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A note on differential-difference analogue of Brück conjecture

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ABSTRACT: In this paper, we prove that for a transcendental entire function $f(z)$ of finite order, $\eta \in \mathbb{C}\backslash\{0\}$ is a constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$, $b(z)$ is an entire function such that $\sigma(b) < \sigma(f)$ and $\lambda(f-b) < \sigma(f)$, if $\Delta_\eta f(z)$ and $f'(z)$ share $a(z)$ CM, where $a(z)$ is an entire function satisfying $\sigma(a) < \sigma(f)$, then

$$\frac{\Delta_\eta f(z) - a(z)}{f'(z) - a(z)} = A, \quad f(z) = b(z) + H(z)\mathrm{e}^{cz},$$

where $a(z)$ and $b(z)$ are entire functions with $\max\{\sigma(a), \sigma(b)\} < 1$, $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H) = \sigma(H) < 1$ and $A, c, \eta \in \mathbb{C}\backslash\{0\}$ are constants satisfying $\mathrm{e}^{c\eta} = 1 + Ac$. Our results are improvements and complements of those in [Bull Korean Math Soc 51 (2014) 1453–1467] and [Commun Korean Math Soc 32 (2017) 361–373].

KEYWORDS: entire function, sharing value, differential-difference equation

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INTRODUCTION

In this paper, we assume that the reader is familiar with the standard symbols and the fundamental results of Nevanlinna theory[1–3]. In addition, we use the notations $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of the zero sequence and the order of growth of meromorphic function $f(z)$, respectively. We also denote by $S(r,f)$ any quantify satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$, outside of a possible exceptional set of finite logarithmic measure. For convenience, we need the following definition.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let a be a constant in the complex plane. We say that $f(z)$ and $g(z)$ share a CM (IM) provided that $f(z)-a$ and $g(z)-a$ have the same zeros counting multiplicities (ignoring multiplicities), and $f(z)$ and $g(z)$ share ∞ CM (IM) provided that $f(z)$ and $g(z)$ have the same poles counting multiplicities (ignoring multiplicities). Using the same method, we can also define $f(z)$ and $g(z)$ share function $a(z)$ CM (IM), where $a(z) \in S(r,f) \cap S(r,g)$.

Definition 1 (Ref. 4) Let $f(z)$ be a meromorphic function in the complex plane. We denote by $\sigma_2(f)$ the order of $\log T(r,f)$, i.e.,

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

$\sigma_2(f)$ is called the hyper-order of $f(z)$.

Brück[5] raised the following conjecture.

Conjecture (Ref. 5) Let $f(z)$ be a non-constant entire function with hyper-order $\sigma_2(f) < \infty$, and $\sigma_2(f) \notin \mathbb{Z}^+$. If $f(z)$ and $f'(z)$ share a finite value a CM, then

$$\frac{f'(z)-a}{f(z)-a} = c,$$

where c is a non-zero constant.

The conjecture has been established in the special case[5] when $a = 0$ or when $f(z)$ is an entire function of finite order[6].

Recently, many results on difference analogues of Brück conjecture were considered in Refs. 7–12. To start with, recall the following results.

Theorem 1 (Ref. 9) *Let $f(z)$ be a meromorphic function of $\sigma(f) < 2$, and η be a non-zero constant.*

If $f(z)$ and $f(z+\eta)$ share a finite value a and ∞ CM, then

$$\frac{f(z+\eta)-a}{f(z)-a}=\tau,$$

for some constant τ.

Heittokangas et al[9] gave the example $f(z)=e^{z^2}+1$ which shows that $\sigma(f)<2$ cannot be relaxed to $\sigma(f)\leqslant 2$.

It is well known that $\Delta_\eta f(z)=f(z+\eta)-f(z)$ (where $\eta\in\mathbb{C}\backslash\{0\}$ is a constant such that $f(z+\eta)-f(z)\not\equiv 0$) is regarded as the difference counterpart of $f'(z)$. For a transcendental entire function $f(z)$ with finite order which has a finite Borel exceptional value, Chen and Yi[7] and Chen[8] proceed to consider the problem that $\Delta_\eta f(z)$ and $f(z)$ share one finite value CM and have obtained the following results.

Theorem 2 (Ref. 7) *Let $f(z)$ be a finite-order transcendental entire function which has a finite Borel exceptional value a, and let η be a constant such that $f(z+\eta)\not\equiv f(z)$. If $\Delta_\eta f(z)$ and $f(z)$ share a CM, then*

$$a=0 \quad \text{and} \quad \frac{f(z+\eta)-f(z)}{f(z)}=c,$$

for some constant c.

Theorem 3 (Ref. 8) *Let $f(z)$ be a transcendental entire function of finite order that is of a finite Borel exceptional value α, and $\eta\in\mathbb{C}$ be a constant such that $f(z+\eta)\not\equiv f(z)$. If $\Delta_\eta f(z)=f(z+\eta)-f(z)$ and $f(z)$ share $a(\neq\alpha)$ CM, then*

$$\frac{\Delta_\eta f(z)-a}{f(z)-a}=\frac{a}{a-\alpha}.$$

After that Liu and Dong[13] considered the differential-difference analogue of Brück conjecture and have obtained the following result.

