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Oscillation for a generalized neutral Emden-Fowler equation with damping and distributed delay

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ABSTRACT: In this study, we consider the generalized neutral Emden-Fowler equation with damping and distributed delay

$$\left[r(t)|z'(t)|^{\alpha-1}z'(t)\right]'+c(t)|z'(t)|^{\alpha-1}z'(t)+\int_a^b q(t,\xi)|x(g(t,\xi))|^{\beta-1}x(g(t,\xi))\,\mathrm{d}\xi=0,$$

where $z(t)=x(t)+p(t)x(\tau(t))$, $\alpha,\beta>0$. By using averaging technique and some analytical skills, we obtain the sufficient conditions to ensure the oscillation for the above equation. Our results improve and generalize some existing results. Finally, two examples are given to show the feasibility of our results.

KEYWORDS: oscillation, damping, distributed delay, Emden-Fowler equation

MSC2010: 34K11 34K40

INTRODUCTION

Since Emden-Fowler equations with generalized forms[1–4] have many applications in various fields of nuclear physics, astrophysics and economics, there is constant interest in obtaining new sufficient conditions for oscillation of solutions of these equations[5–7]. For more details of oscillation of delay differential equations[8–10], neutral differential equations[11–14] and dynamic equations[15–18], see the mentioned references.

In Ref. 19, Baculíková and Džurina studied the oscillation of the second-order neutral differential equation of the form

$$\left[a(t)[z'(t)]^{\alpha}\right]'+q(t)x^{\beta}(\sigma(t))=0, \qquad (1)$$

where $z(t)=x(t)+p(t)x(\tau(t))$, $\alpha,\beta>0$.

In Ref. 20, Liu et al considered the generalized Emden-Fowler equation

$$\left[r(t)|z'(t)|^{\alpha-1}z'(t)\right]'+q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t))=0, \qquad (2)$$

where $t\geqslant t_0, z(t)=x(t)+p(t)x(\tau(t))$. By using averaging technique and Riccati transformation, they obtained some oscillation criteria for $\alpha\geqslant\beta>0$.

Subsequently, by using the similar method, Zeng et al[21] obtained the oscillation for (2) with $\alpha\geqslant\beta>0$ or $0<\alpha\leqslant\beta$. Further, by using the generalized Riccati inequality, Wu et al[22] studied the oscillation of (2) for $\alpha,\beta>0$.

In recent years, the oscillation of differential equations with damping or distributed delay has been studied by many authors. For example, Bohner et al[23] studied the oscillation of the second-order damped nonlinear delay differential equations of Emden-Fowler type

$$\left[a(t)x'(t)\right]'+p(t)x'(t)+ q(t)|x(g(t))|^{\lambda}\operatorname{sgn}x(g(t))=0,\ t\in[t_0,\infty). \qquad (3)$$

Zeng et al[24] studied the oscillation of generalized neutral delay differential equations of Emden-Fowler type with damping

$$\left[r(t)|z'(t)|^{\alpha-1}z'(t)\right]'+c(t)|z'(t)|^{\alpha-1}z'(t) +q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t))=0,\ t\geqslant t_0. \qquad (4)$$

By using integral inequality technique and Riccati transformation, they obtained some oscillation criteria for the case $\alpha\geqslant\beta>0$ or $\beta\geqslant\alpha>0$.

Qin et al[25] studied the second-order differential equation with distributed deviating arguments and a damping term

$$r(x)\psi(y(x))[y(x)+c(x)y(d(x))]' + \int_\alpha^\beta a(x,\theta)f(y(b(x,\theta)))\,\mathrm{d}\theta = 0, \quad (5)$$

and established some oscillation criteria of this equation.

In this study, motivated by the above work, by using averaging technique and some analytical skills, we obtain the sufficient conditions to ensure the oscillation for the following generalized Emden-Fowler equation with damping and distributed delay

$$\left[r(t)|z'(t)|^{\alpha-1}z'(t)\right]' + c(t)|z'(t)|^{\alpha-1}z'(t) + \int_a^b q(t,\xi)|x(g(t,\xi))|^{\beta-1}x(g(t,\xi))\,\mathrm{d}\xi = 0, \quad (6)$$

where $t \geqslant t_0$, $z(t) = x(t) + p(t)x(\tau(t))$.

In the sequel, we always make the following assumptions for (6).

