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New equalities and inequalities of K-g-frames in subspaces

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ABSTRACT: In this paper, we present new types of equalities and inequalities for K-g-frames in Hilbert spaces. Our equalities and inequalities for K-g-frames are different from previous ones for g-frames or fusion frames, owing to the fact that the bounded linear operator K and a parameter λ are involved, and thus allowing several known results to be derivable from our results by proper choices of K and λ.

KEYWORDS: g-Bessel sequence, equality, inequality, K-dual

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INTRODUCTION

Sun[1] introduced a frame, later named the g-frame, in order to deal with existing frames, such as frames[2], pseudo-frames[3], and fusion frames[4,5] as a united single object. Nowadays, g-frames[6–8] are widely studied by many authors. We refer the readers to Refs. 9–11 for more information on g-frames.

In this paper, we obtain some new equalities and inequalities for K-g-frames in Hilbert spaces. The K-g-frame was first introduced by Xiao et al in Ref. 12 to generalize g-frames and K-frames[13]. It should be noted that some properties of K-g-frames are quite different from those of g-frames and K-frames. For more details on K-g-frames, the readers can check Ref. 10.

Although there are a lot of literatures on equalities and inequalities for frames[14,15], fusion frames[16] and g-frames[17,18], we present some new relations involving a parameter λ for K-g-frames, inspired by the work of Poria[19]. Several known results can be obtained by assigning specific values to λ (see Remark 1, Remark 2, and Remark 3). Note that the equalities and inequalities for K-g-frames obtained in this paper contain the bounded linear operator K and a parameter λ, and so are different from previous ones obtained for g-frames or fusion frames (e.g., Refs. 14, 17, 19).

Throughout this paper, we adopt the following notation:s $\mathscr{U}$ and $\mathscr{V}$ are Hilbert spaces, with inner product $\langle\cdot,\cdot\rangle$, and norm $\|\cdot\|$; the identity operator of $\mathscr{U}$ is denoted by $I_{\mathscr{U}}$; $L(\mathscr{U},\mathscr{V})$ denotes the collection of all linear bounded operators from $\mathscr{U}$ to $\mathscr{V}$, if $\mathscr{U}=\mathscr{V}$, then $L(\mathscr{U},\mathscr{V})$ is shortened to $L(\mathscr{U})$. If $K\in L(\mathscr{H})$, then the range and the kernel of K are denoted by $R(K)$ and $N(K)$, respectively.

PRELIMINARIES OF K-G-FRAMES

In this section we mainly recall some preliminaries of K-g-frames in Hilbert spaces.

Definition 1 [Ref. 1] A sequence $\{\Lambda_j\in L(\mathscr{U},\mathscr{V}_j): j\in J\}$ is called a g-frame for $\mathscr{U}$ with respect to (w.r.t.) $\{\mathscr{V}_j: j\in J\}$, if there exist two positive constants A and B such that

$$A\|f\|^2\leqslant\sum_{j\in J}\|\Lambda_j f\|^2\leqslant B\|f\|^2,\quad\forall f\in\mathscr{U}.\qquad(1)$$

A and B are called the lower and the upper frame bounds of $\{\Lambda_j: j\in J\}$. We call $\{\Lambda_j: j\in J\}$ the g-Bessel sequence if only the right-hand inequality of (1) holds. If $\sum_{j\in J}\|\Lambda_j f\|^2=A\|f\|^2$, $A>0$, then $\{\Lambda_j: j\in J\}$ is called a tight g-frame with frame bound A.

Definition 2 [Ref. 12] A sequence $\{\Lambda_j\in L(\mathscr{U},\mathscr{V}_j): j\in J\}$ is called a K-g-frame for $\mathscr{U}$ w.r.t. $\{\mathscr{V}_j: j\in J\}$, if there exist $A,B>0$ such that

$$A\|K^*f\|^2\leqslant\sum_{j\in J}\|\Lambda_j f\|^2\leqslant B\|f\|^2,\quad\forall f\in\mathscr{U}.\qquad(2)$$

We call A and B the lower and the upper frame bounds for the K-g-frame $\{\Lambda_j : j \in J\}$, respectively.

For a g-Bessel sequence $\{\Lambda_j : j \in J\}$ in $\mathscr{U}$ w.r.t. $\{\mathscr{V}_j : j \in J\}$, if there exists $A > 0$ such that

$$\sum_{j\in J} \|\Lambda_j f\|^2 = A\|K^* f\|^2, \quad \forall f \in \mathscr{U}, \tag{3}$$

then we call $\{\Lambda_j : j \in J\}$ a tight K-g-frame with frame bound A.

Assume that $\{\Lambda_j \in L(\mathscr{U}, \mathscr{V}_j) : j \in J\}$ is a g-Bessel sequence in $\mathscr{U}$. Then the frame operator of $\{\Lambda_j : j \in J\}$ is defined as follows.

$$S : \mathscr{U} \to \mathscr{U}, \quad Sf = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \quad \forall f \in \mathscr{U}. \tag{4}$$

Let $\{\Lambda_j \in L(\mathscr{U}, \mathscr{V}_j) : j \in J\}$ be a K-g-frame for $\mathscr{U}$. According to Ref. 12, there exists a g-Bessel sequence $\{\Gamma_j \in L(\mathscr{U}, \mathscr{V}_j) : j \in J\}$ in $\mathscr{U}$ such that

$$Kf = \sum_{j\in J} \Lambda_j^* \Gamma_j f, \quad \forall f \in \mathscr{U}. \tag{5}$$

$\{\Gamma_j\}_{j\in J}$ is called a K-dual of $\{\Lambda_j\}_{j\in J}$. Note that the roles of $\{\Gamma_j\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ are not interchangeable in general[12]. By (4) and the K-dual definition (5), we can obtain the following interesting result.