Theorem 4 (Ref. 13) *Suppose that $f(z)$ is an entire solution of equation*

$$f'(z)-a(z)=e^{P(z)}(f(z+c)-a(z)),$$

where $c\in\mathbb{C}\backslash\{0\}$ is a constant, $P(z)$ is a polynomial and $a(z)$ is an entire function with $\sigma(a)<\sigma(f)$. If $\lambda(f-a)<\sigma(f)$, then $\sigma(f)=1+\deg P(z)$.

Chen and Gao[14] have recently proved the following result.

Theorem 5 (Ref. 14) *Let $f(z)$ be a transcendental entire function of finite order, $\eta\in\mathbb{C}\backslash\{0\}$ be a constant such that $\Delta_\eta f(z)=f(z+\eta)-f(z)\not\equiv 0$, $a(z)$ be an entire function such that $\sigma(a)<1$ and $\lambda(f-a)<\sigma(f)$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a(z)$ CM, then one of the following two cases holds:*

(i) If $a(z)\not\equiv 0$, then

$$\frac{\Delta_\eta f(z)-a(z)}{f'(z)-a(z)}=1, \quad f(z)=a(z)+H(z)e^{cz},$$

where $H(z)\not\equiv 0$ is an entire function with $\lambda(H)=\sigma(H)<1$ and $c\in\mathbb{C}\backslash\{0\}$ is a constant satisfying $e^{c\eta}=1+c$;

(ii) If $a(z)\equiv 0$, then

$$\frac{\Delta_\eta f(z)}{f'(z)}=A, \quad f(z)=H(z)e^{cz},$$

where $H(z)\not\equiv 0$ is an entire function with $\lambda(H)=\sigma(H)<1$, A, $c\in\mathbb{C}\backslash\{0\}$ are constants satisfying $e^{c\eta}=1+Ac$.

RESULTS

Here, we will proceed to consider the differential-difference analogue of Brück conjecture and obtain the accurate expression of the transcendental entire function $f(z)$. The aim of this paper is to improve the results obtained in Theorem 4 and Theorem 5. In fact, we will prove the following result.

Theorem 6 *Let $f(z)$ be a transcendental entire function of finite order, $\eta\in\mathbb{C}\backslash\{0\}$ be a constant such that $\Delta_\eta f(z)=f(z+\eta)-f(z)\not\equiv 0$, $b(z)$ be an entire function such that $\sigma(b)<\sigma(f)$ and $\lambda(f-b)<\sigma(f)$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a(z)$ CM, where $a(z)$ is an entire function satisfying $\sigma(a)<\sigma(f)$, then*

$$\frac{\Delta_\eta f(z)-a(z)}{f'(z)-a(z)}=A, \quad f(z)=b(z)+H(z)e^{cz},$$

where $a(z)$, $b(z)$ are entire functions with $\max\{\sigma(a),\sigma(b)\}<1$, $H(z)\not\equiv 0$ is an entire function with $\lambda(H)=\sigma(H)<1$ and $A,c,\eta\in\mathbb{C}\backslash\{0\}$ are constants satisfying $e^{c\eta}=1+Ac$.

Remark 1 From the assumptions of Theorem 6, we conclude that $\sigma(f)\geqslant 1$. Hence if $\sigma(a)<1$ and $\sigma(b)<1$, we obtain the following corollary.

Corollary 1 *Let $f(z)$ be a transcendental entire function of finite order, $\eta\in\mathbb{C}\backslash\{0\}$ be a constant such that $\Delta_\eta f(z)=f(z+\eta)-f(z)\not\equiv 0$, $b(z)$ be an entire function such that $\sigma(b)<1$ and $\lambda(f-b)<\sigma(f)$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a(z)$ CM, where $a(z)$ is an entire function satisfying $\sigma(a)<1$, then*

$$\frac{\Delta_\eta f(z)-a(z)}{f'(z)-a(z)}=A, \quad f(z)=b(z)+H(z)e^{cz},$$

where $H(z)\not\equiv 0$ is an entire function with $\lambda(H)=\sigma(H)<1$ and $A,c,\eta\in\mathbb{C}\backslash\{0\}$ are constants satisfying $e^{c\eta}=1+Ac$.

Remark 2 From the assumptions of Theorem 6, we conclude that $\sigma(f) \geqslant 1$. Hence if $a(z) \equiv b(z) \not\equiv 0$ and $\sigma(a) < 1$, we obtain the following corollary, which is the conclusion (i) of Theorem 5.

Corollary 2 *Let $f(z)$ be a transcendental entire function of finite order, $\eta \in \mathbb{C}\backslash\{0\}$ be a constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$, $a(z)$ be an entire function such that $\sigma(a) < 1$ and $\lambda(f-a) < \sigma(f)$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a(z) \not\equiv 0$ CM, then*

$$\frac{\Delta_\eta f(z) - a(z)}{f'(z) - a(z)} = A, \quad f(z) = a(z) + H(z)\,\mathrm{e}^{cz},$$

where $H(z) \not\equiv 0$ is an entire function with $\lambda(H) = \sigma(H) < 1$ and $A, c, \eta \in \mathbb{C}\backslash\{0\}$ are constants satisfying $\mathrm{e}^{c\eta} = 1 + Ac$.

