A note on differential-difference analogue of Brück conjecture

Minfeng Chen\textsuperscript{a,}\textsuperscript{*}, Ning Cui\textsuperscript{b}

\textsuperscript{a} School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou 510006, China
\textsuperscript{b} College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, China

\textsuperscript{*}Corresponding author, e-mail: chenminfeng198710@126.com

ABSTRACT: In this paper, we prove that for a transcendental entire function \( f(z) \) of finite order, \( \eta \in \mathbb{C}\setminus\{0\} \) is a constant such that \( \Delta_\eta f(z) = f(z + \eta) - f(z) \neq 0 \), \( b(z) \) is an entire function such that \( \sigma(b) < \sigma(f) \) and \( \lambda(f - b) < \sigma(f) \), if \( \Delta_\eta f(z) \) and \( f'(z) \) share \( a(z) \) CM, where \( a(z) \) is an entire function satisfying \( \sigma(a) < \sigma(f) \), then

\[
\frac{\Delta_\eta f(z) - a(z)}{f'(z) - a(z)} = A, \quad f(z) = b(z) + H(z)e^{cz},
\]

where \( a(z) \) and \( b(z) \) are entire functions with \( \max(\sigma(a), \sigma(b)) < 1 \), \( H(z)(\neq 0) \) is an entire function with \( \lambda(H) = \sigma(H) < 1 \) and \( A, c, \eta \in \mathbb{C}\setminus\{0\} \) are constants satisfying \( e^\eta = 1 + Ac \). Our results are improvements and complements of those in [Bull Korean Math Soc 51 (2014) 1453–1467] and [Commun Korean Math Soc 32 (2017) 361–373].

KEYWORDS: entire function, sharing value, differential-difference equation

MSC2010: 39B32 34M05 30D35

INTRODUCTION

In this paper, we assume that the reader is familiar with the standard symbols and the fundamental results of Nevanlinna theory\textsuperscript{1–3}. In addition, we use the notations \( \lambda(f) \) and \( \sigma(f) \) to denote the exponent of convergence of the zero sequence and the order of growth of meromorphic function \( f(z) \), respectively. We also denote by \( S(r, f) \) any quantity satisfying \( S(r, f) = o(T(r, f)) \), as \( r \to \infty \), outside of a possible exceptional set of finite logarithmic measure. For convenience, we need the following definition.

Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions, and let \( a \) be a constant in the complex plane. We say that \( f(z) \) and \( g(z) \) share \( a \) CM (IM) provided that \( f(z) - a \) and \( g(z) - a \) have the same zeros counting multiplicities (ignoring multiplicities), and \( f(z) \) and \( g(z) \) share \( \infty \) CM (IM) provided that \( f(z) \) and \( g(z) \) have the same poles counting multiplicities (ignoring multiplicities). Using the same method, we can also define \( f(z) \) and \( g(z) \) share function \( a(z) \) CM (IM), where \( a(z) \in S(r, f) \cap S(r, g) \).

**Definition 1** (Ref. 4) Let \( f(z) \) be a meromorphic function in the complex plane. We denote by \( \sigma_2(f) \) the order of \( \log T(r, f) \), i.e.,

\[
\sigma_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.
\]

\( \sigma_2(f) \) is called the hyper-order of \( f(z) \).

Brück\textsuperscript{5} raised the following conjecture.

**Conjecture** (Ref. 5) Let \( f(z) \) be a non-constant entire function with hyper-order \( \sigma_2(f) < \infty \), and \( \sigma_2(f) \notin \mathbb{Z}^+ \). If \( f(z) \) and \( f'(z) \) share a finite value \( a \) CM, then

\[
\frac{f'(z) - a}{f(z) - a} = c,
\]

where \( c \) is a non-zero constant.

The conjecture has been established in the special case\textsuperscript{5} when \( a = 0 \) or when \( f(z) \) is an entire function of finite order\textsuperscript{5}.

Recently, many results on difference analogues of Brück conjecture were considered in Refs. 7–12. To start with, recall the following results.

**Theorem 1** (Ref. 9) Let \( f(z) \) be a meromorphic function of \( \sigma(f) < 2 \), and \( \eta \) be a non-zero constant.
If \( f(z) \) and \( f(z + \eta) \) share a finite value \( a \) and \( \infty \) \( CM \), then
\[
\frac{f(z + \eta) - a}{f(z) - a} = \tau,
\]
for some constant \( \tau \).

Heittokangas et al.\(^9\) gave the example \( f(z) = e^{\frac{z^2}{2}} + 1 \) which shows that \( \sigma(f) < 2 \) cannot be relaxed to \( \sigma(f) \leq 2 \).

It is well known that \( \Delta_{\eta}f(z) = f(z + \eta) - f(z) \) (where \( \eta \in \mathbb{C} \setminus \{0\} \)) is a constant such that \( f(z + \eta) - f(z) \neq 0 \) is regarded as the difference counterpart of \( f'(z) \). For a transcendental entire function \( f(z) \) with finite order which has a finite Borel exceptional value, Chen and Yi\(^7\) and Chen\(^8\) proceed to consider the problem that \( \Delta_{\eta}f(z) \) and \( f(z) \) share one finite value \( CM \) and have obtained the following results.

