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A note on equivalence of some Rotfel'd type theorems

Yaxin Gao, Chaojun Yang, Fangyan Lu*

Department of Mathematics, Soochow University, Suzhou 215006 China

*Corresponding author, e-mail: fylu@suda.edu.cn

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ABSTRACT: In this note, we prove that some of recent Rotfel'd type inequalities are equivalent, which is an extension of Huang, Wang and Zhang [*Linear Multilinear Algebra* **66** (2018) 1626–1632]. Among other results, it is shown that if $f:[0,\infty)\to[0,\infty)$ is a concave function and $A\in\mathbb{M}_2(\mathbb{M}_n)$ is a normal matrix with its numerical range contained in a sector: $S_\alpha=\{z\in\mathbb{C}:\operatorname{Re}z\geqslant 0, |\operatorname{Im}z|\leqslant(\operatorname{Re}z)\tan\alpha\}$ for some $\alpha\in[0,\frac{\pi}{2})$, then $\|f(|A|)\|\leqslant 2\left\|f\left(\frac{\sec\alpha}{2}|A_{11}+A_{22}|\right)\right\|$ for any unitarily invariant norm $\|\cdot\|$. This inequality improves a recent result of Zhao and Ni [*Linear Multilinear Algebra* **66** (2018) 410–417].

KEYWORDS: Rotfel'd theorem, concave function, unitarily invariant norm, numerical range

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INTRODUCTION

Throughout this paper, let $\mathbb{M}_n$ be the set of all $n\times n$ complex matrices. For $A\in\mathbb{M}_n$, its singular values are always arranged in decreasing order: $\sigma_1(A)\geqslant\sigma_2(A)\geqslant\cdots\geqslant\sigma_n(A)$. We denote by $\|A\|$ the unitarily invariant norm of A, and $|A|=(A^*A)^{1/2}$. If A is Hermitian, we enumerate eigenvalues of A in non-increasing order: $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$. Note that tr is the usual trace functional. For two Hermitian matrices $A,B\in\mathbb{M}_n$, we use $A\geqslant B$ ($A\leqslant B$) to mean that $A-B$ is a positive (negative) semidefinite matrix. A matrix $A\in\mathbb{M}_n$ is called accretive-dissipative if in its Cartesian (or Toeptliz) decomposition, $A=\operatorname{Re}A+i\operatorname{Im}A$, the matrices $\operatorname{Re}A$ and $\operatorname{Im}A$ are positive semidefinite, where $\operatorname{Re}A=\frac{1}{2}(A+A^*)$, $\operatorname{Im}A=\frac{1}{2i}(A-A^*)$. From Ref. 18 we know, for the Cartesian decomposition of A, that A is normal if and only if $\operatorname{Re}A\operatorname{Im}A=\operatorname{Im}A\operatorname{Re}A$.

The numerical range of $A\in\mathbb{M}_n$ is defined by

$$W(A)=\{x^*Ax: x\in\mathbb{C}^n,\ x^*x=1\}.$$

For $\alpha\in[0,\frac{\pi}{2})$, S_α and S'_α denote, respectively, the sector regions in the complex plane as follows.

$$S_\alpha=\{z\in\mathbb{C}:\operatorname{Re}z\geqslant 0,\ |\operatorname{Im}z|\leqslant(\operatorname{Re}z)\tan\alpha\}$$

and

$$S'_\alpha=\{z\in\mathbb{C}:\operatorname{Re}z\geqslant 0,\ 0\leqslant\operatorname{Im}z\leqslant(\operatorname{Re}z)\tan\alpha\}.$$

Recent studies on matrices with numerical ranges in a sector can be found in Refs. 5, 6, 11–13, 15, 19 and references therein.

Consider a partitioned matrix $A\in\mathbb{M}_n$ in the form

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \tag{1}$$

where A_{11} and A_{22} are square matrices. By $\mathbb{M}_2(\mathbb{M}_n)$ we mean

$$\mathbb{M}_2(\mathbb{M}_n)=\left\{\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}: A_{ij}\in\mathbb{M}_n,\ i,j=1,2\right\}. \tag{2}$$

Similarly, we can define $\mathbb{M}_n(\mathbb{M}_k)$.

