delta(s))}}{\mathrm{e}^{u(s-\delta(s))} + k_1} - \frac{h_1(s)}{\mathrm{e}^{u(s)}}\Big]\,\mathrm{d}s \\ &\leqslant \int_{\xi-\tau(\xi)}^{\xi} r_1(s)\,\mathrm{d}s \leqslant r_1^+\tau^+. \end{aligned} \tag{12}$$

From (11) and (12), it follows that

$$u(\xi) = u(\xi - \tau(\xi)) + \int_{\xi-\tau(\xi)}^{\xi} \dot{u}(s)\,\mathrm{d}s \leqslant \ln \frac{r_1^+}{b^-} + r_1^+\tau^+ := \rho_1,$$

which yields from (5) that

$$u^* \leqslant \rho_1 + \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$u^* \leqslant \rho_1. \tag{13}$$

In view of the second equation of system (8), we have from (6) that

$$\frac{a_2^-\,\mathrm{e}^{v_*}}{\mathrm{e}^{\rho_1} + k_2} + \frac{h_2^-}{\mathrm{e}^{v_*+\epsilon}} \leqslant \frac{a_2(\varsigma)\,\mathrm{e}^{v(\varsigma-\sigma(\varsigma))}}{\mathrm{e}^{u(\varsigma-\sigma(\varsigma))} + k_2} + \frac{h_2(\varsigma)}{\mathrm{e}^{v(\varsigma)}} = r_2(\varsigma) \leqslant r_2^+.$$

That is,

$$\frac{a_2^-}{\mathrm{e}^{\rho_1} + k_2}\,\mathrm{e}^{2v_*} - r_2^+\,\mathrm{e}^{v_*} + h_2^-\,\mathrm{e}^{-\epsilon} \leqslant 0,$$

which implies that

$$\begin{aligned} \ln\left[\frac{r_2^+ - \sqrt{(r_2^+)^2 - 4ch_2^-\,\mathrm{e}^{-\epsilon}}}{2c}\right] &\leqslant v_* \\ &\leqslant \ln\left[\frac{r_2^+ + \sqrt{(r_2^+)^2 - 4ch_2^-\,\mathrm{e}^{-\epsilon}}}{2c}\right], \end{aligned} \tag{14}$$

where $c = (a_2^-)/(e^{\rho_1} + k_2)$. Letting $\epsilon \to 0$ in (14) leads to

$$\ln f_2^- \leqslant v_* \leqslant \ln f_2^+, \tag{15}$$

where

$$f_2^\pm = \frac{r_2^+ \pm \sqrt{(r_2^+)^2 - 4ch_2^-}}{2c}.$$

Further, by the second equation of system (7), we have

$$\frac{a_2^- e^{v(\zeta-\sigma(\zeta))}}{e^{\rho_1} + k_2} \leqslant \frac{a_2(\zeta) e^{v(\zeta-\sigma(\zeta))}}{e^{u(\zeta-\sigma(\zeta))} + k_2} \leqslant r_2(\zeta) \leqslant r_2^+,$$

which implies that

$$v(\zeta - \sigma(\zeta)) \leqslant \ln \frac{r_2^+(e^{\rho_1} + k_2)}{a_2^-}. \tag{16}$$

Since

$$\int_{\zeta-\sigma(\zeta)}^{\zeta} \dot v(s)\,ds = \int_{\zeta-\sigma(\zeta)}^{\zeta} \lambda\left[r_2(s) - \frac{a_2(s) e^{v(s-\sigma(s))}}{e^{u(s-\sigma(s))} + k_2} - \frac{h_2(s)}{e^{v(s)}}\right] ds$$
$$\leqslant \int_{\zeta-\sigma(\zeta)}^{\zeta} r_2(s)\,ds \leqslant r_2^+ \sigma^+. \tag{17}$$

From (16) and (17), it follows that

$$v(\zeta) = v(\zeta - \sigma(\zeta)) + \int_{\zeta-\sigma(\zeta)}^{\zeta} \dot v(s)\,ds$$
$$\leqslant \ln \frac{r_2^+(e^{\rho_1} + k_2)}{a_2^-} + r_2^+ \sigma^+ := \rho_2,$$

which yields from (5) that

$$v^* \leqslant \rho_2 + \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$v^* \leqslant \rho_2. \tag{18}$$