**Theorem 2 (Ref. 7)** Let \( f(z) \) be a finite-order transcendental entire function which has a finite Borel exceptional value \( a \), and let \( \eta \) be a constant such that \( f(z+\eta) \neq f(z) \). If \( \Delta_{\eta}f(z) \) and \( f(z) \) share a CM, then
\[
a = 0 \quad \text{and} \quad \frac{f(z + \eta) - f(z)}{f(z)} = c,
\]
for some constant \( c \).

**Theorem 3 (Ref. 8)** Let \( f(z) \) be a transcendental entire function of finite order that is of a finite Borel exceptional value \( a \), and \( \eta \in \mathbb{C} \) be a constant such that \( f(z + \eta) \neq f(z) \). If \( \Delta_{\eta}f(z) = f(z + \eta) - f(z) \) and \( f(z) \) share \( a(\neq a) \) CM, then
\[
\frac{\Delta_{\eta}f(z) - a}{f(z) - a} = \frac{a}{a - a}.
\]

After that Liu and Dong\(^13\) considered the differential-difference analogue of Brück conjecture and have obtained the following result.

**Theorem 4 (Ref. 13)** Suppose that \( f(z) \) is an entire solution of equation
\[
f'(z) - a(z) = e^{\eta(z)}(f(z + c) - a(z)),
\]
where \( c \in \mathbb{C} \setminus \{0\} \) is a constant, \( P(z) \) is a polynomial and \( a(z) \) is an entire function with \( \sigma(a) < \sigma(f) \). If \( \lambda(f - a) < \sigma(f) \), then \( \sigma(f) = 1 + \deg P(z) \).

Chen and Gao\(^14\) have recently proved the following result.

**Theorem 5 (Ref. 14)** Let \( f(z) \) be a transcendental entire function of finite order, \( \eta \in \mathbb{C} \setminus \{0\} \) be a constant such that \( \Delta_{\eta}f(z) = f(z + \eta) - f(z) \neq 0 \), \( a(z) \) be an entire function such that \( \sigma(a) < 1 \) and \( \lambda(f - a) < \sigma(f) \). If \( \Delta_{\eta}f(z) \) and \( f'(z) \) share \( a(z) \) CM, then one of the following two cases holds:

1. If \( a(z) \neq 0 \), then
\[
\frac{\Delta_{\eta}f(z) - a(z)}{f'(z) - a(z)} = 1, \quad f(z) = (a(z) + H(z)e^{\eta z},
\]
where \( H(z) \neq 0 \) is an entire function with \( \lambda(H) = \sigma(H) < 1 \) and \( a, c \in \mathbb{C} \setminus \{0\} \) is a constant satisfying \( e^{\eta z} = 1 + c \);

2. If \( a(z) \equiv 0 \), then
\[
\frac{\Delta_{\eta}f(z)}{f'(z)} = A, \quad f(z) = H(z)e^{\eta z},
\]
where \( H(z) \neq 0 \) is an entire function with \( \lambda(H) = \sigma(H) < 1 \), \( a, c \in \mathbb{C} \setminus \{0\} \) are constants satisfying \( e^{\eta z} = 1 + Ac \).

**RESULTS**

Here, we will proceed to consider the differential-difference analogue of Brück conjecture and obtain the accurate expression of the transcendental entire function \( f(z) \). The aim of this paper is to improve the results obtained in Theorem 4 and Theorem 5. In fact, we will prove the following result.

**Theorem 6** Let \( f(z) \) be a transcendental entire function of finite order, \( \eta \in \mathbb{C} \setminus \{0\} \) be a constant such that \( \Delta_{\eta}f(z) = f(z + \eta) - f(z) \neq 0 \), \( b(z) \) be an entire function such that \( \sigma(b) < \sigma(f) \) and \( \lambda(f - b) < \sigma(f) \). If \( \Delta_{\eta}f(z) \) and \( f'(z) \) share \( a(z) \) CM, where \( a(z) \) is an entire function satisfying \( \sigma(a) < \sigma(f) \), then
\[
\frac{\Delta_{\eta}f(z) - a(z)}{f'(z) - a(z)} = A, \quad f(z) = b(z) + H(z)e^{\eta z},
\]
where \( a(z), b(z) \) are entire functions with \( \max\{\sigma(a), \sigma(b)\} < 1 \), \( H(z) \neq 0 \) is an entire function with \( \lambda(H) = \sigma(H) < 1 \) and \( A, c, \eta \in \mathbb{C} \setminus \{0\} \) are constants satisfying \( e^{\eta z} = 1 + Ac \).

**Remark 1** From the assumptions of Theorem 6, we conclude that \( \sigma(f) \geq 1 \). Hence if \( \sigma(a) < 1 \) and \( \sigma(b) < 1 \), we obtain the following corollary.