(H$_1$) $a, b, \alpha > 0$, $\beta > 0$ are all constants. In addition, $b > a$, $p \in C([t_0,\infty),[0,1))$, $c \in C([t_0,\infty),[0,\infty))$, $r \in C^1([t_0,\infty),(0,\infty))$, $r'(t) \geqslant 0$ for $t \in [t_0,\infty)$;

(H$_2$) $\tau \in C^1([t_0,\infty),R)$, $q, g \in C([t_0,\infty) \times [a,b],[0,\infty))$, $q(t,\xi) \not\equiv 0$, $\lim_{t\to\infty} g(t,\xi) = \infty$ uniformly holds on $\xi \in [a,b]$, $0 \leqslant \tau(t) \leqslant t$, $\lim_{t\to\infty} \tau(t) = \infty$; $g(t,\xi) \leqslant t$, $g_t(t,\xi) > 0$, $g_\xi(t,\xi) \geqslant 0$ for $\xi \in [a,b]$, $t \in [t_0,\infty)$;

(H$_3$) $\lim_{t\to\infty} R(t) = \infty$, where

$$R(t) = \int_{t_0}^t \left(r(\eta)\,\mathrm{e}^{\int_{t_0}^\eta \frac{c(s)}{r(s)}\,\mathrm{d}s}\right)^{-1/\alpha} \mathrm{d}\eta;$$

(H$_4$) There exists $\rho \in C^1([t_0,\infty),(0,\infty))$ such that $\rho'(t) \geqslant 0$ and for any $m \in (0,1]$,

$$\int_{t_0}^\infty \left[\rho(t)Q(t) - \frac{r(t)[\rho'(t)]^{\lambda+1}\,\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s}}{(\lambda+1)^{\lambda+1}[m\rho(t)g_t(t,a)]^\lambda}\right]\mathrm{d}t = \infty,$$

where $\lambda = \min\{\alpha,\beta\}$,

$$Q(t) = \mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s} \int_a^b q(t,\xi)[1-p(g(t,\xi))]^\beta\,\mathrm{d}\xi;$$

(H$_5$) $\Pi(t) < \infty$ for $t \in [t_0,\infty)$, where

$$\Pi(t) = \int_t^\infty \left[r(\eta)\,\mathrm{e}^{\int_{t_0}^\eta \frac{c(s)}{r(s)}\,\mathrm{d}s}\right]^{-1/\alpha} \mathrm{d}\eta.$$

MAIN RESULTS

It is clear that (6) is equivalent to the following equation

$$\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)|z'(t)|^{\alpha-1}z'(t)\right]' + \mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s} \times \int_a^b q(t,\xi)|x(g(t,\xi))|^{\beta-1}x(g(t,\xi))\,\mathrm{d}\xi = 0, \quad (7)$$

where $t \geqslant t_0$. Therefore, the oscillation of (6) is equivalent to that of (7). In order to prove the main results, we need the following lemmas.

Lemma 1 *Let θ, A, B be constants, $\theta > 0$, $A \geqslant 0$, $B > 0$. Then, for any $u > 0$,*

$$Au - Bu^{(\theta+1)/\theta} \leqslant \frac{\theta^\theta}{(\theta+1)^{\theta+1}} \frac{A^{\theta+1}}{B^\theta}.$$

The proof of Lemma 1 is easy. So we omit it here.

Lemma 2 *Suppose that* (H$_1$)*,* (H$_2$)*, and* (H$_5$) *hold. Let $x(t)$ be an eventually positive solution for* (7)*. In addition, $z'(t)$ is eventually negative. Then there exist $T > t_0$ and $L > 0$ such that*

$$0 < w_1(t)\Pi^\mu(t) \leqslant L, \quad t \in [T,\infty),$$

where $\mu = \max\{\alpha,\beta\}$,

$$w_1(t) = \mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^\alpha / z^\beta(t).$$

Proof: From (H$_1$), $z(t)$ is eventually positive. Further, by using (7) and (H$_2$), there exists $T > t_0$ such that

$$\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^\alpha\right]' \geqslant 0, \quad t \in [T,\infty). \quad (8)$$

Hence, for any $v > t > T$, we have

$$-z'(v)\left[\mathrm{e}^{\int_{t_0}^v \frac{c(s)}{r(s)}\,\mathrm{d}s} r(v)\right]^{1/\alpha} \geqslant -z'(t)\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)\right]^{1/\alpha},$$

which yields

$$-z'(v) \geqslant -z'(t)\left[\mathrm{e}^{\int_{t_0}^v \frac{c(s)}{r(s)}\,\mathrm{d}s} r(v)\right]^{-1/\alpha}\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)\right]^{1/\alpha}.$$