Remark 3 In Theorem 6, if $b(z) \equiv b$ and $a(z) \equiv a$, we obtain the following corollary.

Corollary 3 *Let $f(z)$ be a transcendental entire function of finite order which has a finite Borel exceptional b, $\eta \in \mathbb{C}\backslash\{0\}$ be a constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a \neq 0$ CM, then*

$$\frac{\Delta_\eta f(z) - a}{f'(z) - a} = A, \quad f(z) = b + H(z)\,\mathrm{e}^{cz},$$

where $H(z) \not\equiv 0$ is an entire function with $\lambda(H) = \sigma(H) < 1$, $A, c, \eta \in \mathbb{C}\backslash\{0\}$ are constants satisfying $\mathrm{e}^{c\eta} = 1 + Ac$.

Examples 1, 2, and 3 below show that Corollaries 1, 2, and 3 are sharp, respectively.

Example 1 (Ref. 14) Suppose that $f(z) = z^2 + \mathrm{e}^{cz}$, where $c \in \mathbb{C}\backslash\{0\}$ is a constant. Then $\lambda(f - z^2) < \sigma(f)$. Let $\eta = 1$ and let c satisfy $\mathrm{e}^c = 1 + \frac{1}{2}c$, we see that $\Delta_\eta f(z) = 2z + 1 + \frac{1}{2}c\,\mathrm{e}^{cz}$ and $f'(z) = 2z + c\,\mathrm{e}^{cz}$. Then $(\Delta_\eta f(z) - 2(z+1))/(f'(z) - 2(z+1)) = \frac{1}{2}$, that is, $\Delta_\eta f(z)$ and $f'(z)$ share $2(z+1)(\not\equiv z^2)$ CM.

Example 2 (Ref. 14) Suppose that $f(z) = z + \mathrm{e}^{cz}$, where $c \in \mathbb{C}\backslash\{0\}$ is a constant. Then $\lambda(f - z) < \sigma(f)$. Let $\eta = 1$ and let c satisfy $\mathrm{e}^c = 1 + c$, we see that $\Delta_\eta f(z) = 1 + c\,\mathrm{e}^{cz} = f'(z)$. Then $(\Delta_\eta f(z) - z)/(f'(z) - z) = 1$, that is, $\Delta_\eta f(z)$ and $f'(z)$ share z CM.

Example 3 Suppose that $f(z) = 1 + \mathrm{e}^{cz}$, where $c \in \mathbb{C}\backslash\{0\}$ is a constant. Then $\lambda(f - 1) < \sigma(f)$. Let $\eta = \log 2$ and let c satisfy $2^c = 1 + c$, we see that $\Delta_\eta f(z) = c\,\mathrm{e}^{cz} = f'(z)$. Then $(\Delta_\eta f(z) - 1)/(f'(z) - 1) = 1$; that is, $\Delta_\eta f(z)$ and $f'(z)$ share 1 CM.

SOME LEMMAS

Lemma 1 (Ref. 15, Corollary 2) *Let $f(z)$ be a transcendental meromorphic function of finite order σ, let k, j $(k > j \geqslant 0)$ be integers. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\sigma - 1 + \varepsilon)}.$$

Lemma 2 (Ref. 16, Theorem 8.2) *Let $f(z)$ be a meromorphic function of finite order σ, let η be a non-zero complex number, and let $\varepsilon > 0$ be a given real constant. Then there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E$, we have*

$$\exp\{-r^{\sigma - 1 + \varepsilon}\} \leqslant \left| \frac{f(z+\eta)}{f(z)} \right| \leqslant \exp\{r^{\sigma - 1 + \varepsilon}\}.$$

Following Hayman (Ref. 17), we define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r \geqslant 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 3 (Ref. 18, Lemma 3.3) *Let $f(z)$ be a transcendental meromorphic function of order $\sigma(f) < 1$, and let $h > 0$. There exists an ε-set E such that*

$$\frac{f'(z+c)}{f(z+c)} \to 0, \quad \frac{f(z+c)}{f(z)} \to 1 \quad \text{as } z \to \infty \text{ in } \mathbb{C}\backslash E,$$

uniformly in c for $|c| \leqslant h$. Further, E may be chosen so that for large $z \notin E$, the function $f(z)$ has no zeros or poles on $|\zeta - z| \leqslant h$.