**Corollary 1** Let \( f(z) \) be a transcendental entire function of finite order, \( \eta \in \mathbb{C} \setminus \{0\} \) be a constant such that \( \Delta_{\eta}f(z) = f(z + \eta) - f(z) \neq 0 \), \( b(z) \) be an entire function such that \( \sigma(b) < 1 \) and \( \lambda(f - b) < \sigma(f) \). If \( \Delta_{\eta}f(z) \) and \( f'(z) \) share \( a(z) \) CM, where \( a(z) \) is an entire function satisfying \( \sigma(a) < 1 \), then
\[
\frac{\Delta_{\eta}f(z) - a(z)}{f'(z) - a(z)} = A, \quad f(z) = b(z) + H(z)e^{\eta z},
\]
where \( H(z) \neq 0 \) is an entire function with \( \lambda(H) = \sigma(H) < 1 \) and \( A, c, \eta \in \mathbb{C} \setminus \{0\} \) are constants satisfying \( e^{\eta z} = 1 + Ac \).
Remark 2 From the assumptions of Theorem 6, we conclude that $\sigma(f) \geq 1$. Hence if $a(z) \equiv b(z) \neq 0$ and $\sigma(a) < 1$, we obtain the following corollary, which is the conclusion (i) of Theorem 5.

**Corollary 2** Let $f(z)$ be a transcendental entire function of finite order, $\eta \in \mathbb{C}\setminus\{0\}$ be a constant such that $\Delta_\eta f(z) = f(z + \eta) - f(z) \neq 0$, $a(z)$ be an entire function such that $\sigma(a) < 1$ and $\lambda(f - a) < \sigma(f)$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a(z) \neq 0$ CM, then
\[
\frac{\Delta_\eta f(z) - a(z)}{f'(z) - a(z)} = A, \quad f(z) = a(z) + H(z)e^{cz},
\]
where $H(z) \neq 0$ is an entire function with $\lambda(H) = \sigma(H) + 1$ and $A, c, \eta \in \mathbb{C}\setminus\{0\}$ are constants satisfying $e^{\nu_0} = 1 + Ac$.

**Remark 3** In Theorem 6, if $b(z) \equiv b$ and $a(z) \equiv a$, we obtain the following corollary.

**Corollary 3** Let $f(z)$ be a transcendental entire function of finite order which has a finite Borel exceptional set $b$, $\eta \in \mathbb{C}\setminus\{0\}$ be a constant such that $\Delta_\eta f(z) = f(z + \eta) - f(z) \neq 0$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a \neq 0$ CM, then
\[
\frac{\Delta_\eta f(z) - a}{f'(z) - a} = A, \quad f(z) = b + H(z)e^{cz},
\]
where $H(z) \neq 0$ is an entire function with $\lambda(H) = \sigma(H) + 1$ and $A, c, \eta \in \mathbb{C}\setminus\{0\}$ are constants satisfying $e^{\nu_0} = 1 + Ac$.

**Examples 1** (Ref. 14) Suppose that $f(z) = z^2 + e^{cz}$, where $c \in \mathbb{C}\setminus\{0\}$ is a constant. Then $\lambda(f - z^2) < \sigma(f)$. Let $\eta = 1$ and let $c$ satisfy $e^{c} = 1 + \frac{1}{2}c$, we see that $\Delta_\eta f(z) = 2z + \frac{1}{2}c e^{cz}$ and $f'(z) = 2z + c e^{cz}$. Then $(\Delta_\eta f(z) - 2(z+1))/(f'(z) - 2(z+1)) = \frac{1}{2}$, that is, $\Delta_\eta f(z)$ and $f'(z)$ share $2(z+1)(\neq z^2)$ CM.

**Example 2** (Ref. 14) Suppose that $f(z) = z + e^{cz}$, where $c \in \mathbb{C}\setminus\{0\}$ is a constant. Then $\lambda(f - z) < \sigma(f)$. Let $\eta = 1$ and let $c$ satisfy $e^{c} = 1 + c$, we see that $\Delta_\eta f(z) = 1 + c e^{cz} = f'(z)$. Then $(\Delta_\eta f(z) - z)/(f'(z) - z) = 1$, that is, $\Delta_\eta f(z)$ and $f'(z)$ share $z$ CM.

**Example 3** Suppose that $f(z) = 1 + e^{cz}$, where $c \in \mathbb{C}\setminus\{0\}$ is a constant. Then $\lambda(f - 1) < \sigma(f)$. Let $\eta = \log 2$ and let $c$ satisfy $2^c = 1 + c$, we see that $\Delta_\eta f(z) = c e^{cz} = f'(z)$. Then $(\Delta_\eta f(z) - 1)/(f'(z) - 1) = 1$; that is, $\Delta_\eta f(z)$ and $f'(z)$ share $1$ CM.

**SOME LEMMATA**

**Lemma 1** (Ref. 15, Corollary 2) Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma$, let $k, j \ (k > j \geq 0)$ be integers. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E$, we have
\[
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.
\]

**Lemma 2** (Ref. 16, Theorem 8.2) Let $f(z)$ be a meromorphic function of finite order $\sigma$, let $\eta$ be a non-zero complex number, and let $\varepsilon > 0$ be a given real constant. Then there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E$, we have
\[
\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left| \frac{f(z + \eta)}{f(z)} \right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.
\]

Following Hayman (Ref. 17), we define an $\varepsilon$-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If $E$ is an $\varepsilon$-set, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets $E$ has finite logarithmic measure, and for almost all real $\theta$ the intersection of $E$ with the ray $\arg z = \theta$ is bounded.

**Lemma 3** (Ref. 18, Lemma 3.3) Let $f(z)$ be a transcendental meromorphic function of order $\sigma(f) < 1$, and let $h > 0$. There exists an $\varepsilon$-set $E$ such that
\[
\frac{f'(z+c)}{f(z+c)} \to 0, \quad \frac{f(z+c)}{f(z)} \to 1
\]
as $z \to \infty$ in $\mathbb{C}\setminus E$,

uniformly in $c$ for $|c| \leq h$. Further, $E$ may be chosen so that for large $z \notin E$, the function $f(z)$ has no zeros or poles on $|z| \leq h$.