Proposition 1 *For any g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ with frame operator S, $\{\Lambda_j\}_{j\in J}$ is an S-dual of $\{\Lambda_j\}_{j\in J}$.*

Proof: It follows from (4) and (5). □

We also need the following lemmas.

Lemma 1 (Ref. 2) *Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are two Hilbert spaces, and $Q \in L(\mathscr{H}_1, \mathscr{H}_2)$ has closed range. Then, there exists a unique bounded operator $Q^+ : \mathscr{H}_2 \to \mathscr{H}_1$, called the pseudo-inverse operator of Q, satisfying*

$$\begin{aligned} N(Q^+) = R(Q)^\perp, \quad R(Q^+) = N(Q)^\perp, \\ QQ^+ = P_{R(Q)}, \quad Q^+Q = P_{R(Q^+)}. \end{aligned} \tag{6}$$

Lemma 2 (Ref. 19) *Suppose that $P, Q \in L(\mathscr{U})$ satisfy $P + Q = I_{\mathscr{U}}$. Then, for any $\lambda \in [0, 1]$, we have*

$$\begin{aligned} P^*P + \lambda(Q^* + Q) &= Q^*Q + (1-\lambda)(P^* + P) + (2\lambda - 1)I_{\mathscr{U}} \\ &\geqslant (1 - (\lambda - 1)^2)I_{\mathscr{U}}. \end{aligned} \tag{7}$$

Lemma 3 *Suppose that $P, Q \in L(\mathscr{U})$ satisfy $P + Q = I_{\mathscr{U}}$. Then for any λ we have*

$$P^*P + \lambda P + Q^* = Q^*Q - (1 + \lambda)Q + (\lambda + 1)I_{\mathscr{U}}. \tag{8}$$

Proof: For any λ, we have

$$\begin{aligned} P^*P + \lambda P + Q^* &= (I_{\mathscr{U}} - Q)^*(I_{\mathscr{U}} - Q) + \lambda(I_{\mathscr{U}} - Q) + Q^* \\ &= Q^*Q - (1 + \lambda)Q + (\lambda + 1)I_{\mathscr{U}}. \end{aligned}$$

□

Lemma 4 *Suppose that $P, Q \in L(\mathscr{U})$ and $P + Q = M$. Then we have*

$$P^*P + M^*M = Q^*Q + M^*P + P^*M.$$

Proof: The result follows from

$$\begin{aligned} Q^*Q + M^*P + P^*M &= (M - P)^*(M - P) + M^*P + P^*M \\ &= P^*P + M^*M. \end{aligned}$$

□

EQUALITIES AND INEQUALITIES OF K-G-FRAMES

For a pair of K-dual $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$, if $R(K)$ is closed, we can obtain the following equalities and inequalities with K^+ and a parameter λ.

Theorem 1 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j \in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t. $\{\mathscr{V}_j : j \in J\}$ and be a K-dual of $\{\Lambda_j\}_{j\in J}$. If $R(K)$ is closed, then we obtain:*

(i) For any $f \in R(K), \{a_j\}_{j\in J} \in l^\infty(J)$, and $\lambda \in [0, 1]$,

$$\begin{aligned} &\Big\| \sum_{j\in J} a_j \Lambda_j^* \Gamma_j K^+ f \Big\|^2 + 2\lambda \operatorname{Re} \sum_{j\in J} (1 - a_j)\langle \Gamma_j K^+ f, \Lambda_j f\rangle \\ &= \Big\| \sum_{j\in J} (1 - a_j)\Lambda_j^* \Gamma_j K^+ f \Big\|^2 \\ &\quad + 2(1 - \lambda) \operatorname{Re} \sum_{j\in J} a_j \langle \Gamma_j K^+ f, \Lambda_j f\rangle + (2\lambda - 1)\|f\|^2 \\ &\geqslant (1 - (\lambda - 1)^2)\|f\|^2. \end{aligned} \tag{9}$$

(ii) For any $f \in R(K), \{a_j\}_{j\in J} \in l^\infty(J)$, $\lambda \in \mathbb{R}$, and $\lambda \in [-1, 3]$,

$$\begin{aligned} &\Big\| \sum_{j\in J} a_j \Lambda_j^* \Gamma_j K^+ f \Big\|^2 + \lambda \sum_{j\in J} a_j \langle \Gamma_j K^+ f, \Lambda_j f\rangle \\ &\quad + \sum_{j\in J} (1 - \bar{a}_j)\langle \Lambda_j f, \Gamma_j K^+ f\rangle \\ &= \Big\| \sum_{j\in J} (1 - a_j)\Lambda_j^* \Gamma_j K^+ f \Big\|^2 \\ &\quad - (1 + \lambda) \sum_{j\in J} (1 - a_j)\langle \Gamma_j K^+ f,\ \Lambda_j f\rangle \\ &\quad + (\lambda + 1)\|f\|^2. \end{aligned} \tag{10}$$

Furthermore, we obtain

$$\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_jK^+f\Big\|^2+\lambda\,\mathrm{Re}\sum_{j\in J}a_j\langle\Gamma_jK^+f,\Lambda_jf\rangle+\mathrm{Re}\sum_{j\in J}(1-\bar{a}_j)\langle\Lambda_jf,\Gamma_jK^+f\rangle \geqslant (1-\tfrac{1}{4}(\lambda-1)^2)\|f\|^2 \quad \textit{or} \quad (\tfrac{3}{4}-\tfrac{1}{4}\lambda)\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_jK^+f\Big\|^2.$$

Proof: For any $f\in\mathscr{U}$ and $\{a_j\}_{j\in J}\in l^\infty(J)$, define $L_1,L_2:\mathscr{U}\to\mathscr{U}$ as follows.

