SHORT REPORT
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Two inequalities of unitarily invariant norms for matrices

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ABSTRACT: In this paper, we present two inequalities of matrix norms. The first one is a generalization of the inequality shown in [J Math Inequal 10 (2016) 1119–1122], and the second one is a refinement of an inequality obtained by Zou [Numer Math J Chinese Univ 38 (2016) 343–349].

KEYWORDS: arithmetic-geometric mean inequality, Kantorovich constant, unitarily invariant norms

MSC2010: 15A60 47A63

INTRODUCTION

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ be any unitarily invariant norm on M_n and suppose that $s_n(A) \leqslant \cdots \leqslant s_1(A)$ are the singular values of A, which are eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in ascending order and repeated according to multiplicity.

Let $A, B \in M_n$ be positive semidefinite. Bhatia and Kittaneh proved in Ref. 1 that

$$\|AB\| \leqslant \tfrac{1}{4}\|(A+B)^2\|. \quad (1)$$

This is an arithmetic-geometric mean inequality for unitarily invariant norms[2]. During the past ten years, several authors discussed inequalities related to arithmetic-geometric mean, for example, see Refs. 3, 4.

Let $A, B \in M_n$ be positive semidefinite and $\alpha \in (0,1)$. Zou and Jiang proved in Ref. 5 that

$$\|AB\|^2 \leqslant \frac{1}{4\alpha(1-\alpha)} \left\|(\alpha A + (1-\alpha)B)^2\right\| \times \left\|((1-\alpha)A + \alpha \mathrm{B})^2\right\|, \quad (2)$$

which is a generalization of inequality (1).

Let $A, B \in M_n$. Lee proved in Ref. 6 that

$$\|A+B\|_{\mathrm{F}} \leqslant 2^{1/4}\big\||A|+|B|\big\|_{\mathrm{F}}, \quad (3)$$

where $\|X\|_{\mathrm{F}}$ is the Frobenius norm of X.

Let $A, B \in M_n$ and $A, B \neq 0$. Zou proved in Ref. 7 that

$$\|A+B\|_{\mathrm{F}} \leqslant \left[2 - \frac{S\left(\|B\|_{\mathrm{F}}/\|A\|_{\mathrm{F}}\right)-1}{S\left(\|B\|_{\mathrm{F}}^2/\|A\|_{\mathrm{F}}^2\right)}\right]^{1/4} \big\||A|+|B|\big\|_{\mathrm{F}}, \quad (4)$$

where $S(t) = t^{1/(t-1)}/e\log t^{1/(t-1)}$, $t>0$, $S(1) = \lim_{t\to 1} S(t) = 1$ is Specht's ratio[8,9]. It was proved in Ref. 10 that $S(\|B\|_{\mathrm{F}}/\|A\|_{\mathrm{F}}) \geqslant 1$, so we know that inequality (4) is a refinement of inequality (3).

In this short note, we obtain a generalization of inequality (2) and we also present an improvement of inequality (4).

MAIN RESULTS

We first show some lemmas used in our proof.

Lemma 1 (Ref. 11) *Let $A, X, B \in M_n$, $1/p+1/q=1$, $p,q>1$, $\alpha \in [0,1]$. If $r \geqslant \max\{1/p, 1/q\}$, then*

$$\big\||A^*XB|^{2r}\big\| \leqslant \big\||T_X(\alpha)|^{rp}\big\|^{1/p}\big\||T_X(1-\alpha)|^{rq}\big\|^{1/q}, \quad (5)$$

where

$$T_X(\alpha) = \alpha AA^*X + (1-\alpha)XBB^*.$$

Lemma 2 (Ref. 1) *Let $A, B \in M_n$ be positive semidefinite. Then*

$$s_j(A^{1/2}(A+B)B^{1/2}) \leqslant \tfrac{1}{2}s_j(A+B)^2,\ j=1,\ldots,n.$$

Lemma 3 (Ref. 6) *Let $A, B \in M_n$. Then*

$$\|A+B\| \leqslant \big\||A|+|B|\big\|^{1/2}\big\||A^*|+|B^*|\big\|^{1/2}.$$

Theorem 1 *Let $A, B \in M_n$ be positive semidefinite and suppose that $1/p+1/q=1$, $p,q>1$, $\alpha \in (0,1)$. If $r \geqslant \max\{1/p, 1/q\}$, then*

$$\big\||AB|^{2r}\big\| \leqslant \left[\frac{1}{4\alpha(1-\alpha)}\right]^r \big\|(\alpha A+(1-\alpha)B)^{2rp}\big\|^{1/p} \times \big\|((1-\alpha)A+\alpha B)^{2rq}\big\|^{1/q}. \quad (6)$$