**Lemma 4** (Ref. 4) Suppose that $f_j(z) \ (j = 1, 2, \ldots, n + 1)$ and $g_j(z) \ (j = 1, 2, \ldots, n) \ (n \geq 1)$ are entire functions satisfying (i) $\sum_{j=1}^{n} f_j(z) e^{2\pi i j z} \equiv f_{n+1}(z)$; (ii) The order of $f_j(z)$ is less than the order of $e^{2\pi i j z}$ for $1 \leq j \leq n+1, 1 \leq k \leq n$; and furthermore, the order of $f_j(z)$ is less than the order of $e^{2\pi i j z}$ for $n \geq 2$ and $1 \leq j \leq n+1, 1 \leq h < k \leq n$. Then $f_j(z) \equiv 0, \ (j = 1, 2, \ldots, n+1)$.

**Lemma 5** Let $f(z)$ be a transcendental entire function of finite order, $\eta \in \mathbb{C}\setminus\{0\}$ be a constant such that $\Delta_\eta f(z) = f(z + \eta) - f(z) \neq 0$, $b(z)$ be an entire
function such that $\sigma(b) < \sigma(f)$ and $\lambda(f-b) < \sigma(f)$. If
\[
\Delta_f f(z) - a(z) = \frac{f'(z) - a(z)}{f(z) - a(z)} = A,
\]
where $A \in \mathbb{C} \setminus \{0\}$ is a constant and $a(z)$ is an entire function such that $\sigma(a) < \sigma(f)$, then
\[
f(z) = b(z) + H(z)e^{cz},
\]
where $b(z)$ is an entire function with $\sigma(b) < 1$, $H(z) \neq 0$ is an entire function with $\lambda(H) = \sigma(H) < 1$ and $A, c, \eta \in \mathbb{C} \setminus \{0\}$ are constants satisfying $e^{c\eta} = 1 + Ac$.

**Proof:** By Hadamard’s factorization theorem (Ref. 4, Theorem 2.5), we obtain
\[
f(z) = b(z) + h(z)e^{\theta(z)}, \tag{1}
\]
where $h(z) \neq 0$ is an entire function, $Q(z)$ is a polynomial with $\deg Q(z) = q \geq 1$, and $h(z), Q(z)$ satisfy
\[
\sigma(h) = \lambda(h) = \lambda(f - b) < \sigma(f) = \deg Q(z). \tag{2}
\]
Note that
\[
\Delta_f f(z) - a(z) = \frac{f'(z) - a(z)}{f(z) - a(z)} = A. \tag{3}
\]
Substituting (1) into (3) yields
\[
h(z + \eta)e^{Q(z+\eta) - Q(z)} - h(z) - A(h'(z) + h(z)Q(z)) = (A d(z) - c(z))e^{-\theta(z)}, \tag{4}
\]
where $c(z) = b(z + \eta) - b(z) - a(z)$ and $d(z) = b'(z) - a(z)$. Since $\sigma(a) < q$ and $\sigma(b) < q$, we see that $\max\{\sigma(c), \sigma(d)\} < q$. If $A d(z) - c(z) \neq 0$, since $\sigma(h) < q$, $\deg(Q(z + \eta) - Q(z)) = q - 1$ and $\max\{\sigma(c), \sigma(d)\} < q$, we see that the order of growth of the left side of (4) is less than $q$, and the order of growth of the right side of (4) is $q$, a contradiction. Therefore $A d(z) - c(z) \equiv 0$, (4) can be rewritten as
\[
e^{Q(z+\eta) - Q(z)} = \left[1 + A \left(\frac{h'(z)}{h(z)} + Q'(z)\right)\right] \frac{h(z)}{h(z + \eta)}. \tag{5}
\]
We claim that $q = 1$. In fact, if it is not true, then $q \geq 2$. If $\sigma(h) < 1$, since $\deg(Q(z+\eta) - Q(z)) = q - 1 \geq 1$, we see that the order of growth of the left side of (5) is $q - 1 \geq 1$, and the order of growth of the right side of (5) is less than 1, a contradiction. Then we have $\sigma(h) \geq 1$.

By Lemma 1, for any given $\varepsilon_1 > 0$, there exists a set $E_1 \subset (1, \infty)$ of finite logarithmic measure, such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_1$, we have
\[
\left|\frac{h'(z)}{h(z)}\right| \leq |z|^\sigma(h) - 1 + \varepsilon_1. \tag{6}
\]