On the other hand, we obtain from the first equation of system (7) that

$$r_1^- \leqslant r_1(\xi) = b(\xi) e^{u(\xi-\tau(\xi))} + \frac{a_1(\xi) e^{v(\xi-\delta(\xi))}}{e^{u(\xi-\delta(\xi))} + k_1} + \frac{h_1(\xi)}{e^{u(\xi)}}$$
$$\leqslant b^+ e^{u^*} + \frac{a_1^+ e^{\rho_2}}{f_1^- + k_1} + \frac{h_1^+}{e^{u^*-\epsilon}},$$

that is,

$$b^+ e^{2u^*} - \left[r_1^- \frac{a_1^+ e^{\rho_2}}{f_1^- + k_1}\right] e^{u^*} + h_1^+ e^{\epsilon}$$
$$= b^+ e^{2u^*} - \mu e^{u^*} + h_1^+ e^{\epsilon} \geqslant 0,$$

which implies that

$$u^* \geqslant \ln\left[\frac{\mu + \sqrt{\mu^2 - 4b^+ h_1^+ e^{\epsilon}}}{2b^+}\right] \quad \text{or}$$
$$u^* \leqslant \ln\left[\frac{\mu - \sqrt{\mu^2 - 4b^+ h_1^+ e^{\epsilon}}}{2b^+}\right]. \tag{19}$$

Letting $\epsilon \to 0$ in (19) leads to

$$u^* \geqslant \ln g_1^+ \quad \text{or} \quad u^* \leqslant \ln g_1^-, \tag{20}$$

where

$$g_1^\pm = \frac{\mu \pm \sqrt{\mu^2 - 4b^+ h_1^+}}{2b^+}.$$

Further, we obtain from the second equation of system (7) that

$$r_2^- \leqslant r_2(\zeta) = \frac{a_2(\zeta) e^{v(\zeta-\sigma(\zeta))}}{e^{u(\zeta-\sigma(\zeta))} + k_2} + \frac{h_2(\zeta)}{e^{v(\zeta)}}$$
$$\leqslant \frac{a_2^+ e^{v^*}}{f_1^- + k_2} + \frac{h_2^+}{e^{v^*-\epsilon}},$$

that is,

$$\frac{a_2^+}{f_1^- + k_2} e^{2v^*} - r_2^- e^{v^*} + h_2^+ e^{\epsilon} = v e^{2v^*} - r_2^- e^{v^*} + h_2^+ e^{\epsilon} \geqslant 0,$$

which implies that

$$v^* \geqslant \ln\left[\frac{r_2^- + \sqrt{(r_2^-)^2 - -4 v h_2^+ e^{\epsilon}}}{2v}\right] \quad \text{or}$$
$$v^* \leqslant \ln\left[\frac{r_2^- \sqrt{(r_2^-)^2 - -4 v h_2^+ e^{\epsilon}}}{2v}\right]. \tag{21}$$

Letting $\epsilon \to 0$ in (21) leads to

$$v^* \geqslant \ln g_2^+ \quad \text{or} \quad v^* \leqslant \ln g_2^-, \tag{22}$$

where

$$g_2^\pm = \frac{r_2^- \pm \sqrt{(r_2^-)^2 - 4 v h_2^+}}{2v}.$$

Clearly, $\ln f_i^\pm$, $\ln g_i^\pm$ and ρ_i are independent of λ, $i = 1, 2$. Let $\varepsilon_i = \frac{1}{4}(\ln g_i^+ - \ln g_i^-)$, $i = 1, 2$, and

$$\Omega_1 = \left\{ z = (u, v)^{\mathrm{T}} \in \mathbb{X} \;\middle|\; \begin{array}{l} \ln f_1^- - 1 < u_* \leqslant u^* < \ln g_1^- + \varepsilon_1, \\ \ln f_2^- - 1 < v_* \leqslant v^* < \ln g_2^- + \varepsilon_2 \end{array} \right\},$$