$$L_1f=\sum_{j\in J}a_j\Lambda_j^*\Gamma_jf,\quad L_2f=\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_jf.$$

We now show that L_1 and L_2 are well defined. For any $f\in\mathscr{U}$ and any finite subset $I\subset J$, we have

$$\begin{aligned}\Big\|\sum_{j\in I}a_j\Lambda_j^*\Gamma_jf\Big\|&=\sup_{g\in\mathscr{U},\|g\|=1}\Big|\Big\langle\sum_{j\in I}a_j\Lambda_j^*\Gamma_jf,g\Big\rangle\Big|\\&=\sup_{g\in\mathscr{U},\|g\|=1}\Big|\sum_{j\in I}a_j\langle\Gamma_jf,\Lambda_jg\rangle\Big|\\&\leqslant\sup_{g\in\mathscr{U},\|g\|=1}\|a\|_\infty\sum_{j\in I}\|\Gamma_jf\|\|\Lambda_jg\|\\&\leqslant\|a\|_\infty\sup_{g\in\mathscr{U},\|g\|=1}\Big(\sum_{j\in I}\|\Gamma_jf\|^2\Big)^{1/2}\Big(\sum_{j\in I}\|\Lambda_jg\|^2\Big)^{1/2}\\&\leqslant\|a\|_\infty\sup_{g\in\mathscr{U},\|g\|=1}\sqrt{B}\|g\|\Big(\sum_{j\in I}\|\Gamma_jf\|^2\Big)^{1/2}\\&=\sqrt{B}\|a\|_\infty\Big(\sum_{j\in I}\|\Gamma_jf\|^2\Big)^{1/2},\end{aligned}$$

where B is the Bessel bound of $\{\Lambda_j\}_{j\in J}$. As $\{\Gamma_j\}_{j\in J}$ is a g-Bessel sequence in $\mathscr{U}$ and I is an arbitrary finite subset of J, L_1 is well defined on $\mathscr{U}$. Similarly we can also show that L_2 is well defined on $\mathscr{U}$.

As $\{\Gamma_j\}_{j\in J}$ is a K-dual of $\{\Lambda_j\}_{j\in J}$, we have

$$L_1+L_2=K. \tag{11}$$

Also, as $R(K)$ is closed, from Lemma 1 and (11), we have

$$L_1K^++L_2K^+=KK^+=P_{R(K)}. \tag{12}$$

It follows that

$$L_1K^+|_{R(K)}+L_2K^+|_{R(K)}=I_{R(K)}. \tag{13}$$

For (i), from Lemma 2 and (13), we obtain

$$\begin{aligned}&(L_1K^+|_{R(K)})^*(L_1K^+|_{R(K)})+\lambda\big[(L_2K^+|_{R(K)})^*+L_2K^+|_{R(K)}\big]\\&=(L_2K^+|_{R(K)})^*(L_2K^+|_{R(K)})+(1-\lambda)\big[(L_1K^+|_{R(K)})^*+L_1K^+|_{R(K)}\big]+(2\lambda-1)I_{R(K)}\\&\geqslant(1-(\lambda-1)^2)I_{R(K)}.\end{aligned}$$

And for any $f\in R(K)$, we have

$$\begin{aligned}&\langle(L_1K^+|_{R(K)})^*(L_1K^+|_{R(K)})f,f\rangle+\lambda\langle((L_2K^+|_{R(K)})^*+L_2K^+|_{R(K)})f,f\rangle\\&=\|L_1K^+f\|^2+\lambda\langle f,L_2K^+f\rangle+\lambda\langle L_2K^+f,f\rangle\\&=\|L_1K^+f\|^2+2\lambda\,\mathrm{Re}\langle L_2K^+f,f\rangle\\&=\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_jK^+f\Big\|^2+2\lambda\,\mathrm{Re}\sum_{j\in J}(1-a_j)\langle\Gamma_jK^+f,\Lambda_jf\rangle,\end{aligned}\tag{14}$$

and

$$\begin{aligned}&\langle(L_2K^+|_{R(K)})^*(L_2K^+|_{R(K)})f,f\rangle+(1-\lambda)\langle((L_1K^+|_{R(K)})^*+L_1K^+|_{R(K)})f,f\rangle+(2\lambda-1)\langle I_{R(K)}f,f\rangle\\&=\|L_2K^+f\|^2+2(1-\lambda)\,\mathrm{Re}\langle L_1K^+f,f\rangle+(2\lambda-1)\|f\|^2\\&=\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_jK^+f\Big\|^2+2(1-\lambda)\,\mathrm{Re}\sum_{j\in J}a_j\langle\Gamma_jK^+f,\Lambda_jf\rangle+(2\lambda-1)\|f\|^2.\end{aligned}\tag{15}$$

Combining with (14) and (15), we know that (9) holds. For (ii), from (13) and Lemma 3, we have for any λ,

$$\begin{aligned}&(L_1K^+|_{R(K)})^*(L_1K^+|_{R(K)})+\lambda(L_1K^+|_{R(K)})+(L_2K^+|_{R(K)})^*\\&=(L_2K^+|_{R(K)})^*(L_2K^+|_{R(K)})-(1+\lambda)(L_2K^+|_{R(K)})+(\lambda+1)I_{R(K)}.\end{aligned}\tag{16}$$

Hence for any $f\in R(K)$, we have

$$\begin{aligned}&\|L_1K^+f\|^2+\lambda\langle(L_1K^+)f,f\rangle+\langle(L_2K^+)^*f,f\rangle\\&=\|(L_2K^+)f\|^2-(1+\lambda)\langle(L_2K^+)f,f\rangle+(\lambda+1)\|f\|^2.\end{aligned}$$

