Two inequalities of unitarily invariant norms for matrices

Xuesha Wu

School of General and International Education, Chongqing College of Electronic Engineering, Chongqing 401331, China

e-mail: xuesha_wu@163.com

ABSTRACT: In this paper, we present two inequalities of matrix norms. The first one is a generalization of the inequality shown in [J Math Inequal 10 (2016) 1119–1122], and the second one is a refinement of an inequality obtained by Zou [Numer Math J Chinese Univ 38 (2016) 343–349].

KEYWORDS: arithmetic-geometric mean inequality, Kantorovich constant, unitarily invariant norms

MSC2010: 15A60 47A63

INTRODUCTION

Let $M_n$ be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ be any unitarily invariant norm on $M_n$ and suppose that $s_n(A) \leq \cdots \leq s_1(A)$ are the singular values of $A$, which are eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in ascending order and repeated according to multiplicity.

Let $A, B \in M_n$ be positive semidefinite. Bhatia and Kittaneh proved in Ref. 1 that

$$\|AB\| \leq \frac{1}{2}\|A + B\|^2. \quad (1)$$

This is an arithmetic-geometric mean inequality for unitarily invariant norms\(^5\). During the past ten years, several authors discussed inequalities related to arithmetic-geometric mean, for example, see Refs. 3, 4.

Let $A, B \in M_n$ be positive semidefinite and $\alpha \in (0, 1)$. Zou and Jiang proved in Ref. 5 that

$$\|AB\|^2 \leq \frac{1}{4\alpha(1-\alpha)} \|\alpha A + (1-\alpha)B\|^2 \times \|((1-\alpha)A + \alpha B)^2\|, \quad (2)$$

which is a generalization of inequality (1). Let $A, B \in M_{n, r}$. Lee proved in Ref. 6 that

$$\|A + B\|_F \leq \sqrt{\frac{1}{4}} \|A + B\|_F, \quad (3)$$

where $\|X\|_F$ is the Frobenius norm of $X$.

Let $A, B \in M_n$ and $A, B \neq 0$. Zou proved in Ref. 7 that

$$\|A + B\|_F \leq \left[2 - \frac{S(\|B\|_F/\|A\|_F) - 1}{S(\|B\|_F^2/\|A\|_F^2)}\right]^{1/4} \|A + B\|_F, \quad (4)$$

where $S(t) = t^{1/(t-1)} / e \log t^{1/(t-1)}$, $t > 0$, $S(1) = \lim_{t \to 1} S(t) = 1$ is Spech's ratio\(^8, 9\). It was proved in Ref. 10 that $S(\|B\|_F/\|A\|_F) \geq 1$, so we know that inequality (4) is a refinement of inequality (3).

In this short note, we obtain a generalization of inequality (2) and we also present an improvement of inequality (4).

MAIN RESULTS

We first show some lemmas used in our proof.

Lemma 1 (Ref. 11) Let $A, X, B \in M_n, 1/p + 1/q = 1, p, q > 1, \alpha \in [0, 1]$. If $r \geq \max\{1/p, 1/q\}$, then

$$\|A^*XB\|^2 \leq \|T_\alpha(\alpha)\|^p \|T_\alpha(1-\alpha)\|^q, \quad (5)$$

where

$$T_\alpha(\alpha) = \alpha AA^*X + (1-\alpha)XBB^*.$$

Lemma 2 (Ref. 1) Let $A, B \in M_n$ be positive semidefinite. Then

$$s_j(A^{1/2}(A+B)B^{1/2}) \leq \frac{1}{4} s_j(A+B)^2, \quad j = 1, \ldots, n.$$

Lemma 3 (Ref. 6) Let $A, B \in M_n$. Then

$$\|A + B\| \leq \|A\| + \|B\|^{1/2}\|A^*\| + \|B^*\|^{1/2}. \quad (6)$$

Theorem 1 Let $A, B \in M_n$ be positive semidefinite and suppose that $1/p + 1/q = 1, p, q > 1, \alpha \in (0, 1)$. If $r \geq \max\{1/p, 1/q\}$, then

$$\|AB\|^2 \leq \left[\frac{1}{4\alpha(1-\alpha)}\right]^{1/2} \|\alpha A + (1-\alpha)B\|^{2r} \times \|((1-\alpha)A + \alpha B)^{2r}\|^{1/4}. \quad (7)$$
Proof: Replacing \( A, B, X \) in (5) with \( A^{1/2}, B^{1/2}, A^{1/2}B^{1/2} \), respectively, we have

\[
\|AB^2\| \leq \left\| AA^{3/2}B^{1/2} + (1-\alpha)A^{1/2}B^{3/2}\right\|_p^{1/p}
\times \left\| (1-\alpha)B^{1/2} + \alpha A^{1/2}B^{3/2}\right\|_q^{1/q}
= \|A^{1/2}Q(\alpha)B^{1/2}\|_p^{1/p}
\times \|A^{1/2}Q(1-\alpha)B^{1/2}\|_q^{1/q},
\]

where

\[ Q(\alpha) = \alpha A +(1-\alpha)B. \]