Proof: Replacing A,B,X in (5) with $A^{1/2}$, $B^{1/2}$, $A^{1/2}B^{1/2}$, respectively, we have

$$\begin{aligned}\left\|\,|AB|^{2r}\right\| &\leqslant \left\|\,\left|\alpha A^{3/2}B^{1/2}+(1-\alpha)A^{1/2}B^{3/2}\right|^{rp}\right\|^{1/p}\\ &\quad\times\left\|\,\left|(1-\alpha)A^{3/2}B^{1/2}+\alpha A^{1/2}B^{3/2}\right|^{rq}\right\|^{1/q}\\ &=\left\|\,\left|A^{1/2}Q(\alpha)B^{1/2}\right|^{rp}\right\|^{1/p}\\ &\quad\times\left\|\,\left|A^{1/2}Q(1-\alpha)B^{1/2}\right|^{rq}\right\|^{1/q}, \qquad (7)\end{aligned}$$

where

$$Q(\alpha)=\alpha A+(1-\alpha)B.$$

By Lemma 2 with $A=\alpha A$ and $B=(1-\alpha)B$, we obtain for $j=1,\dots,n$,

$$s_j\big(A^{1/2}Q(\alpha)B^{1/2}\big)\leqslant\frac{1}{2\sqrt{\alpha(1-\alpha)}}s_j(Q^2(\alpha)).$$

Thus, for $k=1,\dots,n$,

$$\sum_{j=1}^{k}s_j\left(\left|A^{1/2}Q(\alpha)B^{1/2}\right|^{rp}\right)\leqslant\left[\frac{1}{2\sqrt{\alpha(1-\alpha)}}\right]^{rp}\sum_{j=1}^{k}s_j(Q^{2rp}(\alpha)),$$

which implies

$$\left\|\,\left|A^{1/2}Q(\alpha)B^{1/2}\right|^{rp}\right\|\leqslant\left[\frac{1}{2\sqrt{\alpha(1-\alpha)}}\right]^{rp}\left\|Q^{2rp}(\alpha)\right\|.$$

Then

$$\left\|\,\left|A^{1/2}Q(\alpha)B^{1/2}\right|^{rp}\right\|^{1/p}\leqslant\left[\frac{1}{2\sqrt{\alpha(1-\alpha)}}\right]^{r}\left\|Q^{2rp}(\alpha)\right\|^{1/p}. \qquad (8)$$

Similarly, we have

$$\left\|\,\left|A^{1/2}Q(1-\alpha)B^{1/2}\right|^{rq}\right\|^{1/q}\leqslant\left[\frac{1}{2\sqrt{\alpha(1-\alpha)}}\right]^{r}\left\|Q^{2rq}(1-\alpha)\right\|^{1/q}. \qquad (9)$$

It follows from (7), (8) and (9) that

$$\left\|\,|AB|^{2r}\right\|\leqslant\left[\frac{1}{4\alpha(1-\alpha)}\right]^{r}\left\|Q^{2rp}(\alpha)\right\|^{1/p}\times\left\|Q^{2rq}(1-\alpha)\right\|^{1/q}.$$

□

Remark 1 Setting $p=q=2$, $r=1/2$ in (6), we obtain inequality (2).

Theorem 2 *Let $A,B\in M_n$ and $A,B\neq 0$. Then*

$$\|A+B\|_{\mathrm F}\leqslant\left[1+\frac{1}{K^{1/2}\left(\|B\|_{\mathrm F}^2/\|A\|_{\mathrm F}^2\right)}\right]^{1/4}\left\|\,|A|+|B|\,\right\|_{\mathrm F}, \qquad (10)$$

where $K(x)=(1+x)^2/4x$, $x>0$ is Kantorovich constant[12].