It follows that (10) holds. Furthermore, we obtain

$$\begin{aligned}&\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_jK^+f\Big\|^2+\lambda\,\mathrm{Re}\sum_{j\in J}a_j\langle\Gamma_jK^+f,\Lambda_jf\rangle\\&\qquad+\mathrm{Re}\sum_{j\in J}(1-\bar a_j)\langle\Lambda_jf,\Gamma_jK^+f\rangle\\&=\|L_1K^+f\|^2+\lambda\,\mathrm{Re}\langle(L_1K^+)f,f\rangle+\mathrm{Re}\langle(L_2K^+)^*f,f\rangle\\&=\|(L_2K^+)f\|^2-(1+\lambda)\,\mathrm{Re}\langle(L_2K^+)f,f\rangle+(\lambda+1)\|f\|^2\\&=\|(L_2K^+)f-\tfrac{1+\lambda}{2}f\|^2+\Big(\lambda+1-\tfrac{(1+\lambda)^2}{4}\Big)\|f\|^2\\&\geqslant(1-\tfrac14(\lambda-1)^2)\|f\|^2.\end{aligned}\tag{17}$$

From (17) we also obtain

$$\begin{aligned}&\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_jK^+f\Big\|^2+\lambda\,\mathrm{Re}\sum_{j\in J}a_j\langle\Gamma_jK^+f,\Lambda_jf\rangle\\&\qquad+\mathrm{Re}\sum_{j\in J}(1-\bar a_j)\langle\Lambda_jf,\Gamma_jK^+f\rangle\\&=\|(L_2K^+)f\|^2-(1+\lambda)\,\mathrm{Re}\langle(L_2K^+)f,f\rangle+(\lambda+1)\|f\|^2\\&=(1+\lambda)(\|f\|^2-\mathrm{Re}\langle(L_2K^+)f,f\rangle)+\|(L_2K^+)f\|^2\\&=(1+\lambda)(\|f-\tfrac12L_2K^+f\|^2-\tfrac14\|(L_2K^+)f\|^2)\\&\qquad+\|(L_2K^+)f\|^2\\&=(1+\lambda)\|f-\tfrac12L_2K^+f\|^2+(\tfrac34-\tfrac14\lambda)\|(L_2K^+)f\|^2\\&\geqslant(\tfrac34-\tfrac14\lambda)\|(L_2K^+)f\|^2.\end{aligned}$$

□

Remark 1 In Theorem 1, if we take $K=I_{\mathscr{U}}$, and let $\{\Lambda_j\}_{j\in J}$ be a tight g-frame for $\mathscr{U}$ with frame bound A. Then the canonical dual $\{\Gamma_j\}_{j\in J}$ has the form $\{(1/A)\Lambda_j\}_{j\in J}$ as $\Gamma_j=\Lambda_jS^{-1}=(1/A)\Lambda_j$. Then, we can obtain Theorem 2.2 in Ref. 17 from (ii) in Theorem 1, by taking $\lambda=-1$.

Corollary 1 *Let $\{\Lambda_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j\in J\}$, with frame operator S. If $R(S)$ is closed, we obtain:*

(i) For any $f\in R(S)$, $\{a_j\}_{j\in J}\in l^\infty(J)$, and $\lambda\in[0,1]$,

$$\begin{aligned}&\Big\|\sum_{j\in J}a_j\Lambda_j^*\Lambda_jS^+f\Big\|^2+2\lambda\,\mathrm{Re}\sum_{j\in J}(1-a_j)\langle\Lambda_jS^+f,\Lambda_jf\rangle\\&\quad=\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Lambda_jS^+f\Big\|^2\\&\quad+2(1-\lambda)\,\mathrm{Re}\sum_{j\in J}a_j\langle\Lambda_jS^+f,\Lambda_jf\rangle+(2\lambda-1)\|f\|^2\\&\quad\geqslant(1-(\lambda-1)^2)\|f\|^2.\end{aligned}$$

(ii) For any $f\in R(S)$, $\{a_j\}_{j\in J}\in l^\infty(J)$, $\lambda\in\mathbb{R}$, and $\lambda\in[-1,3]$,

$$\begin{aligned}&\Big\|\sum_{j\in J}a_j\Lambda_j^*\Lambda_jS^+f\Big\|^2+\lambda\sum_{j\in J}a_j\langle\Lambda_jS^+f,\Lambda_jf\rangle\\&\qquad+\sum_{j\in J}(1-\bar a_j)\langle\Lambda_jf,\Lambda_jS^+f\rangle\\&=\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Lambda_jS^+f\Big\|^2\\&\quad-(1+\lambda)\sum_{j\in J}(1-a_j)\langle\Lambda_jS^+f,\Lambda_jf\rangle+(\lambda+1)\|f\|^2,\end{aligned}$$

and

$$\begin{aligned}&\Big\|\sum_{j\in J}a_j\Lambda_j^*\Lambda_jS^+f\Big\|^2+\lambda\,\mathrm{Re}\sum_{j\in J}a_j\langle\Lambda_jS^+f,\Lambda_jf\rangle\\&\qquad+\mathrm{Re}\sum_{j\in J}(1-\bar a_j)\langle\Lambda_jf,\Lambda_jS^+f\rangle\\&\geqslant(1-\tfrac14(\lambda-1)^2)\|f\|^2\quad\text{or}\\&\qquad(\tfrac34-\tfrac14\lambda)\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Lambda_jS^+f\Big\|^2.\end{aligned}$$

Proof: From Proposition 1, we know that $\{\Lambda_j\}_{j\in J}$ is an S-dual of $\{\Lambda_j\}_{j\in J}$, where S is the frame operator of $\{\Lambda_j\}_{j\in J}$. Hence we can obtain the results from Theorem 1. □

If we take $\lambda=1/2$ in (i) or $\lambda=0$ in (ii) in Theorem 1, we obtain the following corollary.