Lemma 4 (Ref. 4) *Suppose that $f_j(z)$ $(j = 1, 2, \dots, n+1)$ and $g_j(z)$ $(j = 1, 2, \dots, n)$ $(n \geqslant 1)$ are entire functions satisfying (i) $\sum_{j=1}^{n} f_j(z)\,\mathrm{e}^{g_j(z)} \equiv f_{n+1}(z)$; (ii) The order of $f_j(z)$ is less than the order of $\mathrm{e}^{g_k(z)}$ for $1 \leqslant j \leqslant n+1$, $1 \leqslant k \leqslant n$; and furthermore, the order of $f_j(z)$ is less than the order of $\mathrm{e}^{g_h(z) - g_k(z)}$ for $n \geqslant 2$ and $1 \leqslant j \leqslant n+1$, $1 \leqslant h < k \leqslant n$. Then $f_j(z) \equiv 0$, $(j = 1, 2, \dots, n+1)$.*

Lemma 5 *Let $f(z)$ be a transcendental entire function of finite order, $\eta \in \mathbb{C}\backslash\{0\}$ be a constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$, $b(z)$ be an entire* *function such that* $\sigma(b)<\sigma(f)$ *and* $\lambda(f-b)<\sigma(f)$. *If*

$$\frac{\Delta_\eta f(z)-a(z)}{f'(z)-a(z)}=A,$$

where $A\in\mathbb{C}\backslash\{0\}$ *is a constant and* $a(z)$ *is an entire function such that* $\sigma(a)<\sigma(f)$, *then*

$$f(z)=b(z)+H(z)\,\mathrm{e}^{cz},$$

where $b(z)$ *is an entire function with* $\sigma(b)<1$, $H(z)\not\equiv 0$ *is an entire function with* $\lambda(H)=\sigma(H)<1$ *and* $A,c,\eta\in\mathbb{C}\backslash\{0\}$ *are constants satisfying* $\mathrm{e}^{c\eta}=1+Ac$.

Proof: By Hadamard's factorization theorem (Ref. 4, Theorem 2.5), we obtain

$$f(z)=b(z)+h(z)\,\mathrm{e}^{Q(z)}, \tag{1}$$

where $h(z)\not\equiv 0$ is an entire function, $Q(z)$ is a polynomial with $\deg Q(z)=q\geqslant 1$, and $h(z)$, $Q(z)$ satisfy

$$\sigma(h)=\lambda(h)=\lambda(f-b)<\sigma(f)=\deg Q(z). \tag{2}$$

Note that

$$\frac{\Delta_\eta f(z)-a(z)}{f'(z)-a(z)}=A. \tag{3}$$

Substituting (1) into (3) yields

$$h(z+\eta)\,\mathrm{e}^{Q(z+\eta)-Q(z)}-h(z)-A(h'(z)+h(z)Q'(z))=(Ad(z)-c(z))\,\mathrm{e}^{-Q(z)}, \tag{4}$$

where $c(z)=b(z+\eta)-b(z)-a(z)$ and $d(z)=b'(z)-a(z)$. Since $\sigma(a)<q$ and $\sigma(b)<q$, we see that $\max\{\sigma(c),\sigma(d)\}<q$. If $Ad(z)-c(z)\not\equiv 0$, since $\sigma(h)<q$, $\deg(Q(z+\eta)-Q(z))=q-1$ and $\max\{\sigma(c),\sigma(d)\}<q$, we see that the order of growth of the left side of (4) is less than q, and the order of growth of the right side of (4) is q, a contradiction. Then $Ad(z)-c(z)\equiv 0$, (4) can be rewritten as

$$\mathrm{e}^{Q(z+\eta)-Q(z)}=\left[1+A\left(\frac{h'(z)}{h(z)}+Q'(z)\right)\right]\frac{h(z)}{h(z+\eta)}. \tag{5}$$

We claim that $q=1$. In fact, if it is not true, then $q\geqslant 2$. If $\sigma(h)<1$, since $\deg(Q(z+\eta)-Q(z))=q-1\geqslant 1$, we see that the order of growth of the left side of (5) is $q-1\geqslant 1$, and the order of growth of the right side of (5) is less than 1, a contradiction. Then we have $\sigma(h)\geqslant 1$.

By Lemma 1, for any given $\varepsilon_1>0$, there exists a set $E_1\subset(1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z|=r\notin[0,1]\cup E_1$, we have

$$\left|\frac{h'(z)}{h(z)}\right|\leqslant|z|^{\sigma(h)-1+\varepsilon_1}. \tag{6}$$

By Lemma 2, for any given $\varepsilon_2>0$, there exists a set $E_2\subset(1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, we have

$$\exp\{-r^{\sigma(h)-1+\varepsilon_2}\}\leqslant\left|\frac{h(z+\eta)}{h(z)}\right|\leqslant\exp\{r^{\sigma(h)-1+\varepsilon_2}\}. \tag{7}$$

Set $\varepsilon_3=\max\{\varepsilon_1,\varepsilon_2\}$, $0<\varepsilon_3<\frac{1}{3}(q-\sigma(h))$, there exists $r_0>0$ such that for all z satisfying $|z|=r>r_0$, we have

$$r^{q-1-\varepsilon_3}\leqslant|Q'(z)|\leqslant r^{q-1+\varepsilon_3}. \tag{8}$$