Further, for any $u > t > T$, we have

$$z(t) \geqslant z(t) - z(u) \geqslant \left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)\right]^{1/\alpha} \times (-z'(t)) \int_t^u \left[\mathrm{e}^{\int_{t_0}^v \frac{c(s)}{r(s)}\,\mathrm{d}s} r(v)\right]^{-1/\alpha} \mathrm{d}v. \quad (9)$$

Letting $u \to \infty$ in (9) yields

$$z(t) \geqslant \left[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t) \right]^{1/\alpha} (-z'(t))\Pi(t), \quad t \in [T, \infty).$$

Further,

$$z^{\alpha}(t) \geqslant r(t)\, \mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} (-z'(t))^{\alpha} \Pi^{\alpha}(t),$$
$$z^{\beta}(t) \geqslant \left[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t) \right]^{\beta/\alpha} (-z'(t))^{\beta} \Pi^{\beta}(t).$$

If $\alpha \geqslant \beta$, then $\mu = \alpha$ and

$$\begin{aligned} w_1(t)\Pi^{\mu}(t) &= w_1(t)\Pi^{\alpha}(t) \\ &\leqslant \frac{\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)(-z'(t))^{\alpha}}{z^{\beta}(t)} \frac{z^{\alpha}(t)}{\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)(-z'(t))^{\alpha}} \\ &= z^{\alpha-\beta}(t). \end{aligned}$$

Note that $z(t) > 0$ and $z'(t) < 0$. There exists a constant $L_1 > 0$ such that

$$0 < w_1(t)\Pi^{\mu}(t) \leqslant z^{\alpha-\beta}(t) \leqslant L_1, \quad t \in [T, \infty).$$

If $\alpha < \beta$, then $\mu = \beta$. Therefore, for $t \in [T, \infty)$,

$$\begin{aligned} w_1(t)\Pi^{\mu}(t) &= w_1(t)\Pi^{\beta}(t) \\ &\leqslant \frac{\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)(-z'(t))^{\alpha}}{z^{\beta}(t)} \frac{z^{\beta}(t)}{\left[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t) \right]^{\frac{\beta}{\alpha}} (-z'(t))^{\beta}} \\ &= \left[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)(-z'(t))^{\alpha} \right]^{\frac{\alpha-\beta}{\alpha}}. \end{aligned}$$

From (8), there exists a constant $L_2 > 0$ such that

$$0 < w_1(t)\Pi^{\mu}(t) \leqslant L_2, \quad t \in [T, \infty).$$

Denote $L = \max\{L_1, L_2\}$. Hence, for any $t \in [T, \infty)$,

$$0 < w_1(t)\Pi^{\mu}(t) \leqslant L.$$

This completes the proof of Lemma 2. □

Theorem 1 *Assume that* (H$_1$)–(H$_4$) *hold. Then* (6) *is oscillatory.*

Proof: Suppose that (7) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution for (7). Hence, there exists $T_0 > t_0$ such that $x(t) > 0$ for $t \geqslant T_0$. If $x(t)$ is eventually negative, the proof is similar.

From (H$_2$) and (7), there exists $T_1 \geqslant T_0$ such that $\tau(t) \geqslant T_0$ and $g(t, \xi) \geqslant T_0$ for any $t \geqslant T_1$ and $\xi \in [a, b]$. This, together with (H$_1$) and (7), implies that for $t \geqslant T_1$,

$$x(g(t, \xi)) > 0, \quad x(\tau(t)) > 0, \quad z(t) \geqslant x(t) > 0,$$
$$\left[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)|z'(t)|^{\alpha-1} z'(t) \right]' \leqslant 0. \quad (10)$$

Further, from (10), it is easy to see that $z'(t)$ is eventually nonnegative or eventually negative. Suppose that $z'(t)$ is eventually negative. That is, there exists $T_2 \geqslant T_1$ such that $z'(t) < 0$ for any $t \geqslant T_2$. It follows from (10) that

$$\left[-\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)(-z'(t))^{\alpha} \right]' \leqslant 0, \quad t \geqslant T_2.$$

Denote $K = r(T_2)(-z'(T_2))^{\alpha}\, \mathrm{e}^{\int_{t_0}^{T_2} \frac{c(s)}{r(s)} \mathrm{d}s} > 0$. Hence,

$$-\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)(-z'(t))^{\alpha} \leqslant -K, \quad t \geqslant T_2.$$