In the late 1960s, Rotfel'd proved a famous trace inequality: let $A,B\geqslant 0$ and let f be a non-negative concave function on $[0,\infty)$. Then

$$\operatorname{tr}f(A+B)\leqslant\operatorname{tr}f(A)+\operatorname{tr}f(B).$$

Lee extended the Rotfel'd theorem to a partitioned positive semidefinite matrix[10].

Theorem 1 *Let $A\in\mathbb{M}_n$ be a positive semidefinite matrix partitioned as in (1) and let $f:[0,\infty)\to[0,\infty)$ be a concave function. Then*

$$\|f(A)\|\leqslant\|f(A_{11})\|+\|f(A_{22})\|.$$

As a further extension of the classic Rotfel'd theorem, Zhang[16] extended Theorem 1 to matrices with $W(A)\subseteq S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$ as follows.

Theorem 2 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function and let $A\in\mathbb{M}_n$ with $W(A)\subseteq S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$ be partitioned as in (1). Then*

$$\|f(|A|)\|\leqslant\|f(|A_{11}|)\|+\|f(|A_{22}|)\| + 2\big(\|f(\tan(\alpha)|A_{11}|)\|+\|f(\tan(\alpha)|A_{22}|)\|\big).$$

Later, Fu and Liu[6] obtained another generalization of Theorem 1 as follows.

Theorem 3 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function and let $A\in\mathbb{M}_n$ with $W(A)\subseteq S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$ be partitioned as in (1). Then*

$$\|f(|A|)\|\leqslant\left\|f\left(\sec^2(\alpha)|A_{11}|\right)\right\|+\left\|f\left(\sec^2(\alpha)|A_{22}|\right)\right\|.$$

Hou and Zhang[7] considered the case: $W(A)\subseteq S'_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$. They derived the following result.

Theorem 4 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function and let $A\in\mathbb{M}_n$ with $W(A)\subseteq S'_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$ be partitioned as in (1). Then*

$$\begin{aligned}\|f(|A|)\|\leqslant{}&\|f(|A_{11}|)\|+\|f(|A_{22}|)\|\\&+\|f(\tan(\alpha)|A_{11}|)\|+\|f(\tan(\alpha)|A_{22}|)\|.\end{aligned}$$

Let A be normal and $W(A)\subseteq S_\alpha$, $\alpha\in[0,\frac{\pi}{2})$. Zhao and Ni[17] derived the following result.

Theorem 5 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function and $A\in\mathbb{M}_n$ be normal with $W(A)\subseteq S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$, and let A be partitioned as in (1). Then*

$$\begin{aligned}\|f(|A|)\|\leqslant{}&\|f(|A_{11}|)\|+\|f(|A_{22}|)\|\\&+\|f(\tan(\alpha)|A_{11}|)\|+\|f(\tan(\alpha)|A_{22}|)\|.\end{aligned}$$

Huang et al[8] derived the following inequality.

Theorem 6 *Let $A\in\mathbb{M}_n$ be partitioned as in (1) and let $f:[0,\infty)\to[0,\infty)$ be a concave function. If $A+A^*\geqslant 0$, then*

$$\left\|f\left(\tfrac{A+A^*}{2}\right)\right\|\leqslant\|f(|A_{11}|)\|+\|f(|A_{22}|)\|.$$

Recently, Yang et al[15] presented a new refinement of Rotfel'd type inequality as follows.

Theorem 7 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function and $A\in\mathbb{M}_n$ be normal with $W(A)\subseteq S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$ and let A be partitioned as in (1). Then*

$$\|f(|A|)\|\leqslant\|f(\sec(\alpha)|A_{11}|)\|+\|f(\sec(\alpha)|A_{22}|)\|.$$

Zhao and Ni presented an extension of Rotfel'd theorem as follows[17].

Theorem 8 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function, and $A\in\mathbb{M}_2(\mathbb{M}_n)$ be a positive semidefinite matrix, and let A be partitioned as in (2). Then*

$$\|f(2A)\|\leqslant 2\|f(A_{11})+f(A_{22})\|.$$

In this note, we show that Theorems 1–7 are equivalent, which is an an extension of Huang et al[8]. In addition, we present a new inequality that can be viewed as a generalization of Theorem 8.

MAIN RESULTS

We observe that. If $f:[0,\infty)\to[0,\infty)$ is concave, then

$$0\leqslant A\leqslant B\implies\|f(A)\|\leqslant\|f(B)\|.\tag{3}$$

Before we give the main results, let us present the following lemmas that will be useful later.