From (5), we see that $\big(1+A\big(h'(z)/h(z)+Q'(z)\big)\big)h(z)/h(z+\eta)$ is an entire function. Then for all z satisfying $|z|=r>r_0$ and $|z|=r\notin[0,1]\cup E_1\cup E_2$, for the above given ε_3, from (6)–(8), we have

$$\begin{aligned}&\left|\left(1+A\left(\frac{h'(z)}{h(z)}+Q'(z)\right)\right)\frac{h(z)}{h(z+\eta)}\right|\\ &\quad\leqslant\left(1+|A|\left(\left|\frac{h'(z)}{h(z)}\right|+|Q'(z)|\right)\right)\left|\frac{h(z)}{h(z+\eta)}\right|\\ &\quad\leqslant(1+|A|(r^{\sigma(h)-1+\varepsilon_3}+r^{q-1+\varepsilon_3}))\exp\{r^{\sigma(h)-1+\varepsilon_3}\}\\ &\quad\leqslant|A|r^{\sigma(h)+q-2+2\varepsilon_3}\exp\{r^{\sigma(h)-1+\varepsilon_3}\}<\exp\{r^{q-1}\},\end{aligned}$$

that is,

$$\begin{aligned}&T\left(r,\left(1+A\left(\frac{h'(z)}{h(z)}+Q'(z)\right)\right)\frac{h(z)}{h(z+\eta)}\right)\\ &\quad=m\left(r,\left(1+A\left(\frac{h'(z)}{h(z)}+Q'(z)\right)\right)\frac{h(z)}{h(z+\eta)}\right)\\ &\quad<r^{q-1}.\end{aligned}$$

The above inequality yields

$$\sigma\left(\left(1+A\left(\frac{h'(z)}{h(z)}+Q'(z)\right)\right)\frac{h(z)}{h(z+\eta)}\right)<q-1.$$

It follows from $\deg(Q(z+\eta)-Q(z))=q-1$ that (5) is a contradiction. Then we must have $q=1$,

$$f(z)=b(z)+H(z)\,\mathrm{e}^{cz},$$

where $c\in\mathbb{C}\backslash\{0\}$ is a constant and $H(z)\not\equiv 0$ is an entire function with $\lambda(H)=\sigma(H)<1$. It follows from (5) that

$$\frac{h(z+\eta)}{h(z)}\,\mathrm{e}^{c\eta}=1+A\left(\frac{h'(z)}{h(z)}+c\right). \tag{9}$$

If $h(z) \not\equiv 0$ is a polynomial, then

$$\frac{h'(z)}{h(z)} \to 0, \quad \frac{h(z+\eta)}{h(z)} \to 1, \quad z \to \infty. \tag{10}$$

It follows from (9) and (10) that $e^{c\eta} = 1 + Ac$. If $h(z) \not\equiv 0$ is a transcendental entire function with $\sigma(h) < 1$, from Lemma 3, we also have $e^{c\eta} = 1 + Ac$. □

PROOF OF Theorem 6

Proof: From the assumptions of Theorem 6, we see that (1) and (2) are still valid. Since $\Delta_\eta f(z)$ and $f'(z)$ share $a(z)$ CM, we have

$$\frac{\Delta_\eta f(z) - a(z)}{f'(z) - a(z)} = e^{P(z)}, \tag{11}$$

where $P(z)$ is a polynomial. It follows from (2) and (11) that

$$\deg P(z) \leqslant \deg Q(z). \tag{12}$$

Substituting (1) into (11) yields

$$\begin{aligned} h(z+\eta)\,e^{Q(z+\eta)-Q(z)} - h(z) + c(z)\,e^{-Q(z)} \\ = (h'(z) + h(z)Q'(z) + d(z)\,e^{-Q(z)})\,e^{P(z)}, \end{aligned} \tag{13}$$

where $c(z) = b(z+\eta) - b(z) - a(z)$ and $d(z) = b'(z) - a(z)$. Since $\sigma(a) < \sigma(f)$ and $\sigma(b) < \sigma(f)$, we see that $\max\{\sigma(c), \sigma(d)\} < \sigma(f)$. In what follows, we consider two cases: $1 \leqslant \deg P(z) < \deg Q(z)$ and $\deg P(z) = \deg Q(z)$. Set

$$\begin{aligned} P(z) &= a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0, \\ Q(z) &= b_q z^q + b_{q-1} z^{q-1} + \cdots + b_0, \end{aligned} \tag{14}$$

where $a_p (\neq 0), \ldots, a_0$, $b_q (\neq 0), \ldots, b_0$ are constants, p, q are positive integers.

Case 1. Suppose that $1 \leqslant p < q$. Then (13) can be rewritten as

$$\begin{aligned} h(z+\eta)\,e^{Q(z+\eta)-Q(z)} - h(z) - (h'(z) + h(z)Q'(z))\,e^{P(z)} \\ = (d(z)\,e^{P(z)} - c(z))\,e^{-Q(z)}. \end{aligned} \tag{15}$$

If $d(z)\,e^{P(z)} - c(z) \not\equiv 0$, since $\sigma(h) < q$, $\deg(Q(z+\eta) - Q(z)) = q-1$ and $\sigma(e^{P(z)}) = \deg P(z) = p < q$, we see that the order of growth of the left side of (5) is less than q, and the order of growth of the right side of (5) is q, a contradiction. If $d(z)\,e^{P(z)} - c(z) \equiv 0$, then (5) can be rewritten as

$$h(z+\eta)\,e^{Q(z+\eta)-Q(z)} - h(z) = (h'(z) + h(z)Q'(z))\,e^{P(z)}. \tag{16}$$

Next, we discuss two subcases: $1 \leqslant \deg P(z) < \deg Q(z) - 1$ and $1 \leqslant \deg P(z) = \deg Q(z) - 1$.