This yields

$$-z'(t) \geqslant K^{1/\alpha} \left[r(t)\, \mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} \right]^{-1/\alpha}, \quad t \geqslant T_2.$$

Integrating this inequality from T_2 to t, we obtain

$$z(t) \leqslant z(T_2) - K^{1/\alpha}[R(t) - R(T_2)], \quad t \geqslant T_2.$$

It follows from (H$_3$) that $\lim_{t\to\infty} z(t) = -\infty$. This contradicts (10). Therefore $z'(t)$ is eventually nonnegative. That is, there exists $T_3 \geqslant T_2$ such that $z'(t) \geqslant 0$ and $z'(g(t, \xi)) \geqslant 0$ for any $t \geqslant T_3$. Further, (7) yields

$$\left[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)(z'(t))^{\alpha} \right]' + \mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} \times \int_a^b q(t, \xi) x^{\beta}(g(t, \xi))\, \mathrm{d}\xi = 0, \quad t \geqslant T_3. \quad (11)$$

From (10), for $t \geqslant T_3$, we have

$$\begin{aligned} &\left[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} r(t)(z'(t))^{\alpha} \right]' \\ &= \mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)} \mathrm{d}s} \Big[c(t)(z'(t))^{\alpha} + r'(t)(z'(t))^{\alpha} \\ &\qquad + r(t)\alpha(z'(t))^{\alpha-1} z''(t) \Big] \leqslant 0, \end{aligned}$$

which, together with (H$_1$) and (10), yields

$$z''(t) \leqslant 0, \quad t \geqslant T_3. \quad (12)$$

Hence, there exists $T_4 \geqslant T_3$ such that

$$z'(t) > 0,\ t \geqslant T_4; \quad \text{or} \quad z'(t) \equiv 0,\ t \geqslant T_4.$$

In fact, if $z'(t)\equiv 0$ for $t\geqslant T_4$, then it follows from (7) that $q(t,\xi)\equiv 0$ for $t\geqslant T_4$ and $\xi\in[a,b]$. This contradicts (H_2). Hence

$$z'(t)>0, \quad t\geqslant T_4.$$

It follows from $z(t)\geqslant x(t)$ that

$$x(t)=z(t)-p(t)x(\tau(t))\geqslant z(t)-p(t)z(\tau(t)) \geqslant [1-p(t)]z(t), \quad t\geqslant T_4.$$

Therefore

$$x^\beta(t)\geqslant[1-p(t)]^\beta z^\beta(t), \quad t\geqslant T_4.$$

From (H_1) and (H_2), there exists $T_5\geqslant T_4$ such that

$$x^\beta(g(t,\xi))\geqslant[1-p(g(t,\xi))]^\beta z^\beta(g(t,\xi)), \quad t\geqslant T_5,\ \xi\in[a,b].$$

From (H_2), for $t\geqslant T_5$,

$$\int_a^b q(t,\xi)x^\beta(g(t,\xi))\,\mathrm{d}\xi \geqslant \int_a^b q(t,\xi)[1-p(g(t,\xi))]^\beta z^\beta(g(t,\xi))\,\mathrm{d}\xi \geqslant z^\beta(g(t,a))\int_a^b q(t,\xi)[1-p(g(t,\xi))]^\beta\,\mathrm{d}\xi. \quad (13)$$

It follows from (11) and (13) that for $t\geqslant T_5$,

$$\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}r(t)(z'(t))^\alpha\right]'+\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s} \times z^\beta(g(t,a))\int_a^b q(t,\xi)[1-p(g(t,\xi))]^\beta\,\mathrm{d}\xi\leqslant 0.$$

Hence,

$$\frac{\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}r(t)(z'(t))^\alpha\right]'}{z^\beta(g(t,a))}\leqslant -Q(t), \quad t\geqslant T_5. \quad (14)$$

Denote

$$w(t)=\rho(t)\frac{\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}r(t)(z'(t))^\alpha}{z^\beta(g(t,a))}\geqslant 0,\ t\geqslant T_5. \quad (15)$$