Lemma 1 (Ref. 3) *Let $A\in\mathbb{M}_n$. Then*

$$\lambda_j(\mathrm{Re}A)\leqslant\sigma_j(A),\quad j=1,2,\dots,n.$$

The above inequality implies that there exists a unitary matrix $U\in\mathbb{M}_n$ such that

$$\mathrm{Re}A\leqslant U|A|U^*.$$

Zhao and Ni[17] presented a decomposition lemma as follows.

Lemma 2 *Let $A\in\mathbb{M}_2(\mathbb{M}_n)$ be a positive semidefinite matrix, and let A be partitioned as in (2). Then there exist unitary matrices $U,V\in\mathbb{M}_2(\mathbb{M}_n)$ such that*

$$A=\frac{1}{2}\left\{U\begin{bmatrix}A_{11}+A_{22}&0\\0&0\end{bmatrix}U^*+V\begin{bmatrix}0&0\\0&A_{11}+A_{22}\end{bmatrix}V^*\right\}.$$

The next lemma was obtained by Aujla and Bourin[2].

Lemma 3 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function, and $A,B\in\mathbb{M}_n$ be positive semidefinite matrices. Then there exist unitary matrices $U,V\in\mathbb{M}_n$ such that*

$$f(A+B)\leqslant Uf(A)U^*+Vf(B)V^*.$$

Bourin and Lee[4] obtained the following important inequality.

Lemma 4 *Let $A,B\geqslant 0$ and $f:[0,\infty)\to[0,\infty)$ be a concave function. Then*

$$\|f(A+B)\|\leqslant\|f(A)+f(B)\|.$$

The following lemma was obtained by Yang et al[15].

Lemma 5 *Let $A=R+iS$ be the Cartesian decomposition of A with $W(A)\subseteq S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$. If $RS=SR$ (i.e., A is normal), then*

$$|A|\leqslant\sec(\alpha)R.$$

Now we are ready to give the first main result.

Theorem 9 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function and $A\in\mathbb{M}_2(\mathbb{M}_n)$ with $W(A)\subseteq S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$, and let A be partitioned as in (2). If $A=R+iS$ is the Cartesian decomposition of A with $RS=SR$, then*

$$\|f(|A|)\|\leqslant 2\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}+A_{22}|\right)\right\|. \tag{4}$$

Proof: We suppose $f(0)=0$, the general case follows directly by using Lee's approach[10]. Consider the Cartesian decomposition $A=R+iS$, where

$$R=\begin{bmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{bmatrix}\quad\text{and}\quad S=\begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}.$$

As $RS=SR$, It follows from Lemma 5 that $|A|\leqslant\sec(\alpha)R$. This gives

$$\begin{aligned}
|A| &\leqslant \frac{\sec\alpha}{2}\left\{U_1\begin{bmatrix}R_{11}+R_{22} & 0\\ 0 & 0\end{bmatrix}U_1^*+V_1\begin{bmatrix}0 & 0\\ 0 & R_{11}+R_{22}\end{bmatrix}V_1^*\right\}\\
&\leqslant \frac{\sec\alpha}{2}\left\{U_1U_2\begin{bmatrix}|R_{11}+R_{22}+i(S_{11}+S_{22})| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*\right.\\
&\qquad\left.+V_1V_2\begin{bmatrix}0 & 0\\ 0 & |R_{11}+R_{22}+i(S_{11}+S_{22})|\end{bmatrix}V_2^*V_1^*\right\}\\
&= \frac{\sec\alpha}{2}\left\{U_1U_2\begin{bmatrix}|A_{11}+A_{22}| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*\right.\\
&\qquad\left.+V_1V_2\begin{bmatrix}0 & 0\\ 0 & |A_{11}+A_{22}|\end{bmatrix}V_2^*V_1^*\right\}\\
&= U_1U_2\begin{bmatrix}\frac{\sec\alpha}{2}|A_{11}+A_{22}| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*\\
&\qquad+V_1V_2\begin{bmatrix}0 & 0\\ 0 & \frac{\sec\alpha}{2}|A_{11}+A_{22}|\end{bmatrix}V_2^*V_1^*,
\end{aligned}$$

where the first and the second equalities are obtained by Lemma 2 and Lemma 1, with corresponding unitary matrices U_1,V_1, and U_2,V_2, respectively.
By (3), Lemma 3 and the triangle inequality,