Subcase 1.1. Suppose that $1 \leqslant p < q-1$. Then (6) can be rewritten as

$$e^{Q(z+\eta)-Q(z)} = \left[1 + \left(\frac{h'(z)}{h(z)} + Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}. \tag{17}$$

If $\sigma(h) < 1$, since $\deg(Q(z+\eta) - Q(z)) = q-1 \geqslant 1$ and $\deg P(z) < q-1$, we know that the order of growth of the left-hand side of (17) is $q-1$, and the order of growth of the right-hand side of (17) is less than $q-1$, a contradiction. Then we have $\sigma(h) \geqslant 1$.

For any given ε_4, $0 < \varepsilon_3 \leqslant \varepsilon_4 < \min\left\{\frac{1}{3}(q - \sigma(h)), \frac{1}{3}(q-1-p)\right\}$, there exists $r_1 > 0$ such that for all z satisfying $|z| = r > r_1$, we have

$$\left|e^{P(z)}\right| \leqslant \exp\{r^{p+\varepsilon_4}\}. \tag{18}$$

From (17), we see that $[1 + (h'(z)/h(z) + Q'(z))\,e^{P(z)}]\,h(z)/h(z+\eta)$ is an entire function. Then for all z satisfying $|z| = r > r_1$ and $|z| = r \notin [0,1] \cup E_1 \cup E_2$, by (6)–(8) and (18), we have

$$\begin{aligned} &\left|\left[1 + \left(\frac{h'(z)}{h(z)} + Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}\right| \\ &\quad \leqslant \left[1 + \left(\left|\frac{h'(z)}{h(z)}\right| + |Q'(z)|\right)|e^{P(z)}|\right]\left|\frac{h(z)}{h(z+\eta)}\right| \\ &\quad \leqslant \left[1 + (r^{\sigma(h)-1+\varepsilon_4} + r^{q-1+\varepsilon_4})\exp\{r^{p+\varepsilon_4}\}\right] \\ &\qquad \times \exp\{r^{\sigma(h)-1+\varepsilon_4}\} \\ &\quad \leqslant r^{\sigma(h)+q-2+2\varepsilon_4}\exp\{r^{p+\varepsilon_4} + r^{\sigma(h)-1+\varepsilon_4}\} \\ &\quad < \exp\{r^{q-1}\}, \end{aligned}$$

that is,

$$\begin{aligned} &T\left(r, \left[1 + \left(\frac{h'(z)}{h(z)} + Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}\right) \\ &\quad = m\left(r, \left[1 + \left(\frac{h'(z)}{h(z)} + Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}\right) \\ &\quad < r^{q-1}. \end{aligned}$$

The above inequality yields

$$\sigma\left(\left[1 + \left(\frac{h'(z)}{h(z)} + Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}\right) < q-1.$$

It follows from $\deg(Q(z+\eta) - Q(z)) = q-1$ that (17) is a contradiction.

Subcase 1.2. Suppose that $1 \leqslant p = q-1$. It follows from (14) that

$$\left.\begin{aligned} P(z) &= a_{q-1} z^{q-1} + P_{q-2}(z), \\ Q(z+\eta) - Q(z) &= q\eta b_q z^{q-1} + Q_{q-2}(z), \end{aligned}\right\} \tag{19}$$

where $a_{q-1}(\neq 0)$, $b_q(\neq 0)$ are constants, $P_{q-2}(z)$, $Q_{q-2}(z)$ are polynomials, $\deg P_{q-2}(z) \leqslant q-2$, $\deg Q_{q-2}(z) \leqslant q-2$. In what follows, we consider two subcases: $a_{q-1} = q\eta b_q$ and $a_{q-1} \neq q\eta b_q$.

Subcase 1.2.1. If $a_{q-1} = q\eta b_q$, then (16) can be rewritten as

$$e^{-P(z)} = \frac{h(z+\eta)}{h(z)} e^{Q(z+\eta)-Q(z)-P(z)} - \left[\frac{h'(z)}{h(z)} + Q'(z)\right]. \tag{20}$$

It follows from $a_{q-1} = q\eta b_q$ that $\deg(Q(z+\eta) - Q(z) - P(z)) = \deg(Q_{q-2}(z) - P_{q-2}(z)) \leqslant q-2$. Using similar reasoning as in the proof of Subcase 1.1, we obtain

$$\sigma\left[\frac{h(z+\eta)}{h(z)} e^{Q(z+\eta)-Q(z)-P(z)} - \left(\frac{h'(z)}{h(z)} + Q'(z)\right)\right] < q-1.$$

It follows from $\deg(-P(z)) = q-1 \geqslant 1$ that (20) is a contradiction.