Corollary 2 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j\in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t.$\{\mathscr{V}_j : j\in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. If $R(K)$ is closed, then for any $f\in R(K)$ and $\{a_j\}_{j\in J}\in l^\infty(J)$, we have*

$$\begin{aligned}&\sum_{j\in J}(1-\bar a_j)\langle\Lambda_jf,\Gamma_jK^+f\rangle+\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_jK^+f\Big\|^2=\\&\sum_{j\in J}a_j\langle\Gamma_jK^+f,\Lambda_jf\rangle+\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_jK^+f\Big\|^2,\end{aligned}\tag{18}$$

$$\mathrm{Re}\sum_{j\in J}(1-\bar a_j)\langle\Lambda_jf,\Gamma_jK^+f\rangle+\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_jK^+f\Big\|^2\geqslant\tfrac34\|f\|^2,\tag{19}$$

$$\mathrm{Re}\sum_{j\in J}(1-a_j)\langle\Gamma_jK^+f,\Lambda_jf\rangle+\tfrac14\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_jK^+f\Big\|^2\geqslant0.\tag{20}$$

Proof: (19) is trivial. If we take $\lambda = 0$ in (ii) in Theorem 1, then from (10) and (13), we have for any $f \in R(K)$,

$$\begin{aligned}
&\Big\|\sum_{j\in J} a_j\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + \sum_{j\in J}(1-\bar{a}_j)\langle\Lambda_j f, \Gamma_j K^+ f\rangle \\
&= \Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j K^+ f\Big\|^2 \\
&\quad -\sum_{j\in J}(1-a_j)\langle\Gamma_j K^+ f, \Lambda_j f\rangle + \|f\|^2 \\
&= \Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j K^+ f\Big\|^2 - \langle L_2K^+ f, f\rangle + \|f\|^2 \\
&= \Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j K^+ f\Big\|^2 - \langle L_2K^+ |_{R(K)} f, f\rangle + \|f\|^2 \\
&= \Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + \langle (I_{R(K)} - L_2K^+ |_{R(K)}) f, f\rangle \\
&= \Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + \langle L_1K^+ |_{R(K)} f, f\rangle \\
&= \Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + \sum_{j\in J} a_j\langle\Gamma_j K^+ f, \Lambda_j f\rangle.
\end{aligned}$$

Hence (18) holds. For (20), from (13) we have for any $f \in R(K)$,

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in J}(1-a_j)\langle\Gamma_j K^+ f, \Lambda_j f\rangle + \Big\|\sum_{j\in J} a_j\Lambda_j^*\Gamma_j K^+ f\Big\|^2 \\
&\quad = \mathrm{Re}\langle L_2K^+ |_{R(K)} f, f\rangle + \|L_1K^+ f\|^2 \\
&\quad = \|L_1K^+ f\|^2 - \mathrm{Re}\langle L_1K^+ |_{R(K)} f, f\rangle + \|f\|^2 \\
&\quad = \|L_1K^+ f\|^2 - \mathrm{Re}\langle L_1K^+ f, f\rangle + \|f\|^2 \\
&\quad = \|f - \tfrac{1}{2}L_1K^+ f\|^2 + \tfrac{3}{4}\|L_1K^+ f\|^2 \\
&\quad \geqslant \tfrac{3}{4}\|L_1K^+ f\|^2 = \tfrac{3}{4}\Big\|\sum_{j\in J} a_j\Lambda_j^*\Gamma_j K^+ f\Big\|^2.
\end{aligned}$$

Hence (20) holds. □

Remark 2 In Corollary 2, if we let $K = I_{\mathscr{U}}$, we conclude that $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathscr{U}$, with an alternate dual g-frame $\{\Gamma_j\}_{j\in J}$. Furthermore, if we take $\Lambda_j f = w_i \pi_{\mathscr{W}_i} f$, $\forall j \in J$, where $\{\mathscr{W}_i\}_{i\in I}$ is a sequence of closed subspaces in $\mathscr{U}$ and $\{w_i\}_{i\in I}$ is a family of positive weights, then it follows that $\{(\mathscr{W}_i, w_i)\}_{i\in I}$ is a fusion frame for $\mathscr{U}$. We now obtain Theorem 4.2 in Ref. 16 from (18) in Corollary 2.

If we take $\{a_j\}_{j\in J}$ in some particular case, then from Theorem 1 we obtain the following result.

Corollary 3 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j \in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t. $\{\mathscr{V}_j : j \in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. If $R(K)$ is closed, then:*

(i) For any $f \in R(K)$, $I \subset J$, and $\lambda \in [0,1]$,

$$\begin{aligned}
&\Big\|\sum_{j\in I^c}\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + 2\lambda\,\mathrm{Re}\sum_{j\in I}\langle\Gamma_j K^+ f, \Lambda_j f\rangle \\
&\quad = \Big\|\sum_{j\in I}\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + 2(1-\lambda)\,\mathrm{Re}\sum_{j\in I^c}\langle\Gamma_j K^+ f, \Lambda_j f\rangle \\
&\qquad + (2\lambda-1)\|f\|^2 \\
&\quad \geqslant (1-(\lambda-1)^2)\|f\|^2.
\end{aligned}$$