$$\begin{aligned}
\|f(|A|)\| &\leqslant \left\|U_3U_1U_2\begin{bmatrix}f\left(\frac{\sec\alpha}{2}|A_{11}+A_{22}|\right) & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*U_3^*\right.\\
&\qquad\left.+V_3V_1V_2\begin{bmatrix}0 & 0\\ 0 & f\left(\frac{\sec\alpha}{2}|A_{11}+A_{22}|\right)\end{bmatrix}V_2^*V_1^*V_3^*\right\|\\
&\leqslant \left\|\begin{bmatrix}f\left(\frac{\sec\alpha}{2}|A_{11}+A_{22}|\right) & 0\\ 0 & 0\end{bmatrix}\right\|\\
&\qquad+\left\|\begin{bmatrix}0 & 0\\ 0 & f\left(\frac{\sec\alpha}{2}|A_{11}+A_{22}|\right)\end{bmatrix}\right\|\\
&= 2\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}+A_{22}|\right)\right\|,
\end{aligned}$$

where U_3, V_3 are unitary matrices in Lemma 3. □

Remark 1 In Theorem 9, we can present another form of (4) as

$$\|f(2|A|)\|\leqslant 2\,\|f\,(\sec(\alpha)|A_{11}+A_{22}|)\|. \tag{5}$$

For a positive semidefinite matrix A, (5) and Lemma 4 give

$$\|f(2A)\|\leqslant 2\|f(A_{11}+A_{22})\|\leqslant 2\|f(A_{11})+f(A_{22})\|.$$

Thus Theorem 9 can be considered as a natural generalization of Theorem 8.

Remark 2 Putting $f(t)=t$ in Theorem 9, we obtain the inequalities

$$\begin{aligned}
\|f(|A|)\| &= \||A|\|\\
&\leqslant 2\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}+A_{22}|\right)\right\|\\
&= \sec(\alpha)\,\||A_{11}+A_{22}|\| = \sec(\alpha)\,\|A_{11}+A_{22}\|\\
&\leqslant \sec(\alpha)\,(\|A_{11}\|+\|A_{22}\|)\\
&= \sec(\alpha)\,(\||A_{11}|\|+\||A_{22}|\|)\\
&= \|f(\sec(\alpha)|A_{11}|)\|+\|f(\sec(\alpha)|A_{22}|)\|.
\end{aligned}$$

Under this condition, Theorem 9 is a refinement of Theorem 7.

We borrow an example from Ref. 15 to show that the equality in (4) may happen.

Example 1 Let $f(t)=t$ be concave and

$$A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix},\quad \alpha\in[0,\frac{\pi}{2}).$$

By simple calculation, we have

$$\sigma_1(A)=\sigma_2(A)=1.$$

Specifying the unitarily invariant norm in this example to the trace norm $\|\cdot\|_{\mathrm{tr}}$. Thus we have $\||A|\|_{\mathrm{tr}}=\sigma_1(A)+\sigma_2(A)=2$ and $\||A_{11}+A_{22}|\|_{\mathrm{tr}}=2\cos\alpha$, which leads to

$$\|f(|A|)\|_{\mathrm{tr}}=\||A|\|_{\mathrm{tr}}=2\left\|\tfrac{\sec\alpha}{2}\,|A_{11}+A_{22}|\right\|_{\mathrm{tr}}=2.$$

Corollary 1 *Let $f:[0,\infty)\to[0,\infty)$ be a concave function, and $A\in\mathbb{M}_2(\mathbb{M}_n)$ with $W(A)\subseteq S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$, and let A be partitioned as in (2). If $A=R+iS$ is the Cartesian decomposition of A with $RS=SR$, then*

$$\|f(|A|)\|\leqslant 2\left(\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}|\right)\right\|+\left\|f\left(\tfrac{\sec\alpha}{2}|A_{22}|\right)\right\|\right).$$