By Lemma 2, for any given $\varepsilon_2 > 0$, there exists a set $E_2 \subset (1, \infty)$ of finite logarithmic measure, such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_2$, we have
\[
\exp\{-r^{\sigma(h) - 1 + \varepsilon_2}\} \leq \left|\frac{h(z + \eta)}{h(z)}\right| \leq \exp\{r^{\sigma(h) - 1 + \varepsilon_2}\}. \tag{7}
\]
Set $\varepsilon_3 = \max\{\varepsilon_1, \varepsilon_2\}, 0 < \varepsilon_3 < \frac{1}{2}(q - \sigma(h))$, there exists $r_0 > 0$ such that for all $z$ satisfying $|z| = r > r_0$, we have
\[
r^{q-1-\varepsilon_3} \leq |Q'(z)| \leq r^{q-1+\varepsilon_3}. \tag{8}
\]
From (5), we see that $(1 + A(h'(z)/h(z) + Q'(z)))h(z)/h(z + \eta)$ is an entire function. Then for all $z$ satisfying $|z| = r > r_0$ and $|z| = r \notin [0, 1] \cup E_1 \cup E_2$, for the above given $\varepsilon_3$, from (6)–(8), we have
\[
\left|\left(1 + A\left(\frac{h'(z)}{h(z)} + Q'(z)\right)\right)\frac{h(z)}{h(z + \eta)}\right| \leq \left(1 + |A|\left(\frac{h'(z)}{h(z)} + |Q'(z)|\right)\right)\frac{h(z)}{h(z + \eta)} \leq (1 + |A|\{r^{\sigma(h) - 1 + \varepsilon_3} + r^{q-1-\varepsilon_3}\})\exp\{r^{\sigma(h) - 1 + \varepsilon_3}\} \leq |A|\{r^{\sigma(h) - q - 2 + 2\varepsilon_3} + \exp\{r^{\sigma(h) - 1 + \varepsilon_3}\}\} < \exp\{r^{q-1}\},
\]
that is,
\[
T\left(r, \left(1 + A\left(\frac{h'(z)}{h(z)} + Q'(z)\right)\right)\frac{h(z)}{h(z + \eta)}\right) = m\left(r, \left(1 + A\left(\frac{h'(z)}{h(z)} + Q'(z)\right)\right)\frac{h(z)}{h(z + \eta)}\right) < r^{q-1}.
\]
The above inequality yields
\[
\sigma\left(1 + A\left(\frac{h'(z)}{h(z)} + Q'(z)\right)\right)\frac{h(z)}{h(z + \eta)} < q - 1.
\]
It follows from $\deg(Q(z + \eta) - Q(z)) = q - 1$ that (5) is a contradiction. Then we must have $q = 1$,
\[
f(z) = b(z) + H(z)e^{cz},
\]
where $c \in \mathbb{C} \setminus \{0\}$ is a constant and $H(z) \neq 0$ is an entire function with $\lambda(H) = \sigma(H) < 1$. It follows from (5) that
\[
\frac{h(z + \eta)}{h(z)}e^{\eta} = 1 + A\left(\frac{h'(z)}{h(z)} + c\right). \tag{9}
\]
If \( h(z) \neq 0 \) is a polynomial, then
\[
\frac{h'(z)}{h(z)} \to 0, \quad \frac{h(z + \eta)}{h(z)} \to 1, \quad z \to \infty. \tag{10}
\]
It follows from (9) and (10) that \( e^{\eta} = 1 + Ac. \) If \( h(z) \neq 0 \) is a transcendental entire function with \( \sigma(h) < 1, \) from Lemma 3, we also have \( e^{\eta} = 1 + Ac. \) □

**PROOF OF Theorem 6**

*Proof:* From the assumptions of Theorem 6, we see that (1) and (2) are still valid. Since \( \Delta \eta f(z) \) and \( f'(z) \) share \( a(z) \) CM, we have
\[
\frac{\Delta \eta f(z) - a(z)}{f'(z) - a(z)} = e^{P(z)}, \tag{11}
\]
where \( P(z) \) is a polynomial. It follows from (2) and (11) that
\[
\deg P(z) \leq \deg Q(z). \tag{12}
\]
Substituting (1) into (11) yields
\[
h(z + \eta) e^{Q(z + \eta) - Q(z)} - h(z) + c(z) e^{-Q(z)} = (h'(z) + h(z) Q'(z) + d(z) e^{-Q(z)}) e^{P(z)}, \tag{13}
\]
where \( c(z) = b(z + \eta) - b(z) - a(z) \) and \( d(z) = b'(z) - a(z). \) Since \( \sigma(a) < \sigma(f) \) and \( \sigma(b) < \sigma(f), \) we see that \( \max(\sigma(c), \sigma(d)) < \sigma(f). \) In what follows, we consider two cases: 1 \( \leq \deg P(z) < \deg Q(z) \) and \( \deg P(z) = \deg Q(z). \)

\[
P(z) = a_\eta z^\eta + a_{\eta - 1} z^{\eta - 1} + \cdots + a_0, \quad Q(z) = b_\eta z^\eta + b_{\eta - 1} z^{\eta - 1} + \cdots + b_0, \tag{14}
\]
where \( a_\eta (\neq 0), \ldots, a_0, b_\eta (\neq 0), \ldots, b_0 \) are constants, \( p, q \) are positive integers.