Subcase 1.2.2. If $a_{q-1} \neq q\eta b_q$, it follows from (16) and (19) that

$$\left(\frac{h'(z)}{h(z)} + Q'(z)\right) e^{a_{q-1}z^{q-1}} = \frac{h(z+\eta)}{h(z)} e^{q\eta b_q z^{q-1} + Q_{q-2}(z) - P_{q-2}(z)} - e^{-P_{q-2}(z)}. \tag{21}$$

Without loss of generality, we assume that $q|\eta b_q| \leqslant |a_{q-1}|$. Set $\arg a_{q-1} = \theta_1$ and $\arg(\eta b_q) = \theta_2$. For the above given ε_3 and for all z satisfying $|z| = r > r_2$ and $|z| = r \notin [0,1] \cup E_1 \cup E_2$, $z = r e^{i\theta_0}$, where θ_0 is a real constant such that $\cos((q-1)\theta_0 + \theta_1) = 1$, by (6)–(8), we have

$$\begin{aligned}
\left|\left[\frac{h'(z)}{h(z)} + Q'(z)\right] e^{a_{q-1}z^{q-1}}\right| &\geqslant \left[|Q'(z)| - \left|\frac{h'(z)}{h(z)}\right|\right] |e^{a_{q-1}z^{q-1}}| \\
&\geqslant (r^{q-1-\varepsilon_3} - r^{\sigma(h)-1+\varepsilon_3}) \exp\{|a_{q-1}| r^{q-1}\} \\
&\geqslant r^{q-1-2\varepsilon_3}(1+o(1)) \exp\{|a_{q-1}| r^{q-1}\} \\
&\geqslant \exp\{|a_{q-1}| r^{q-1}\},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{h(z+\eta)}{h(z)} e^{q\eta b_q z^{q-1} + Q_{q-2}(z) - P_{q-2}(z)} - e^{-P_{q-2}(z)}\right| \\
&\quad \leqslant \left|\frac{h(z+\eta)}{h(z)}\right| |e^{q\eta b_q z^{q-1} + Q_{q-2}(z) - P_{q-2}(z)}| + |e^{-P_{q-2}(z)}| \\
&\quad \leqslant \exp\{r^{\sigma(h)-1+\varepsilon_3}\} \\
&\qquad \times \exp\{q|\eta b_q| \cos((q-1)\theta_0 + \theta_2) r^{q-1} + O(r^{q-2})\} \\
&\quad \leqslant \exp\{q|\eta b_q| \cos((q-1)\theta_0 + \theta_2) r^{q-1} + o(r^{q-1})\},
\end{aligned}$$

that is,

$$\exp\{|a_{q-1}| r^{q-1}\} \leqslant \exp\{q|\eta b_q| \cos((q-1)\theta_0 + \theta_2) r^{q-1} + o(r^{q-1})\}. \tag{22}$$

We claim that $q|\eta b_q| \cos((q-1)\theta_0 + \theta_2) < |a_{q-1}|$. In fact, if $q|\eta b_q| = |a_{q-1}|$, it follows from $a_{q-1} \neq q\eta b_q$ that $\cos((q-1)\theta_0 + \theta_2) \neq 1$, then $\cos((q-1)\theta_0 + \theta_2) < 1$. Thus $q|\eta b_q| \cos((q-1)\theta_0 + \theta_2) < q|\eta b_q| = |a_{q-1}|$. If $q|\eta b_q| < |a_{q-1}|$, then $q|\eta b_q| \cos((q-1)\theta_0 + \theta_2) \leqslant q|\eta b_q| < |a_{q-1}|$. For any given ε_5, $0 < \varepsilon_5 < \frac{1}{3}\big(|a_{q-1}| - q|\eta b_q| \cos((q-1)\theta_0 + \theta_2)\big)$, it follows from (22) that

$$\begin{aligned}
&\exp\left\{|a_{q-1}| r^{q-1}\right\} \\
&\quad \leqslant \exp\left\{q|\eta b_q| \cos((q-1)\theta_0 + \theta_2) r^{q-1} + o(r^{q-1})\right\} \\
&\quad < \exp\{(|a_{q-1}| - \varepsilon_5) r^{q-1}\}.
\end{aligned}$$

This is a contradiction.

Case 2. Suppose that $p = q$. For a_q and b_q, we consider three subcases: (2.1) $a_q = b_q$; (2.2) $a_q = -b_q$; (2.3) $a_q \neq b_q$ and $a_q \neq -b_q$.

Subcase 2.1. Suppose that $a_q = b_q$. Then (13) can be rewritten as

$$(h'(z) + h(z)Q'(z)) e^{P(z)} - c(z) e^{-Q(z)} = h(z+\eta) e^{Q(z+\eta)-Q(z)} - h(z) - d(z) e^{P(z)-Q(z)}. \tag{23}$$

Since $\sigma(h) < q$, $\deg(Q(z+\eta) - Q(z)) = q-1$, $\max\{\sigma(c), \sigma(d)\} < q$ and $\deg(P(z) - Q(z)) \leqslant q-1$, we have $\sigma(h'(z) + h(z)Q'(z)) < q$ and $\sigma(h(z+\eta) e^{Q(z+\eta)-Q(z)} - h(z) - d(z) e^{P(z)-Q(z)}) < q$.