Proof: It follows from Theorem 9 that there exists unitary matrices $U,V\in\mathbb{M}_2(\mathbb{M}_n)$ such that

$$\begin{aligned}
\|f(|A|)\| &\leqslant 2\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}+A_{22}|\right)\right\|\\
&\leqslant 2\left\|f\left(\tfrac{\sec\alpha}{2}(U|A_{11}|U^*+V|A_{22}|V^*)\right)\right\|\\
&\leqslant 2\left\|Uf\left(\tfrac{\sec\alpha}{2}|A_{11}|\right)U^*+Vf\left(\tfrac{\sec\alpha}{2}|A_{22}|\right)V^*\right\|\\
&\leqslant 2\left(\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}|\right)\right\|+\left\|f\left(\tfrac{\sec\alpha}{2}|A_{22}|\right)\right\|\right),
\end{aligned}$$

where the third inequality is from Lemma 4. □

We can also obtain Corollary 1 by utilizing Theorem 7 as follows.

$$\|f(|A|)\| \leqslant \|f(\sec(\alpha)|A_{11}|)\| + \|f(\sec(\alpha)|A_{22}|)\| \leqslant 2\left(\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}|\right)\right\| + \left\|f\left(\tfrac{\sec\alpha}{2}|A_{22}|\right)\right\|\right).$$

Note that matrix A is accretive-dissipative if and only if $W(\mathrm{e}^{-\pi/4}A) \subset S_{\pi/4}$. Let $\alpha = \pi/4$ be in Corollary 1, we obtain

$$\|f(|A|)\| \leqslant 2\left(\left\|f\left(\tfrac{\sqrt{2}}{2}|A_{11}|\right)\right\| + \left\|f\left(\tfrac{\sqrt{2}}{2}|A_{22}|\right)\right\|\right),$$

which coincides with (3.1) in Ref. 16. If we put $\alpha = \pi/4$ in Theorem 9, then

$$\|f(|A|)\| \leqslant 2\left\|f\left(\tfrac{\sqrt{2}}{2}|A_{11} + A_{22}|\right)\right\|.$$

We give a refinement of Corollary 1 without the normality assumption on A in the following theorem.

Theorem 10 *Let $f : [0,\infty) \to [0,\infty)$ be a concave function, and $A \in \mathbb{M}_n$ with $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$, and let A be partitioned as in (1). Then*

$$\|f(|A|)\| \leqslant 2\left(\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}|\right)\right\| + \left\|f\left(\tfrac{\sec\alpha}{2}|A_{22}|\right)\right\|\right).$$

Proof: Let $A = U|A|$ be the polar decomposition of A, and $A = R + iS$ be the Cartesian decomposition of A with R, S being partitioned as in (1). Thus by Ref. 4, there exist unitary matrices U_1, V_1 such that

$$R = \left[U_1\begin{bmatrix}R_{11} & 0\\ 0 & 0\end{bmatrix}U_1^* + V_1\begin{bmatrix}0 & 0\\ 0 & R_{22}\end{bmatrix}V_1^*\right],$$

which gives

$$\begin{aligned}
|A| &\leqslant \frac{\sec\alpha}{2}(R + U^*RU)\\
&= \frac{\sec\alpha}{2}\left\{U_1\begin{bmatrix}R_{11} & 0\\ 0 & 0\end{bmatrix}U_1^* + V_1\begin{bmatrix}0 & 0\\ 0 & R_{22}\end{bmatrix}V_1^*\right.\\
&\quad \left. + U^*U_1\begin{bmatrix}R_{11} & 0\\ 0 & 0\end{bmatrix}U_1^*U + U^*V_1\begin{bmatrix}0 & 0\\ 0 & R_{22}\end{bmatrix}V_1^*U\right\}\\
&\leqslant \frac{\sec\alpha}{2}\left\{U_1U_2\begin{bmatrix}|R_{11}+iS_{11}| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*\right.\\
&\quad + V_1V_2\begin{bmatrix}0 & 0\\ 0 & |R_{22}+iS_{22}|\end{bmatrix}V_2^*V_1^*\\
&\quad + U^*U_1U_2\begin{bmatrix}|R_{11}+iS_{11}| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*U\\
&\quad \left. + U^*V_1V_2\begin{bmatrix}0 & 0\\ 0 & |R_{22}+iS_{22}|\end{bmatrix}V_2^*V_1^*U\right\}\\
&= \frac{\sec\alpha}{2}\left\{U_1U_2\begin{bmatrix}|A_{11}| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^* + V_1V_2\begin{bmatrix}0 & 0\\ 0 & |A_{22}|\end{bmatrix}V_2^*V_1^*\right.\\
&\quad \left. + U^*U_1U_2\begin{bmatrix}|A_{11}| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*U + U^*V_1V_2\begin{bmatrix}0 & 0\\ 0 & |A_{22}|\end{bmatrix}V_2^*V_1^*U\right\}\\
&= U_1U_2\begin{bmatrix}\frac{\sec\alpha}{2}|A_{11}| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^* + V_1V_2\begin{bmatrix}0 & 0\\ 0 & \frac{\sec\alpha}{2}|A_{22}|\end{bmatrix}V_2^*V_1^*\\
&\quad + U^*U_1U_2\begin{bmatrix}\frac{\sec\alpha}{2}|A_{11}| & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*U\\
&\quad + U^*V_1V_2\begin{bmatrix}0 & 0\\ 0 & \frac{\sec\alpha}{2}|A_{22}|\end{bmatrix}V_2^*V_1^*U,
\end{aligned}$$