Further,

$$w^{\frac{\alpha+1}{\alpha}}(t) =\frac{\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}\rho(t)r(t)\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}\rho(t)r(t)\right]^{\frac{1}{\alpha}}(z'(t))^{\alpha+1}}{z^{\frac{\beta(\alpha+1)}{\alpha}}(g(t,a))}, \quad t\geqslant T_5, \quad (16)$$

and

$$w^{\frac{\beta+1}{\beta}}(t) =\frac{\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}\rho(t)r(t)\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}\rho(t)r(t)\right]^{\frac{1}{\beta}}(z'(t))^{\frac{\alpha(\beta+1)}{\beta}}}{z^{\beta+1}(g(t,a))}, \quad t\geqslant T_5. \quad (17)$$

In addition, for $t\geqslant T_5$,

$$w'(t)=\frac{\rho'(t)}{\rho(t)}w(t)+\rho(t)\frac{\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}r(t)(z'(t))^\alpha\right]'}{z^\beta(g(t,a))} -\rho(t)\frac{\beta\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}r(t)[z'(t)]^\alpha z'(g(t,a))g_t(t,a)}{z^{\beta+1}(g(t,a))}. \quad (18)$$

If $\alpha=\beta$, then it follows from (H_2), (12), (14), (16), and (18) that for $t\geqslant T_5$,

$$\begin{aligned} w'(t)&=\frac{\rho'(t)}{\rho(t)}w(t)+\rho(t)\frac{\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}r(t)(z'(t))^\alpha\right]'}{z^\beta(g(t,a))} -\rho(t)\frac{\alpha\,\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}r(t)[z'(t)]^\alpha z'(g(t,a))g_t(t,a)}{z^{\alpha+1}(g(t,a))}\\ &\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t) -w^{\frac{\alpha+1}{\alpha}}(t)\frac{\alpha z'(g(t,a))g_t(t,a)}{\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}\rho(t)r(t)\right]^{\frac{1}{\alpha}}z'(t)}\\ &\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t) -w^{\frac{\alpha+1}{\alpha}}(t)\frac{\alpha g_t(t,a)}{\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}\rho(t)r(t)\right]^{\frac{1}{\alpha}}}. \end{aligned} \quad (19)$$

If $\alpha<\beta$, then it follows from (16) and (18) that for $t\geqslant T_5$,

$$w'(t)=\frac{\rho'(t)}{\rho(t)}w(t)+\rho(t)\frac{\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}r(t)(z'(t))^\alpha\right]'}{z^\beta(g(t,a))} -w^{\frac{\alpha+1}{\alpha}}(t)\frac{\beta z'(g(t,a))g_t(t,a)[z(g(t,a))]^{\frac{\beta-\alpha}{\alpha}}}{\left[\mathrm{e}^{\int_{t_0}^t \frac{c(s)}{r(s)}\mathrm{d}s}\rho(t)r(t)\right]^{\frac{1}{\alpha}}z'(t)}. \quad (20)$$

Note that $z'(t)>0$ for $t\geqslant T_4$. From (H_2), we have

$$[z(g(t,a))]^{\frac{\beta-\alpha}{\alpha}}\geqslant[z(g(T_5,a))]^{\frac{\beta-\alpha}{\alpha}}=m_1,\ t\geqslant T_5. \quad (21)$$

Obviously, $m_1>0$. From (12), (14), (20), and (21), it follows that for $t\geqslant T_5$,

$$w'(t)\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-w^{\frac{\alpha+1}{\alpha}}(t)\frac{\beta z'(g(t,a))g_t(t,a)[z(g(t,a))]^{\frac{\beta-\alpha}{\alpha}}}{\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}\rho(t)r(t)\right]^{\frac{1}{\alpha}}z'(t)}\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-w^{\frac{\alpha+1}{\alpha}}(t)m_1\frac{\alpha g_t(t,a)}{\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}\rho(t)r(t)\right]^{\frac{1}{\alpha}}}.\quad(22)$$

If $\alpha>\beta$, it follows from (17) and (18), for $t\geqslant T_5$,

$$w'(t)=\frac{\rho'(t)}{\rho(t)}w(t)+\rho(t)\frac{\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}r(t)(z'(t))^{\alpha}\right]'}{z^{\beta}(g(t,a))}-w^{\frac{\beta+1}{\beta}}(t)\frac{\beta z'(g(t,a))g_t(t,a)[z'(t)]^{\frac{\beta-\alpha}{\beta}}}{\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}\rho(t)r(t)\right]^{\frac{1}{\beta}}z'(t)}.\quad(23)$$