$$\Omega_2 = \left\{ z = (u, v)^{\mathrm{T}} \in \mathbb{X} \;\middle|\; \begin{array}{l} u_* \in (\ln f_1^- - 1, \ln f_1^+ + 1), \\ u^* \in (\ln g_1^+ - \varepsilon_1, \rho_1 + 1), \\ \ln f_2^- - 1 < v_* \leqslant v^* < \ln g_2^- + \varepsilon_2 \end{array} \right\},$$

$$\Omega_3 = \left\{ z=(u,v)^{\mathrm{T}} \in \mathbb{X} \;\middle|\; \begin{array}{l} u_* \in (\ln f_1^- - 1, \ln f_1^+ + 1), \\ u^* \in (\ln g_1^+ - \varepsilon_1, \rho_1 + 1), \\ v_* \in (\ln f_2^- - 1, \ln f_2^+ + 1), \\ v^* \in (\ln g_2^+ - \varepsilon_2, \rho_2 + 1) \end{array} \right\},$$

$$\Omega_4 = \left\{ z=(u,v)^{\mathrm{T}} \in \mathbb{X} \;\middle|\; \begin{array}{l} \ln f_1^- - 1 < u_* \leqslant u^* < \ln g_1^- + \varepsilon_1, \\ v_* \in (\ln f_2^- - 1, \ln f_2^+ + 1), \\ v^* \in (\ln g_2^+ + \varepsilon_2, \rho_2 + 1) \end{array} \right\}.$$

Then Ω_1, Ω_2, Ω_3, and Ω_4 are bounded open subsets of $\mathbb{X}$, $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, $i,j = 1,2,3,4$. Hence Ω_1, Ω_2, Ω_3, and Ω_4 satisfy condition (a) of Lemma 4.

Now we show that condition (b) of Lemma 4 holds, i.e., we prove that $QNz \neq 0$ for all $z = (u,v)^{\mathrm{T}} \in \partial\Omega_i \cap \mathrm{Ker}L = \partial\Omega_i \cap \mathbb{R}^2$, $i = 1,2,3,4$. If it is not true, then there exists at least one constant vector $z_0 = (u_0, v_0)^{\mathrm{T}} \in \partial\Omega_i$, $i = 1,2,3,4$, such that

$$0 = m\left[r_1(t) - b(t)\,\mathrm{e}^{u_0} - \frac{a_1(t)\,\mathrm{e}^{v_0}}{\mathrm{e}^{u_0} + k_1} - \frac{h_1(t)}{\mathrm{e}^{u_0}} \right],$$
$$0 = m\left[r_2(t) - \frac{a_2(t)\,\mathrm{e}^{v_0}}{\mathrm{e}^{u_0} + k_2} - \frac{h_2(t)}{\mathrm{e}^{v_0}} \right].$$

As with the arguments as those in (10), (13), (15), (18), (20), and (22), it follows that

$$\ln f_1^- \leqslant u_0 \leqslant \ln g_1^-, \qquad \ln f_2^- \leqslant v_0 \leqslant \ln g_2^-,$$

or

$$\ln g_1^+ \leqslant u_0 \leqslant \ln f_1^+, \qquad \ln f_2^- \leqslant v_0 \leqslant \ln g_2^-,$$

or

$$\ln g_1^+ \leqslant u_0 \leqslant \ln f_1^+, \qquad \ln g_2^+ \leqslant v_0 \leqslant \ln f_2^+,$$

or

$$\ln f_1^- \leqslant u_0 \leqslant \ln g_1^-, \qquad \ln g_2^+ \leqslant v_0 \leqslant \ln f_2^+.$$

Then $z_0 \in \Omega_1 \cap \mathbb{R}^2$ or $z_0 \in \Omega_2 \cap \mathbb{R}^2$ or $z_0 \in \Omega_3 \cap \mathbb{R}^2$ or $z_0 \in \Omega_4 \cap \mathbb{R}^2$. This contradicts the fact that $z_0 \in \partial\Omega_i$, $i = 1,2,3,4$. This proves that condition (b) of Lemma 4 holds.