(ii) For any $f \in R(K)$, $\lambda \in \mathbb{R}$, and $\lambda \in [-1,3]$, $I \subset J$,

$$\begin{aligned}
&\Big\|\sum_{j\in I^c}\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + \lambda\sum_{j\in I^c}\langle\Gamma_j K^+ f, \Lambda_j f\rangle + \sum_{j\in I}\langle\Lambda_j f, \Gamma_j K^+ f\rangle \\
&\quad = \Big\|\sum_{j\in I}\Lambda_j^*\Gamma_j K^+ f\Big\|^2 \\
&\qquad -(1+\lambda)\sum_{j\in I}\langle\Gamma_j K^+ f, \Lambda_j f\rangle + (\lambda+1)\|f\|^2.
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
&\Big\|\sum_{j\in I^c}\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + \lambda\,\mathrm{Re}\sum_{j\in I^c}\langle\Gamma_j K^+ f, \Lambda_j f\rangle \\
&\qquad + \mathrm{Re}\sum_{j\in I}\langle\Lambda_j f, \Gamma_j K^+ f\rangle \\
&\quad \geqslant (1-\tfrac{1}{4}(\lambda-1)^2)\|f\|^2 \text{ or } (\tfrac{3}{4}-\tfrac{1}{4}\lambda)\Big\|\sum_{j\in I}\Lambda_j^*\Gamma_j K^+ f\Big\|^2.
\end{aligned}$$

Proof: For any subset $I \subset J$, if we take $\{a_j\}_{j\in J}$ in Theorem 1 as

$$a_j = \begin{cases} 0, & j \in I, \\ 1, & j \in I^c, \end{cases} \tag{21}$$

then the results follow from Theorem 1. □

Remark 3 In Corollary 3, if we take $K = I_{\mathscr{U}}$, $\Gamma_j = \Lambda_j$, $\forall j \in J$, and $\lambda = 1/2$ in (i), or $\lambda = -1$ in (ii), then we obtain Theorem 3.2 in Ref. 19 for the case of Parseval g-frames. Furthermore, if we take $\lambda = 1/2$ in (i), or $\lambda = 0$ in (ii), we obtain the version of Parseval g-frames inequality for Corollary 3.3 in Ref. 19. Also, if we take $K = I_{\mathscr{U}}$, $\Lambda_j f = \langle f, f_j\rangle e_j$, $\forall j \in J$, where $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathscr{U}$, then $\{\Lambda_j\}_{j\in J}$ being an $I_{\mathscr{U}}$-g-frame for $\mathscr{U}$ implies that $\{f_j\}_{j\in J}$ is a frame for $\mathscr{U}$. Now letting $\lambda = 1/2$ in (i), and $\lambda = -1$ in (ii), we respectively obtain Theorem 3.2 in Ref. 14 and Theorem 2.2 in Ref. 15 from Corollary 3.

Now we present a parallel result to Theorem 1 in which the elements are restricted to $R(K^+)$.

Theorem 2 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j \in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t.$\{\mathscr{V}_j : j \in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. If $R(K)$ is closed, then we have:*
(i) For $f \in R(K^+)$, $\{a_j\}_{j\in J} \in l^\infty(J)$, and $\lambda \in [0,1]$,

$$\Big\|\sum_{j\in J} a_j K^+\Lambda_j^*\Gamma_j f\Big\|^2 + 2\lambda\,\mathrm{Re}\sum_{j\in J}(1-a_j)\langle \Gamma_j f, \Lambda_j (K^+)^* f\rangle = \Big\|\sum_{j\in J}(1-a_j)K^+\Lambda_j^*\Gamma_j f\Big\|^2 + 2(1-\lambda)\,\mathrm{Re}\sum_{j\in J} a_j\langle \Gamma_j f, \Lambda_j (K^+)^* f\rangle + (2\lambda-1)\|f\|^2 \geqslant (1-(\lambda-1)^2)\|f\|^2. \tag{22}$$

(ii) For any $f \in R(K^+)$, $\{a_j\}_{j\in J} \in l^\infty(J)$, $\lambda \in \mathbb{R}$, and $\lambda \in [-1,3]$,

$$\Big\|\sum_{j\in J} a_j K^+\Lambda_j^*\Gamma_j f\Big\|^2 + \lambda\sum_{j\in J} a_j\langle \Gamma_j f, \Lambda_j (K^+)^* f\rangle + \sum_{j\in J}(1-\bar{a}_j)\langle \Lambda_j (K^+)^* f, \Gamma_j f\rangle = \Big\|\sum_{j\in J}(1-a_j)K^+\Lambda_j^*\Gamma_j f\Big\|^2 - (1+\lambda)\sum_{j\in J}(1-a_j)\langle \Gamma_j f, \Lambda_j (K^+)^* f\rangle + (\lambda+1)\|f\|^2.$$

Furthermore, we obtain

$$\Big\|\sum_{j\in J} a_j K^+\Lambda_j^*\Gamma_j f\Big\|^2 + \lambda\,\mathrm{Re}\sum_{j\in J} a_j\langle \Gamma_j f, \Lambda_j (K^+)^* f\rangle + \mathrm{Re}\sum_{j\in J}(1-\bar{a}_j)\langle \Lambda_j (K^+)^* f, \Gamma_j f\rangle \geqslant (1-\tfrac{1}{4}(\lambda-1)^2)\|f\|^2 \quad or \quad (\tfrac{3}{4}-\tfrac{1}{4}\lambda)\Big\|\sum_{j\in J}(1-a_j)K^+\Lambda_j^*\Gamma_j f\Big\|^2.$$

Proof: Let L_1 and L_2 be defined as in Theorem 1. Combining with Lemma 1 and (11), we have

$$K^+L_1 + K^+L_2 = K^+K = P_{R(K^+)}.$$

It follows that

$$K^+L_1|_{R(K^+)} + K^+L_2|_{R(K^+)} = I_{R(K^+)}. \tag{23}$$

Then, by the same method as in Theorem 1, we can show that (i) and (ii) hold. □

From Theorem 2 we also obtain several corollaries as follows.