where the first inequality is obtained by the previous equality[1] and the second inequality is obtained by Lemma 1 with unitary matrices U_1, V_1, and U_2, V_2, respectively.

By (3) and Lemma 3, we have

$$\begin{aligned}
\|f(|A|)\| \leqslant \Bigg\| & U_3U_1U_2\begin{bmatrix}f\left(\frac{\sec\alpha}{2}|A_{11}|\right) & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*U_3^*\\
& + V_3V_1V_2\begin{bmatrix}0 & 0\\ 0 & f\left(\frac{\sec\alpha}{2}|A_{22}|\right)\end{bmatrix}V_2^*V_1^*V_3^*\\
& + U_3U^*U_1U_2\begin{bmatrix}f\left(\frac{\sec\alpha}{2}|A_{11}|\right) & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*UU_3^*\\
& + V_3U^*V_1V_2\begin{bmatrix}0 & 0\\ 0 & f\left(\frac{\sec\alpha}{2}|A_{22}|\right)\end{bmatrix}V_2^*V_1^*UV_3^*\Bigg\|,
\end{aligned}$$

$$\begin{aligned}
\|f(|A|)\| &\leqslant 2\left\|\begin{bmatrix}f\left(\frac{\sec\alpha}{2}|A_{11}|\right) & 0\\ 0 & 0\end{bmatrix}\right\| + 2\left\|\begin{bmatrix}0 & 0\\ 0 & f\left(\frac{\sec\alpha}{2}|A_{22}|\right)\end{bmatrix}\right\|\\
&= 2\left(\left\|f\left(\tfrac{\sec\alpha}{2}|A_{11}|\right)\right\| + \left\|f\left(\tfrac{\sec\alpha}{2}|A_{22}|\right)\right\|\right),
\end{aligned}$$

in which U_3, V_3 correspond to the unitary matrices in Lemma 3. □

Letting $f(t) = t$ in Theorem 10, we obtain the following corollary.

Corollary 2 *Let A be partitioned as in (2) with $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$. If A is normal, then*

$$\|A\| \leqslant \sec(\alpha)\,\|A_{11} + A_{22}\|. \tag{6}$$

Next we shall extend inequality (6) to a higher number of blocks. First of all, let us introduce some relevant conceptions.

A matrix $A = (A_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ is said to be positive partial transpose (PPT) if A is positive semidefinite, and its partial transpose $A^\tau = (A_{ji})_{i,j=1}^n$ is also positive semidefinite.

In Ref. 9, Kuai defined a new conception called sectorial partial transpose (SPT). A matrix $A = (A_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ is said to be SPT if $W(A) \subseteq S_\alpha$ and $W(A^\tau) \subseteq S_\alpha$.

Lemma 6 (Ref. 9) *If A is SPT, then* ReA *is PPT.*

Lemma 7 (Ref. 19) *Let $A \in \mathbb{M}_n$ be such that $W(A) \subseteq S_\alpha$. Then*

$$\|A\| \leqslant \sec(\alpha)\,\|\mathrm{Re}A\|.$$

Lemma 8 (Ref. 14) *Let $A = (A_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ be a PPT matrix. Then*

$$\|A\| \leqslant \left\| \sum_{i=1}^n A_{ii} \right\|.$$

We note that the following theorem is an extension of Corollary 2 and Lemma 8 to sector matrices.