**Case 1.** Suppose that \( 1 \leq p < q. \) Then (13) can be rewritten as
\[
h(z + \eta) e^{Q(z + \eta) - Q(z)} - h(z) = (h'(z) + h(z) Q'(z)) e^{P(z)}.
\]
If \( d(z) e^{P(z)} - c(z) \neq 0, \) since \( \sigma(h) < q, \) \( \deg(Q(z + \eta) - Q(z)) = q - 1 \) and \( \sigma(e^{P(z)}) = \deg P(z) = p < q, \) we see that the order of growth of the left side of (5) is less than \( q, \) and the order of growth of the right side of (5) is \( q, \) a contradiction. If \( d(z) e^{P(z)} - c(z) \equiv 0, \) then (5) can be rewritten as
\[
h(z + \eta) e^{Q(z + \eta) - Q(z)} - h(z) = (h'(z) + h(z) Q'(z)) e^{P(z)}. \tag{15}
\]
Next, we discuss two subcases: 1 \( \leq \deg P(z) < \deg Q(z) - 1 \) and 1 \( \leq \deg P(z) = \deg Q(z) - 1. \)

**Subcase 1.1.** Suppose that \( 1 \leq p < q - 1. \) Then (6) can be rewritten as
\[
e^{Q(z + \eta) - Q(z)} = \left[ 1 + \frac{h'(z)}{h(z)} + Q'(z) \right] e^{P(z)} \frac{h(z)}{h(z + \eta)}. \tag{17}
\]
If \( \sigma(h) < 1, \) since \( \deg(Q(z + \eta) - Q(z)) = q - 1 \) \( \geq 1 \) and \( \deg P(z) < q - 1, \) we know that the order of growth of the left-hand side of (17) is \( q - 1, \) and the order of growth of the right-hand side of (17) is less than \( q - 1, \) a contradiction. Then we have \( \sigma(h) \geq 1. \)

For any given \( \varepsilon_4, 0 < \varepsilon_4 \leq \varepsilon_4 < \min \left\{ \frac{1}{3}(q - \sigma(h)), \frac{1}{3}(q - 1 - p) \right\}, \) there exists \( r_1 > 0 \) such that for all \( z \) satisfying \( |z| = r > r_1, \)
\[
|e^{P(z)}| \leq \exp(r^{\sigma + \varepsilon_4}). \tag{18}
\]
From (17), we see that \( [1 + \left( \frac{h'(z)}{h(z)} + Q'(z) \right) e^{P(z)}] h(z)/h(z + \eta) \) is an entire function. Then for all \( z \) satisfying \( |z| = r > r_1 \) and \( |z| = r \notin [0, 1] \cup E_1 \cup E_2, \) by (6)–(8) and (18), we have
\[
\left[ 1 + \left( \frac{h'(z)}{h(z)} + Q'(z) \right) e^{P(z)} \right] \frac{h(z)}{h(z + \eta)} \leq \left[ 1 + \left( \frac{h'(z)}{h(z)} + |Q'(z)| \right) e^{P(z)} \right] \frac{h(z)}{h(z + \eta)}
\]
\[
\leq \left[ 1 + (r^{\sigma(h) - 1} + r^{1 - \varepsilon_4}) \exp(r^{\sigma + \varepsilon_4}) \right]
\times \exp(r^{\sigma(h) - 1 + \varepsilon_4})
\]
\[
\leq r^{\sigma(h) + q - 2 + 2\varepsilon_4} \exp(r^{\sigma + \varepsilon_4} + r^{\sigma(h) - 1 + \varepsilon_4})
\]
\[
< \exp(r^{q - 1}),
\]
that is,
\[
T \left( r, \left[ 1 + \left( \frac{h'(z)}{h(z)} + Q'(z) \right) e^{P(z)} \right] \frac{h(z)}{h(z + \eta)} \right)
\]
\[
= m \left( r, \left[ 1 + \left( \frac{h'(z)}{h(z)} + Q'(z) \right) e^{P(z)} \right] \frac{h(z)}{h(z + \eta)} \right)
\]
\[
< r^{q - 1}.
\]
The above inequality yields
\[
\sigma \left( \left[ 1 + \left( \frac{h'(z)}{h(z)} + Q'(z) \right) e^{P(z)} \right] \frac{h(z)}{h(z + \eta)} \right) < q - 1.
\]
It follows from \( \deg(Q(z + \eta) - Q(z)) = q - 1 \) that (17) is a contradiction.

**Subcase 1.2.** Suppose that \( 1 \leq p = q - 1. \) It follows from (14) that
\[
P(z) = a_{q - 1} z^{q - 1} + P_{q - 2}(z),
\]
\[
Q(z + \eta) - Q(z) = q \eta b_q z^{q - 1} + P_{q - 2}(z), \tag{19}
\]
where \( b_q (\neq 0), \ldots, b_0 \) are constants, \( q \) is a positive integer.
where \( a_{q-1} \neq 0, b_q \neq 0 \) are constants, \( P_{q-2}(z), Q_{q-2}(z) \) are polynomials, \( \deg P_{q-2}(z) \leq q - 2 \), \( \deg Q_{q-2}(z) \leq q - 2 \). In what follows, we consider two subcases: \( a_{q-1} = q \eta b_q \) and \( a_{q-1} \neq q \eta b_q \).

**Subcase 1.2.1.** If \( a_{q-1} = q \eta b_q \), then (16) can be rewritten as

\[
e^{-p(z)} = \frac{h(z + \eta)}{h(z)} e^{Q(z+\eta)-Q(z)-P(z)} \left[ \frac{h'(z)}{h(z)} + Q'(z) \right].
\]

(20)

It follows from \( a_{q-1} = q \eta b_q \) that \( \deg Q(z+\eta) - Q(z) - P(z) = \deg Q_{q-2}(z) - P_{q-2}(z) \leq q - 2 \). Using similar reasoning as in the proof of Subcase 1.1, we obtain

\[
\sigma \left[ \frac{h(z + \eta)}{h(z)} e^{Q(z+\eta)-Q(z)-P(z)} \left( \frac{h'(z)}{h(z)} + Q'(z) \right) \right] < q - 1.
\]

It follows from \( \deg(-P(z)) = q - 1 \geq 1 \) that (20) is a contradiction.