Finally, we will show that condition (c) of Lemma 4 is satisfied. Consider the homotopy

$$H(\iota, z) = \iota QNz + (1-\iota)\Phi z, \quad (\iota, z) \in [0,1] \times \mathbb{R}^2,$$

where

$$\Phi z = \Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \bar{r}_1 - \bar{b}\,\mathrm{e}^u - \frac{\bar{h}_1}{\mathrm{e}^u} \\ \bar{r}_2 - \frac{\bar{a}_2\,\mathrm{e}^v}{\mathrm{e}^{\rho_1} + k_2} - \frac{\bar{h}_2}{\mathrm{e}^v} \end{pmatrix}.$$

From the above discussion, it is easy to verify that $H(\iota, z) \neq 0$ on $\partial\Omega_i \cap \mathrm{Ker}(L)$, $\forall \iota \in [0,1]$, $i = 1,2,3,4$. Further, $\Phi z = 0$ has four distinct solutions:

$$(u_1^*, v_1^*)^{\mathrm{T}} = (\ln u^-, \ln v^-)^{\mathrm{T}},\ (u_2^*, v_2^*)^{\mathrm{T}} = (\ln u^+, \ln v^-)^{\mathrm{T}},$$
$$(u_3^*, v_3^*)^{\mathrm{T}} = (\ln u^+, \ln v^+)^{\mathrm{T}},\ (u_4^*, v_4^*)^{\mathrm{T}} = (\ln u^-, \ln v^+)^{\mathrm{T}},$$

where

$$u^{\pm} = \frac{\bar{r}_1 \pm \sqrt{(\bar{r}_1)^2 - 4\bar{b}\bar{h}_1}}{2\bar{b}},\ v^{\pm} = \frac{\bar{r}_2 \pm \sqrt{(\bar{r}_2)^2 - \frac{4\bar{a}_2\bar{h}_2}{\mathrm{e}^{\rho_1}+k_2}}}{\frac{2\bar{a}_2}{\mathrm{e}^{\rho_1}+k_2}}.$$

It is easy to verify that

$$\ln f_1^- < \ln u^- < \ln g_1^- < \ln g_1^+ < \ln u^+ < \rho_1,$$
$$\ln f_2^- < \ln v^- < \ln g_2^- < \ln g_2^+ < \ln v^+ < \rho_2.$$

Hence

$$(u_1^*, v_1^*)^{\mathrm{T}} \in \Omega_1, \quad (u_2^*, v_2^*)^{\mathrm{T}} \in \Omega_2,$$
$$(u_3^*, v_3^*)^{\mathrm{T}} \in \Omega_3, \quad (u_4^*, v_4^*)^{\mathrm{T}} \in \Omega_4.$$

A direct computation yields

$$\begin{aligned} &\deg(\Phi, \Omega_i \cap \mathrm{Ker}(L), 0) \\ &= \mathrm{sgn}\begin{vmatrix} -\bar{b}\,\mathrm{e}^{u_i^*} + \frac{\bar{h}_1}{\mathrm{e}^{u_i^*}} & 0 \\ 0 & -\frac{\bar{a}_2}{\mathrm{e}^{\rho_1}+k_2}\,\mathrm{e}^{v_i^*} + \frac{\bar{h}_2}{\mathrm{e}^{v_i^*}} \end{vmatrix} \\ &= \mathrm{sgn}\left[\left(-\bar{b}\,\mathrm{e}^{u_i^*} + \frac{\bar{h}_1}{\mathrm{e}^{u_i^*}}\right)\left(-\frac{\bar{a}_2}{\mathrm{e}^{\rho_1}+k_2}\,\mathrm{e}^{v_i^*} + \frac{\bar{h}_2}{\mathrm{e}^{v_i^*}}\right)\right] \\ &= \mathrm{sgn}\left[\left(\bar{r}_1 - 2\bar{b}\,\mathrm{e}^{u_i^*}\right)\left(\bar{r}_2 - \frac{2\bar{a}_2}{\mathrm{e}^{\rho_1}+k_2}\,\mathrm{e}^{v_i^*}\right)\right], \end{aligned}$$