Theorem 11 *Let $A = (A_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ be an SPT matrix. Then*

$$\|A\| \leqslant \sec(\alpha) \left\| \sum_{i=1}^n A_{ii} \right\|.$$

Proof: As A is SPT, we obtain by Lemma 6 that ReA is PPT. Hence we have

$$\begin{aligned}
\|A\| &\leqslant \sec(\alpha)\|\mathrm{Re}A\| && \text{(by Lemma 7)}\\
&\leqslant \sec(\alpha) \left\| \sum_{i=1}^n \mathrm{Re}A_{ii} \right\| && \text{(by Lemma 8)}\\
&= \sec(\alpha) \left\| \mathrm{Re}\left(\sum_{i=1}^n A_{ii} \right) \right\| \leqslant \sec(\alpha) \left\| \sum_{i=1}^n A_{ii} \right\|.
\end{aligned}$$

□

Next we give our second main result, which proves the equivalence of some recent Rotfel'd type theorems.

Theorem 12 *Let $f : [0, \infty) \to [0, \infty)$ be a concave function and $A \in \mathbb{M}_n$ be partitioned as in (1). The following statements are equivalent.*

(a) If $A \in \mathbb{M}_n$ be a positive semidefinite matrix, then [10]

$$\|f(A)\| \leqslant \|f(A_{11})\| + \|f(A_{22})\|.$$

(b) If $A + A^* \geqslant 0$, then [8]

$$\left\| f\left(\tfrac{A+A^*}{2}\right) \right\| \leqslant \|f(|A_{11}|)\| + \|f(|A_{22}|)\|.$$

(c) If $W(A) \subseteq S'_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$, then [7]

$$\begin{aligned}
\|f(|A|)\| &\leqslant \|f(|A_{11}|)\| + \|f(|A_{22}|)\|\\
&\quad + \|f(\tan(\alpha)|A_{11}|)\| + \|f(\tan(\alpha)|A_{22}|)\|.
\end{aligned}$$

(d) If A is normal and $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$, then [17]

$$\begin{aligned}
\|f(|A|)\| &\leqslant \|f(|A_{11}|)\| + \|f(|A_{22}|)\|\\
&\quad + \|f(\tan(\alpha)|A_{11}|)\| + \|f(\tan(\alpha)|A_{22}|)\|.
\end{aligned}$$

(e) If $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$, then [19]

$$\begin{aligned}
\|f(|A|)\| &\leqslant \|f(|A_{11}|)\| + \|f(|A_{22}|)\|\\
&\quad + 2(\|f(\tan(\alpha)|A_{11}|)\| + \|f(\tan(\alpha)|A_{22}|)\|).
\end{aligned}$$

(f) If A is normal and $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$, then [15]

$$\|f(|A|)\| \leqslant \|f(\sec(\alpha)|A_{11}|)\| + \|f(\sec(\alpha)|A_{22}|)\|.$$

(g) If $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$, then [6]

$$\|f(|A|)\| \leqslant \left\| f(\sec^2(\alpha)|A_{11}|) \right\| + \left\| f(\sec^2(\alpha)|A_{22}|) \right\|.$$

Proof: The equivalence from (a)–(e) was shown by Huang et al[8], we thus only need to prove (b) $\Longrightarrow$ (f), (b) $\Longrightarrow$ (g), (f) $\Longrightarrow$ (a), and (g) $\Longrightarrow$ (a).

(b) $\Longrightarrow$ (f): Consider the Cartesian decomposition $A = R + iS$. It follows from Lemma 5 that

$$|A| \leqslant \sec(\alpha) R.$$

Then, by (3) and (b), we have

$$\begin{aligned}
\|f(|A|)\| &\leqslant \|f(\sec(\alpha)R)\|\\
&= \left\| f\left(\frac{(\sec(\alpha)A) + (\sec(\alpha)A)^*}{2} \right) \right\|\\
&\leqslant \|f(\sec(\alpha)|A_{11}|)\| + \|f(\sec(\alpha)|A_{22}|)\|.
\end{aligned}$$

(b) $\Longrightarrow$ (g): For the Cartesian decomposition $A = R + iS$, it follows from Ref. 5 that there exists a unitary matrix $U \in \mathbb{M}_n$ such that

$$|A| \leqslant \sec^2(\alpha)\, URU^*,$$

and the rest of the proof is the same as above.

(f) $\Longrightarrow$ (a): For a positive semidefinite matrix A, we have $\alpha = 0$ in (f), which implies (a) directly. Similarly, we obtain (g) $\Longrightarrow$ (a). □

Apparently, Theorem 12 is an extension of Huang et al[8] (Theorem 3.1).

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