**Subcase 1.2.2.** If \( a_{q-1} \neq q \eta b_q \), it follows from (16) and (19) that

\[
\left( \frac{h'(z)}{h(z)} + Q'(z) \right) e^{\sigma(z)-r(z)}
\]

\[= \frac{h(z + \eta)}{h(z)} e^{Q(z+\eta)-Q(z)-P(z)} e^{-P(z)}. \]

(21)

Without loss of generality, we assume that \( q | \eta b_q | \leq |a_{q-1}| \). Set \( \arg a_{q-1} = \theta_1 \) and \( \arg(\eta b_q) = \theta_2 \). For the above given \( \varepsilon_3 \) and for all \( z \) satisfying \( |z| = r > r_2 \) and \( |z| = r \notin [0,1] \cup E_1 \cup E_2 \), \( |z| = r e^{i\theta_0} \), where \( \theta_0 \) is a real constant such that \( \cos((q-1)\theta_0 + \theta_1) = 1 \), by (6)–(8), we have

\[
\left[ \frac{h'(z)}{h(z)} + Q'(z) \right] e^{\sigma(z)-r(z)}
\]

\[\geq \left| Q'(z) \right| - \left| \frac{h'(z)}{h(z)} \right| e^{\sigma(z)-r(z)}
\]

\[\geq \left( r^{q-1} - \frac{1}{r^{q-1}} + \frac{1}{r^{q-1}} \right) e^{\sigma(z)-r(z)}
\]

\[\geq r^{q-1} \exp\{ |a_{q-1}| r^{q-1} \}
\]

\[\geq \exp\{ |a_{q-1}| r^{q-1} \},
\]

and

\[
\left| \frac{h(z + \eta)}{h(z)} e^{Q(z+\eta)-Q(z)-P(z)} e^{-P(z)} \right|
\]

\[\leq \left| \frac{h(z + \eta)}{h(z)} \right| e^{Q(z+\eta)-Q(z)-P(z)} e^{-P(z)}
\]

\[\leq \exp\{ \sigma(h(z+\varepsilon)) \}
\]

\[\times \exp\{ q |\eta b_q | \cos((q-1)\theta_0 + \theta_2) r^{q-1} + O(r^{q-2}) \}
\]

\[\leq \exp\{ q |\eta b_q | \cos((q-1)\theta_0 + \theta_2) r^{q-1} + o(r^{q-1}) \},
\]

that is,

\[
\exp\{ |a_{q-1}| r^{q-1} \}
\]

\[\leq \exp\{ q |\eta b_q | \cos((q-1)\theta_0 + \theta_2) r^{q-1} + o(r^{q-1}) \}.
\]

(22)

We claim that \( q |\eta b_q | \cos((q-1)\theta_0 + \theta_2) < |a_{q-1}| \). In fact, if \( q |\eta b_q | = |a_{q-1}| \), it follows from \( a_{q-1} \neq q \eta b_q \) that \( \cos((q-1)\theta_0 + \theta_2) \neq 1 \), then \( \cos((q-1)\theta_0 + \theta_2) < 1 \). Thus \( q |\eta b_q | \cos((q-1)\theta_0 + \theta_2) < q |\eta b_q | = |a_{q-1}| \). If \( q |\eta b_q | < |a_{q-1}| \), then \( q |\eta b_q | \cos((q-1)\theta_0 + \theta_2) \leq q |\eta b_q | < |a_{q-1}| \). For any given \( \varepsilon_3 \), \( 0 < \varepsilon_3 < \frac{1}{2} \left( |a_{q-1}| - q |\eta b_q | \cos((q-1)\theta_0 + \theta_2) \right) \), it follows from (22) that

\[
\exp\{ |a_{q-1}| r^{q-1} \}
\]

\[\leq \exp\{ q |\eta b_q | \cos((q-1)\theta_0 + \theta_2) r^{q-1} + o(r^{q-1}) \}
\]

\[\exp\{ |a_{q-1}| r^{q-1} \}.
\]

This is a contradiction.

**Case 2.** Suppose that \( p = q \). For \( a_q \) and \( b_q \), we consider three subcases: (2.1) \( a_q = b_q \); (2.2) \( a_q = -b_q \); (2.3) \( a_q \neq b_q \) and \( a_q \neq -b_q \).

**Subcase 2.1.** Suppose that \( a_q = b_q \). Then (13) can be rewritten as

\[
(h'(z) + h(z)Q'(z)) e^{p(z)} = c(z) e^{Q(z)}
\]

\[= h(z + \eta) e^{Q(z+\eta)-Q(z)-h(z)} - d(z) e^{p(z)-Q(z)}.
\]

(23)

Since \( \sigma(h) < q \), \( \deg Q(z + \eta) - Q(z) = q - 1 \), \( \max\{ \sigma(c), \sigma(d) \} < q \) and \( \deg P(z) - Q(z) \leq q - 1 \), we have \( h'(z) + h(z)Q'(z) < q \) and \( h(z + \eta) e^{Q(z+\eta)-Q(z) - h(z) - d(z)} e^{p(z)-Q(z)} < q \).