where $i = 1,2,3,4$. Thus

$$\begin{aligned} &\deg\left(\Phi, \Omega_1 \cap \mathrm{Ker}(L), 0\right) \\ &\quad = \mathrm{sign}\left[\left(\bar{r}_1 - 2\bar{b}u^-\right)\left(\bar{r}_2 - \frac{2\bar{a}_2}{\mathrm{e}^{\rho_1}+k_2}v^-\right)\right] = 1, \\ &\deg\left(\Phi, \Omega_2 \cap \mathrm{Ker}(L), 0\right) \\ &\quad = \mathrm{sign}\left[\left(\bar{r}_1 - 2\bar{b}u^+\right)\left(\bar{r}_2 - \frac{2\bar{a}_2}{\mathrm{e}^{\rho_1}+k_2}v^-\right)\right] = -1, \\ &\deg\left(\Phi, \Omega_3 \cap \mathrm{Ker}(L), 0\right) \\ &\quad = \mathrm{sign}\left[\left(\bar{r}_1 - 2\bar{b}u^+\right)\left(\bar{r}_2 - \frac{2\bar{a}_2}{\mathrm{e}^{\rho_1}+k_2}v^+\right)\right] = 1, \\ &\deg\left(\Phi, \Omega_4 \cap \mathrm{Ker}(L), 0\right) \\ &\quad = \mathrm{sign}\left[\left(\bar{r}_1 - 2\bar{b}u^-\right)\left(\bar{r}_2 - \frac{2\bar{a}_2}{\mathrm{e}^{\rho_1}+k_2}v^+\right)\right] = -1. \end{aligned}$$

By the invariance property of homotopy, we have for $i = 1,2,3,4$,

$$\begin{aligned} \deg\left(JQN, \Omega_i \cap \mathrm{Ker}(L), 0\right) &= \deg\left(QN, \Omega_i \cap \mathrm{Ker}(L), 0\right) \\ &= \deg\left(\Phi, \Omega_i \cap \mathrm{Ker}(L), 0\right) \neq 0, \end{aligned}$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree, and J is the identity mapping since $\mathrm{Im}(Q) = \mathrm{Ker}(L)$. Clearly, all the conditions of Lemma 4 are satisfied. Hence system (3) has at least four almost periodic solutions, that is, system (1) has at least four positive almost periodic solutions. □

Corollary 1 *Assume that* (H_1) *and* (H_2) *hold. Suppose further that* r_i, b, a_i, τ, δ, σ, *and* h_i *of system* (1) *are continuous nonnegative periodic functions with periods* α_i, β, ρ_i, υ, η, ξ, *and* θ_i, *respectively, then system* (1) *has at least four positive almost periodic solutions,* $i = 1, 2$.

In Corollary 1, let $\alpha_i = \beta = \rho_i = \upsilon = \eta = \xi = \theta_i = \omega$, $i = 1, 2$. Then we obtain the following.

Corollary 2 *Assume that* (H_1) *and* (H_2) *hold. Suppose further that* r_i, b, a_i, τ, δ, σ, *and* h_i *of system* (1) *are continuous nonnegative* ω*-periodic functions, then system* (1) *has at least four positive* ω*-periodic solutions,* $i = 1, 2$.

Remark 1 By Corollary 1, it is easy to obtain the existence of at least four positive almost periodic solutions of system (2) in Example 1, although the positive periodic solution of system (2) is nonexistent.

AN EXAMPLE WITH COMPUTER SIMULATIONS

Example 2 Consider the following harvesting predator-prey model with modified Leslie-Gower Holling-type II schemes:

$$\begin{aligned}\dot{x}(t) &= x(t)\Big[1 - \big(1+\cos^2(\sqrt{2}t)\big)\,x\,(t - |\cos(2t)|) \\ &\qquad - \frac{0.1|\sin(\sqrt{3}t)|y(t-1)}{x(t-1)+4}\Big] - 0.02, \\ \dot{y}(t) &= y(t)\Big[1 - \frac{\big(0.5+|\cos(\sqrt{3}t)|\big)\,y(t-1)}{x(t-1)+1}\Big] - 0.1.\end{aligned} \tag{23}$$

Then system (23) has at least four positive almost periodic solutions.