Corollary 4 *Let $\{\Lambda_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j \in J\}$ with frame operator S. If $R(S)$ is closed, then we have:*
(i) For $f \in R(S^+)$, $\{a_j\}_{j\in J} \in l^\infty(J)$, and $\lambda \in [0,1]$,

$$\Big\|\sum_{j\in J} a_j S^+\Lambda_j^*\Lambda_j f\Big\|^2 + 2\lambda\,\mathrm{Re}\sum_{j\in J}(1-a_j)\langle \Lambda_j f, \Lambda_j (S^+)^* f\rangle = \Big\|\sum_{j\in J}(1-a_j)S^+\Lambda_j^*\Lambda_j f\Big\|^2 + 2(1-\lambda)\,\mathrm{Re}\sum_{j\in J} a_j\langle \Lambda_j f, \Lambda_j (S^+)^* f\rangle + (2\lambda-1)\|f\|^2 \geqslant (1-(\lambda-1)^2)\|f\|^2.$$

(ii) For any $f \in R(S^+)$, $\{a_j\}_{j\in J} \in l^\infty(J)$, $\lambda \in \mathbb{R}$, and $\lambda \in [-1,3]$,

$$\Big\|\sum_{j\in J} a_j S^+\Lambda_j^*\Lambda_j f\Big\|^2 + \lambda\sum_{j\in J} a_j\langle \Lambda_j f, \Lambda_j (S^+)^* f\rangle + \sum_{j\in J}(1-\bar{a}_j)\langle \Lambda_j (S^+)^* f, \Lambda_j f\rangle = \Big\|\sum_{j\in J}(1-a_j)S^+\Lambda_j^*\Lambda_j f\Big\|^2 - (1+\lambda)\sum_{j\in J}(1-a_j)\langle \Lambda_j f, \Lambda_j (S^+)^* f\rangle + (\lambda+1)\|f\|^2,$$

and

$$\Big\|\sum_{j\in J} a_j S^+\Lambda_j^*\Lambda_j f\Big\|^2 + \lambda\,\mathrm{Re}\sum_{j\in J} a_j\langle \Lambda_j f, \Lambda_j (S^+)^* f\rangle + \mathrm{Re}\sum_{j\in J}(1-\bar{a}_j)\langle \Lambda_j (S^+)^* f, \Lambda_j f\rangle \geqslant (1-\tfrac{1}{4}(\lambda-1)^2)\|f\|^2 \quad or \quad (\tfrac{3}{4}-\tfrac{1}{4}\lambda)\Big\|\sum_{j\in J}(1-a_j)S^+\Lambda_j^*\Lambda_j f\Big\|^2.$$

Proof: The results follow from Proposition 1 and Theorem 2. □

Corollary 5 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j \in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t.$\{\mathscr{V}_j : j \in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. If $R(K)$ is closed, then for any $f \in R(K^+)$ and $\{a_j\}_{j\in J} \in l^\infty(J)$, we have*

$$\sum_{j\in J}(1-a_j)\langle \Gamma_j f, \Lambda_j (K^+)^* f\rangle + \Big\|\sum_{j\in J} a_j K^+\Lambda_j^*\Gamma_j f\Big\|^2 = \sum_{j\in J}\bar{a}_j\langle \Lambda_j (K^+)^* f, \Gamma_j f\rangle + \Big\|\sum_{j\in J}(1-a_j)K^+\Lambda_j^*\Gamma_j f\Big\|^2,$$

$$\mathrm{Re}\sum_{j\in J}(1-a_j)\langle\Gamma_j f,\Lambda_j(K^+)^*f\rangle+\Big\|\sum_{j\in J}a_jK^+\Lambda_j^*\Gamma_j f\Big\|^2 \geqslant \tfrac{3}{4}\|f\|^2,$$

$$\mathrm{Re}\sum_{j\in J}(1-a_j)\langle\Gamma_j f,\Lambda_j(K^+)^*f\rangle+\tfrac{1}{4}\Big\|\sum_{j\in J}a_jK^+\Lambda_j^*\Gamma_j f\Big\|^2\geqslant 0.$$

Proof: The proof is similar to that of Corollary 2, so we omit it. □

Corollary 6 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j\in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t.$\{\mathscr{V}_j : j\in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. If $R(K)$ is closed, then we have:*
(i) For any $f\in R(K^+), I\subset J$, and $\lambda\in[0,1]$,

$$\begin{aligned}\Big\|\sum_{j\in I^c}K^+\Lambda_j^*\Gamma_j f\Big\|^2+2\lambda\,\mathrm{Re}\sum_{j\in I}\langle\Gamma_j f,\Lambda_j(K^+)^*f\rangle\\ =\Big\|\sum_{j\in I}K^+\Lambda_j^*\Gamma_j f\Big\|^2\\ +2(1-\lambda)\,\mathrm{Re}\sum_{j\in I^c}\langle\Gamma_j f,\Lambda_j(K^+)^*f\rangle+(2\lambda-1)\|f\|^2\\ \geqslant(1-(\lambda-1)^2)\|f\|^2.\end{aligned}\quad(24)$$

(ii) For any $f\in R(K^+)$, $\lambda\in\mathbb{R}$, and $\lambda\in[-1,3]$,

$$\begin{aligned}\Big\|\sum_{j\in I^c}K^+\Lambda_j^*\Gamma_j f\Big\|^2+\lambda\sum_{j\in I^c}\langle\Gamma_j f,\Lambda_j(K^+)^*f\rangle\\ +\sum_{j\in I}\langle\Lambda_j(K^+)^*f,\Gamma_j f\rangle\\ =\Big\|\sum_{j\in I}K^+\Lambda_j^*\Gamma_j f\Big\|^2\\ -(1+\lambda)\sum_{j\in I}\langle\Gamma_j f,\Lambda_j(K^+)^*f\rangle+(\lambda+1)\|f\|^2,\end{aligned}$$

and

$$\begin{aligned}\Big\|\sum_{j\in I^c}K^+\Lambda_j^*\Gamma_j f\Big\|^2+\lambda\,\mathrm{Re}\sum_{j\in I^c}\langle\Gamma_j f,\Lambda_j(K^+)^*f\rangle\\ +\mathrm{Re}\sum_{j\in I}\langle\Lambda_j(K^+)^*f,\Gamma_j f\rangle\\ \geqslant(1-\tfrac{1}{4}(\lambda-1)^2)\|f\|^2\ \text{or}\ (\tfrac{3}{4}-\tfrac{1}{4}\lambda)\Big\|\sum_{j\in I}K^+\Lambda_j^*\Gamma_j f\Big\|^2.\end{aligned}$$