By Lemma 2 with \( A = \alpha A \) and \( B = (1-\alpha)B \), we obtain for \( j = 1, \ldots, n \),

\[ s_j(A^{1/2}Q(\alpha)B^{1/2}) \leq \frac{1}{2\sqrt{\alpha (1-\alpha)}} s_j(Q^2(\alpha)). \]

Thus, for \( k = 1, \ldots, n \),

\[
\sum_{j=1}^k s_j\left(\|A^{1/2}Q(\alpha)B^{1/2}\|_p^p\right) \leq \left[\frac{1}{2\sqrt{\alpha (1-\alpha)}}\right]^p \sum_{j=1}^k s_j(Q^{2p}(\alpha)),
\]

which implies

\[
\|A^{1/2}Q(\alpha)B^{1/2}\|_p^p \leq \left[\frac{1}{2\sqrt{\alpha (1-\alpha)}}\right]^p \|Q^{2p}(\alpha)\|_p^p.
\]

Then

\[
\|A^{1/2}Q(\alpha)B^{1/2}\|_q^q \leq \left[\frac{1}{2\sqrt{\alpha (1-\alpha)}}\right]^q \|Q^{2q}(1-\alpha)\|_q^q.
\]

Similarly, we have

\[
\|A^{1/2}Q(1-\alpha)B^{1/2}\|_q^q \leq \left[\frac{1}{2\sqrt{\alpha (1-\alpha)}}\right]^q \|Q^{2q}(1-\alpha)\|_q^q.
\]

It follows from (7), (8) and (9) that

\[
\|AB\|^{2r} \leq \left[\frac{1}{4\alpha (1-\alpha)}\right]^{r} \|Q^{2r}(\alpha)\|_p^{1/p}
\times \|Q^{2r}(1-\alpha)\|_q^{1/q}.
\]

\[ \square \]

Remark 1 Setting \( p = q = 2, r = 1/2 \) in (6), we obtain inequality (2).
Since $\text{tr}|A||B| \geq 0$, inequality (15) implies
\[
\left\| |A^*| + |B^*| \right\|_F \leq \left[ 1 + \frac{1}{K^{1/2}} \left( \frac{\|B\|_F^2}{\|A\|_F^2} \right) \right]^{1/2} \| |A| + |B| \|_F. \tag{16}
\]
Lemma 3 and (16) complete the proof.

Remark 2 Let $x > 0$, $s \in [0, 1/2]$. It was pointed out in Ref. 12 that $S(x') \leq K^s(x)$. Hence we have
\[
S \left( \frac{\|B\|_F}{\|A\|_F} \right) \leq K^{1/2} \left( \frac{\|B\|_F^2}{\|A\|_F^2} \right)^{1/2},
\]
which implies
\[
1 + \frac{1}{S \left( \frac{\|B\|_F}{\|A\|_F} \right)} \geq 1 + \frac{1}{K^{1/2} \left( \frac{\|B\|_F^2}{\|A\|_F^2} \right)}.
\]
On the other hand, by small calculations, we obtain
\[
2 - \frac{S \left( \frac{\|B\|_F}{\|A\|_F} \right) - 1}{S \left( \frac{\|B\|_F}{\|A\|_F} \right)} = \left( 1 + \frac{1}{S \left( \frac{\|B\|_F}{\|A\|_F} \right)} \right)^{-1} = 1 - \frac{1}{S \left( \frac{\|B\|_F}{\|A\|_F} \right)} = \frac{S \left( \frac{\|B\|_F}{\|A\|_F} \right) - S \left( \frac{\|B\|_F}{\|A\|_F} \right)}{S \left( \frac{\|B\|_F}{\|A\|_F} \right)} \geq 0.
\]
It follows that
\[
2 - \frac{S \left( \frac{\|B\|_F}{\|A\|_F} \right) - 1}{S \left( \frac{\|B\|_F}{\|A\|_F} \right)} \geq 1 + \frac{1}{K^{1/2} \left( \frac{\|B\|_F^2}{\|A\|_F^2} \right)}.
\]
\begin{align*}
\text{Acknowledgements:} & \quad \text{The author wishes to express her heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript.} \\
& \quad \text{11. Zou L (2019) Unification of the arithmetic-geometric mean and Hölder inequalities for unitarily invariant norms. Linear Algebra Appl 562, 154–162.} \\
\end{align*}