Proof: Corresponding to system (1), we have $r_1 = r_2 \equiv 1$, $b(s) = 1 + \cos^2(\sqrt{2}s)$, $a_1(s) = 0.1|\sin(\sqrt{3}s)|$, $a_2(s) = 0.5 + |\cos(\sqrt{3}s)|$, $k_1 = 4$, $k_2 = 1$, $\forall s \in \mathbb{R}$. By an easy calculation, we obtain that

$$\rho_1 \approx 1, \quad \rho_2 \approx 3, \quad \mu > 0.5.$$

So (H_2) in Theorem 1 holds. By Theorem 1, system (23) admits at least four positive almost periodic solutions (x_i, y_i), see Fig. 1 and Fig. 2. □

CONCLUSIONS

By using a fixed point theorem of coincidence degree theory, some criteria for the multiplicity of positive almost periodic solutions to a kind of delayed harvesting predator-prey model with modified Leslie-Gower Holling-type II schemes are obtained. Theorem 1 gives the sufficient conditions for the multiplicity of positive almost periodic solutions of system (1). The method used in this paper provides a possible method to study the multiplicity of positive almost periodic solutions of the models in biological populations.

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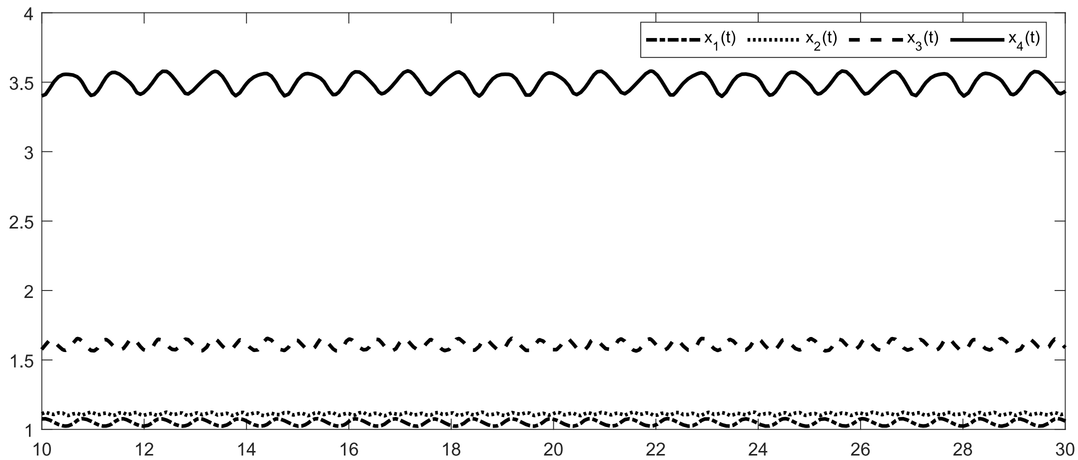

Fig. 1 Four positive almost periodic solutions of state variable x of system (23).

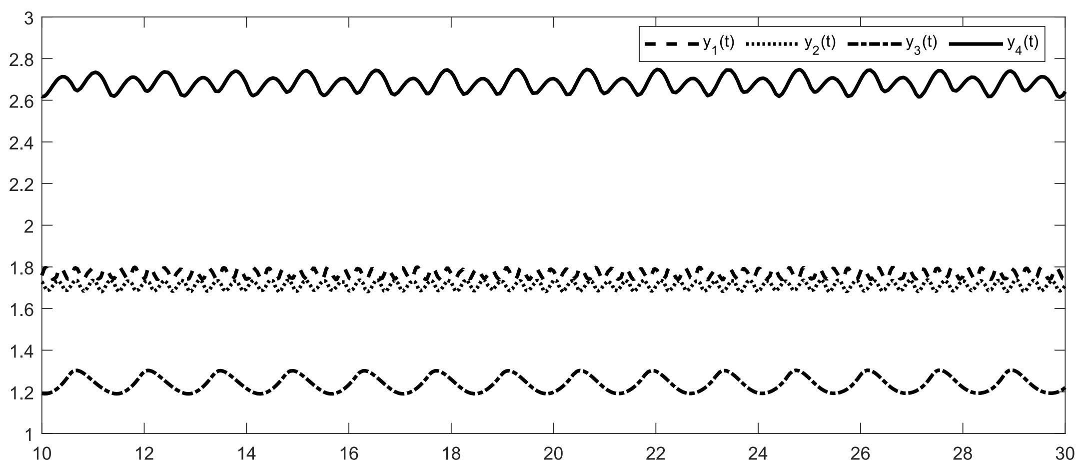

Fig. 2 Four positive almost periodic solutions of state variable y of system (23).

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