Proof: It follows from Theorem 2 if we take $\{a_j\}_{j\in J}$ as defined in (21). □

Remark 4 Note that if we take $K=I_{\mathscr{U}}$, then Theorem 1 and Theorem 2 are the same, and according to Remark 1, Remark 2, and Remark 3 we know from Theorem 2 that many known equalities and inequalities in Refs. 14–17, 19 can also be obtained.

Lastly, we provide some new equalities for a K-dual $pair\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$.

Theorem 3 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j\in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t.$\{\mathscr{V}_j : j\in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. Then for any $f\in\mathscr{U}$ and $\{a_j\}_{j\in J}\in l^\infty(J)$, we obtain*

$$\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j f\Big\|^2+2\,\mathrm{Re}\sum_{j\in J}a_j\langle\Gamma_j f,\Lambda_jKf\rangle =\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_j f\Big\|^2+\|Kf\|^2.\quad(25)$$

Proof: Let L_1 and L_2 be defined as in Theorem 1. As $\{\Gamma_j\}_{j\in J}$ is a K-dual of $\{\Lambda_j\}_{j\in J}$, (11) holds. Regarding L_1 and L_2 as P and Q, respectively, in Lemma 4, for any $f\in\mathscr{U}$, now we have

$$\langle L_1^*L_1f,f\rangle+\langle K^*Kf,f\rangle =\langle L_2^*L_2f,f\rangle+\langle K^*L_1f,f\rangle+\langle L_1^*Kf,f\rangle,$$

that is

$$\|L_1f\|^2+\|Kf\|^2=\|L_2f\|^2+\langle L_1f,Kf\rangle+\langle Kf,L_1f\rangle.$$

Then we have

$$\begin{aligned}\Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_j f\Big\|^2+\|Kf\|^2\\ =\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j f\Big\|^2+\sum_{j\in J}a_j\langle\Gamma_j f,\ \Lambda_jKf\rangle\\ +\sum_{j\in J}\bar{a}_j\langle\Lambda_jKf,\Gamma_j f\rangle.\end{aligned}\quad(26)$$

Now, (25) follows from (26). □

From different aspects of Theorem 3, we can obtain the following four corollaries.

Corollary 7 *Let $\{\Lambda_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j\in J\}$, with frame operator S. Then for any $f\in\mathscr{U}$ and $\{a_j\}_{j\in J}\in l^\infty(J)$, we obtain*

$$\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Lambda_j f\Big\|^2+2\,\mathrm{Re}\sum_{j\in J}a_j\langle\Lambda_j f,\Lambda_jSf\rangle =\Big\|\sum_{j\in J}a_j\Lambda_j^*\Lambda_j f\Big\|^2+\|Sf\|^2.$$

Proof: For any g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ with frame operator S, $\{\Lambda_j\}_{j\in J}$ is an S-dual of $\{\Lambda_j\}_{j\in J}$. Hence the result follows from Theorem 3. □

Corollary 8 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j \in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t.$\{\mathscr{V}_j : j \in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. Then for any $f \in \mathscr{U}$ and any subset $I \subset J$, we obtain*

$$\Big\|\sum_{j\in I}\Lambda_j^*\Gamma_j f\Big\|^2 + 2\,\mathrm{Re}\sum_{j\in I^c}\langle \Gamma_j f, \Lambda_j K f\rangle = \Big\|\sum_{j\in I^c}\Lambda_j^*\Gamma_j f\Big\|^2 + \|Kf\|^2.$$

Proof: The result follows by taking $\{a_j\}_{j\in J}$ as in (21). □

Corollary 9 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j \in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t.$\{\mathscr{V}_j : j \in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. If $R(K)$ is closed, then for any $f \in R(K)$ and $\{a_j\}_{j\in J} \in l^\infty(J)$, we obtain*

$$\Big\|\sum_{j\in J}(1-a_j)\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + 2\,\mathrm{Re}\sum_{j\in J} a_j\langle \Gamma_j K^+ f, \Lambda_j f\rangle = \Big\|\sum_{j\in J}a_j\Lambda_j^*\Gamma_j K^+ f\Big\|^2 + \|f\|^2.$$

Proof: The proof is similar to that of Theorem 1, so we omit it. □

Corollary 10 *Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathscr{U}$ w.r.t $\{\mathscr{V}_j : j \in J\}$. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence in $\mathscr{U}$ w.r.t.$\{\mathscr{V}_j : j \in J\}$, and be a K-dual of $\{\Lambda_j\}_{j\in J}$. If $R(K)$ is closed, then for any $f \in R(K^+)$ and $\{a_j\}_{j\in J} \in l^\infty(J)$, we obtain*

$$\Big\|\sum_{j\in J}(1-a_j)K^+\Lambda_j^*\Gamma_j f\Big\|^2 + 2\,\mathrm{Re}\sum_{j\in J} a_j\langle \Gamma_j f, \Lambda_j (K^+)^* f\rangle = \Big\|\sum_{j\in J}a_j K^+\Lambda_j^*\Gamma_j f\Big\|^2 + \|f\|^2.$$

Proof: The proof is similar to that of Theorem 2, so we omit it. □

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