Noting that $e^{P(z)}$, $e^{-Q(z)}$ and $e^{P(z)+Q(z)}$ are of regular growth, and $\sigma(e^{P(z)}) = \sigma(e^{-Q(z)}) = \sigma(e^{P(z)+Q(z)}) = q$, it follows from Lemma 4 and (23) that

$$h'(z) + h(z)Q'(z) \equiv 0.$$

If $h'(z) + h(z)Q'(z) \equiv 0$, suppose that $h(z)$ is a polynomial. Then $h(z) \equiv 0$, it contradicts $h(z) \not\equiv 0$. If $h'(z) + h(z)Q'(z) \equiv 0$, suppose that $h(z)$ is a transcendental entire function. Then $h(z) = c\, e^{-Q(z)}$, $c \in \mathbb{C} \setminus \{0\}$, that is, $\sigma(h) = q$, it contradicts $\sigma(h) < q$. Then we see that $h'(z) + h(z)Q'(z) \equiv 0$ is absurd.

Subcase 2.2. Suppose that $a_q = -b_q$. Then (13) can be rewritten as

$$\big[(h'(z) + h(z)Q'(z)) e^{P(z)+Q(z)} - c(z)\big] e^{-Q(z)} + d(z) e^{P(z)-Q(z)} = h(z+\eta) e^{Q(z+\eta)-Q(z)} - h(z). \tag{24}$$

Since $\sigma(h) < q$, $\deg(Q(z+\eta)-Q(z)) = q-1$, $\max\{\sigma(c),\sigma(d)\} < q$ and $\deg(P(z)+Q(z)) \leqslant q-1$, we have $\sigma((h'(z)+h(z)Q'(z))\,\mathrm{e}^{P(z)+Q(z)}-c(z)) < q$ and $\sigma(h(z+\eta)\,\mathrm{e}^{Q(z+\eta)-Q(z)}-h(z)) < q$.

Noting that $\mathrm{e}^{-Q(z)}$, $\mathrm{e}^{P(z)-Q(z)}$ and $\mathrm{e}^{-P(z)}$ are of regular growth, and $\sigma(\mathrm{e}^{-Q(z)}) = \sigma(\mathrm{e}^{-P(z)}) = \sigma(\mathrm{e}^{P(z)-Q(z)}) = q$, it follows from Lemma 4 and (24) that

$$h(z+\eta)\,\mathrm{e}^{Q(z+\eta)-Q(z)}-h(z) \equiv 0.$$

Making use of the above identity, we obtain

$$\mathrm{e}^{Q(z)-Q(z+\eta)} \equiv \frac{h(z+\eta)}{h(z)}. \qquad (25)$$

Combining with (7) and (2), we conclude that the order of growth of the left-hand side of (25) is $q-1$, and the order of growth of the right-hand side of (25) is less than $q-1$, a contradiction.

Subcase 2.3. Suppose that $a_q \neq b_q$ and $a_q \neq -b_q$. Then (13) can be rewritten as

$$(h'(z)+h(z)Q'(z))\,\mathrm{e}^{P(z)}-c(z)\,\mathrm{e}^{-Q(z)}+d(z)\,\mathrm{e}^{P(z)-Q(z)} = h(z+\eta)\,\mathrm{e}^{Q(z+\eta)-Q(z)}-h(z). \qquad (26)$$

Since $\sigma(h) < q$, $\deg(Q(z+\eta)-Q(z)) = q-1$ and $\max\{\sigma(c),\sigma(d)\} < q$, we have $\sigma(h'(z)+h(z)Q'(z)) < q$ and $\sigma(h(z+\eta)\,\mathrm{e}^{Q(z+\eta)-Q(z)}-h(z)) < q$.

Noting that $\mathrm{e}^{\pm P(z)}$, $\mathrm{e}^{\pm Q(z)}$ and $\mathrm{e}^{P(z)\pm Q(z)}$ are of regular growth, and $\sigma(\mathrm{e}^{\pm P(z)}) = \sigma(\mathrm{e}^{\pm Q(z)}) = \sigma(\mathrm{e}^{P(z)\pm Q(z)}) = q$, it follows from Lemma 4 and (26) that

$$h'(z)+h(z)Q'(z) \equiv 0,$$
$$h(z+\eta)\,\mathrm{e}^{Q(z+\eta)-Q(z)}-h(z) \equiv 0.$$

Using similar reasoning as above, we also obtain a contradiction.

Thus $P(z)$ can only be a constant, so is $\mathrm{e}^{P(z)}$. Set $\mathrm{e}^{P(z)} \equiv A$, where A is a non-zero constant. It follows from (11) that

$$\frac{\Delta_\eta f(z)-a(z)}{f'(z)-a(z)} = A.$$

By Lemma 5, we have

$$f(z) = b(z)+H(z)\,\mathrm{e}^{cz},$$

where $b(z)$ is an entire function with $\sigma(b) < 1$, $H(z) \not\equiv 0$ is an entire function with $\lambda(H) = \sigma(H) < 1$ and $A, c, \eta \in \mathbb{C}\backslash\{0\}$ are constants satisfying $\mathrm{e}^{c\eta} = 1+Ac$. This completes the proof of Theorem 6. □

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