Proof: By definition of inner product of matrices and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\operatorname{tr}|A^*||B^*|=(|A^*|,|B^*|)&\leqslant(|A^*|,|A^*|)^{1/2}(|B^*|,|B^*|)^{1/2}\\ &=\|A\|_{\mathrm F}\|B\|_{\mathrm F}. \qquad (11)\end{aligned}$$

Note that

$$2K^{1/2}\left(\frac{\|B\|_{\mathrm F}^2}{\|A\|_{\mathrm F}^2}\right)\|A\|_{\mathrm F}\|B\|_{\mathrm F}=\|A\|_{\mathrm F}^2+\|B\|_{\mathrm F}^2. \qquad (12)$$

It follows from (11) and (12) that

$$2\operatorname{tr}|A^*||B^*|+2\left(K^{1/2}\left(\frac{\|B\|_{\mathrm F}^2}{\|A\|_{\mathrm F}^2}\right)-1\right)\|A\|_{\mathrm F}\|B\|_{\mathrm F}\leqslant\|A\|_{\mathrm F}^2+\|B\|_{\mathrm F}^2,$$

which is equivalent to

$$\left\|\,|A^*|+|B^*|\,\right\|_{\mathrm F}\leqslant\left\{2\left\|\,|A|+|B|\,\right\|_{\mathrm F}^2-2\left[K^{1/2}\left(\frac{\|B\|_{\mathrm F}^2}{\|A\|_{\mathrm F}^2}\right)-1\right]\|A\|_{\mathrm F}\|B\|_{\mathrm F}-4\operatorname{tr}|A|\,|B|\right\}^{1/2}. \qquad (13)$$

Meanwhile, we also have

$$\begin{aligned}\|A\|_{\mathrm F}\|B\|_{\mathrm F}&=\frac{1}{2K^{1/2}\left(\frac{\|B\|_{\mathrm F}^2}{\|A\|_{\mathrm F}^2}\right)}\times\left[\|A\|_{\mathrm F}^2+\|B\|_{\mathrm F}^2+2\operatorname{tr}|A|\,|B|-2\operatorname{tr}|A|\,|B|\right]\\ &=\frac{1}{2K^{1/2}\left(\frac{\|B\|_{\mathrm F}^2}{\|A\|_{\mathrm F}^2}\right)}\left\|\,|A|+|B|\,\right\|_{\mathrm F}^2-\frac{1}{K^{1/2}\left(\frac{\|B\|_{\mathrm F}^2}{\|A\|_{\mathrm F}^2}\right)}\operatorname{tr}|A|\,|B|. \qquad (14)\end{aligned}$$

It follows from (13) and (14) that

$$\left\|\,|A^*|+|B^*|\,\right\|_{\mathrm F}\leqslant\left\{\left(1+\frac{1}{K^{1/2}\left(\frac{\|B\|_{\mathrm F}^2}{\|A\|_{\mathrm F}^2}\right)}\right)\times\left(\left\|\,|A|+|B|\,\right\|_{\mathrm F}^2-2\operatorname{tr}|A|\,|B|\right)\right\}^{1/2}. \qquad (15)$$

Since $\mathrm{tr}|A||B| \geqslant 0$, inequality (15) implies

$$\big\||A^*|+|B^*|\big\|_{\mathrm{F}} \leqslant \left[1+\frac{1}{K^{1/2}\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)}\right]^{1/2} \big\||A|+|B|\big\|_{\mathrm{F}}. \quad (16)$$

Lemma 3 and (16) complete the proof. □

Remark 2 Let $x > 0$, $s \in [0, 1/2]$. It was pointed out in Ref. 12 that $S(x^s) \leqslant K^s(x)$. Hence we have

$$S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right) \leqslant K^{1/2}\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)$$

which implies

$$1+\frac{1}{S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)} \geqslant 1+\frac{1}{K^{1/2}\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)}.$$

On the other hand, by small calculations, we obtain

$$\begin{aligned} 2-\frac{S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)-1}{S\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)} - \left(1+\frac{1}{S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)}\right) &= 1-\frac{S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)-1}{S\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)} - \frac{1}{S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)} \\ &= \frac{\left(S\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)-S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)\right)\left(S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)-1\right)}{S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)S\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)} \\ &\geqslant 0. \end{aligned}$$

It follows that

$$2-\frac{S\left(\frac{\|B\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\right)-1}{S\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)} \geqslant 1+\frac{1}{K^{1/2}\left(\frac{\|B\|_{\mathrm{F}}^2}{\|A\|_{\mathrm{F}}^2}\right)},$$

thus inequality (10) is a refinement of (4).

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