Noting that \( e^{p(z)} \), \( e^{Q(z)} \) and \( e^{p(z)-Q(z)} \) are of regular growth, and \( \sigma(e^{p(z)}) = \sigma(e^{p(z)-Q(z)}) = \sigma(e^{p(z)+Q(z)}) = q \), it follows from Lemma 4 and (23) that

\[h'(z) + h(z)Q'(z) \equiv 0.
\]

If \( h'(z) + h(z)Q'(z) \equiv 0 \), suppose that \( h(z) \) is a polynomial. Then \( h(z) \equiv 0 \), it contradicts \( h(z) \neq 0 \). If \( h'(z) + h(z)Q'(z) \equiv 0 \), suppose that \( h(z) \) is a transcendental entire function. Then \( h(z) = c e^{\alpha z} \), \( c \in C \setminus \{0\} \), that is, \( \sigma(h) = q \), it contradicts \( \sigma(h) < q \). Then we see that \( h'(z) + h(z)Q'(z) \equiv 0 \) is absurd.

**Subcase 2.2.** Suppose that \( a_q = -b_q \). Then (13) can be rewritten as

\[
\left( h'(z) + h(z)Q'(z) \right) e^{p(z)+Q(z)} - c(z) e^{Q(z)}
\]

\[+ d(z) e^{p(z)-Q(z)} = h(z + \eta) e^{Q(z+\eta)-Q(z) - h(z)}.
\]

(24)
Since $\sigma(h) < q$, $\deg(Q(z + \eta) - Q(z)) = q - 1$, $\max\{\sigma(c), \sigma(d)\} < q$ and $\deg(P(z + Q(z))) < q - 1$, we have $\sigma((h'(z) + h(z)Q'(z))e^{P(z)Q(z)} - c(z)) < q$ and $\sigma(h(z + \eta)e^{Q(z+\eta) - Q(z)} - h(z)) < q$.

Noting that $e^{-Q(z)}$, $e^{P(z)Q(z)}$, and $e^{-P(z)}$ are of regular growth, and $\sigma(e^{-Q(z)}) = \sigma(e^{-P(z)}) = \sigma(e^{P(z)\pm Q(z)}) = q$, it follows from Lemma 4 and (24) that

$$h(z + \eta)e^{Q(z+\eta) - Q(z)} - h(z) \equiv 0.$$  

Making use of the above identity, we obtain

$$e^{Q(z) - Q(z+\eta)} \equiv \frac{h(z + \eta)}{h(z)}.$$  

(25)

Combining with (7) and (2), we conclude that the order of growth of the left-hand side of (25) is $q - 1$, and the order of growth of the right-hand side of (25) is less than $q - 1$, a contradiction.

**Subcase 2.3.** Suppose that $a_0 \neq b_0$ and $a_0 \neq -b_0$. Then (13) can be rewritten as

$$(h'(z) + h(z)Q'(z))e^{P(z)} - c(z)e^{-Q(z)} + d(z)e^{P(z)Q(z)} = h(z + \eta)e^{Q(z+\eta) - Q(z)} - h(z).$$

(26)

Since $\sigma(h) < q$, $\deg(Q(z + \eta) - Q(z)) = q - 1$, and $\max\{\sigma(c), \sigma(d)\} < q$, we have $\sigma(h'(z) + h(z)Q'(z)) < q$ and $\sigma(h(z + \eta)e^{Q(z+\eta) - Q(z)} - h(z)) < q$.

Noting that $e^{\pm P(z)}$, $e^{\pm Q(z)}$, and $e^{\pm P(z)\pm Q(z)}$ are of regular growth, and $\sigma(e^{\pm P(z)}) = \sigma(e^{\pm Q(z)}) = \sigma(e^{\pm P(z)\pm Q(z)}) = q$, it follows from Lemma 4 and (26) that

$$h'(z) + h(z)Q'(z) \equiv 0,$$

$$h(z + \eta)e^{Q(z+\eta) - Q(z)} - h(z) \equiv 0.$$  

Using similar reasoning as above, we also obtain a contradiction.

Thus $P(z)$ can only be a constant, so is $e^{P(z)}$. Set $e^{P(z)} \equiv A$, where $A$ is a non-zero constant. It follows from (11) that

$$\frac{\Delta_1 f(z) - a(z)}{f'(z) - a(z)} = A.$$  

By Lemma 5, we have

$$f(z) = b(z) + H(z)e^{cz},$$

where $b(z)$ is an entire function with $\sigma(b) < 1$, $H(z) \neq 0$ is an entire function with $\lambda(H) = \sigma(H) < 1$ and $A, c, \eta \in \mathbb{C} \setminus \{0\}$ are constants satisfying $e^{cz} = 1 + Ac$. This completes the proof of Theorem 6. □

**Acknowledgements:** This study was partly supported by the National Natural Science Foundation for Young Scientists of China (Nos: 11701524, 11801093) and by the National Natural Science Foundation of Guangdong Province (No: 2016A030313686). The authors would like to thank the referee for their thorough reviewing with constructive suggestions and comments to the paper.

**REFERENCES**