From (12), we have

$$[z'(t)]^{\frac{\beta-\alpha}{\beta}}\geqslant[z'(T_5)]^{\frac{\beta-\alpha}{\beta}}=m_2,\quad t\geqslant T_5.\quad(24)$$

Further, $m_2>0$. From (12), (14), (23), and (24), for $t\geqslant T_5$, we have

$$w'(t)\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-w^{\frac{\beta+1}{\beta}}(t)\frac{\beta g_t(t,a)[z'(t)]^{\frac{\beta-\alpha}{\beta}}}{\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}\rho(t)r(t)\right]^{\frac{1}{\beta}}}\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-w^{\frac{\beta+1}{\beta}}(t)m_2\frac{\beta g_t(t,a)}{\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}\rho(t)r(t)\right]^{\frac{1}{\beta}}}.\quad(25)$$

From (19), (22), (25), and Lemma 1, there exists $m\in(0,1]$ such that

$$w'(t)\leqslant-\rho(t)Q(t)+\frac{\rho'(t)}{\rho(t)}w(t)-w^{\frac{\lambda+1}{\lambda}}(t)\frac{\lambda m g_t(t,a)}{\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}\rho(t)r(t)\right]^{\frac{1}{\lambda}}}\leqslant-\rho(t)Q(t)+\frac{r(t)(\rho'(t))^{\lambda+1}e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}}{(\lambda+1)^{\lambda+1}(mg_t(t,a)\rho(t))^{\lambda}}.\quad(26)$$

Integrating (26) from T_5 to t, we get

$$w(t)\leqslant w(T_5)-\int_{T_5}^t\left[\rho(s)Q(s)-\frac{r(s)[\rho'(s)]^{\lambda+1}e^{\int_{t_0}^s\frac{c(k)}{r(k)}\,dk}}{(\lambda+1)^{\lambda+1}[mg_s(s,a)\rho(s)]^{\lambda}}\right]ds.\quad(27)$$

Letting $t\to\infty$ in (27), it follows from (H_4) that $\lim_{t\to\infty}w(t)=-\infty$. This contradicts (15). Therefore, (7) is oscillatory. This completes the proof. □

Choosing $\rho(t)=1$ in Theorem 1, we get the results.

Corollary 1 *Assume that* (H_1)–(H_3) *hold. If*

$$\int_{t_0}^{\infty}\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}\int_a^b q(t,\xi)[1-p(g(t,\xi))]^{\beta}\,d\xi\right]dt=\infty,$$

then (6) *is oscillatory.*

Theorem 2 *Assume that* (H_1)*,* (H_2)*,* (H_4)*, and* (H_5) *hold. If*

(i) $p'(t)\geqslant0$*,* $\tau'(t)\geqslant0$*,* $g(t,\xi)\leqslant\tau(t)$*, for* $t\in[t_0,\infty)$*,* $\xi\in[a,b]$*;*

(ii) for any $M\in(0,1]$*,*

$$\int_{t_0}^{\infty}\left[\Pi^{\mu}(t)Q_1(t)-\frac{\mu^{2\mu+1}}{(\beta M)^{\mu}\Pi(t)(\mu+1)^{\mu+1}\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}r(t)\right]^{\frac{1}{\alpha}}}\right]dt=\infty,$$

where $\mu=\max\{\alpha,\beta\}$ *and*

$$Q_1(t)=[1-p(t)]^{\beta}e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}\int_a^b q(t,\xi)\,d\xi,$$

then (6) *is oscillatory.*

Proof: Suppose that (7) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution for (7). Hence, there exists $T_0>t_0$ such that $x(t)>0$ for $t\geqslant T_0$. If $x(t)$ is eventually negative, the proof is similar.

From (H_2) and (7), there exists $T_1\geqslant T_0$ such that $\tau(t)\geqslant T_0$ and $g(t,\xi)\geqslant T_0$ for any $t\geqslant T_1$ and $\xi\in[a,b]$. This, together with (H_1) and (7), implies that for $t\geqslant T_1$

$$x(g(t,\xi))>0,\quad x(\tau(t))>0,\quad z(t)\geqslant x(t)>0,\quad\left[e^{\int_{t_0}^t\frac{c(s)}{r(s)}\,ds}r(t)|z'(t)|^{\alpha-1}z'(t)\right]'\leqslant0.\quad(28)$$

From (28), it is easy to see that $z'(t)$ is eventually nonnegative or eventually negative.

Case 1. Assume that $z'(t)$ is eventually nonnegative. By using the same method as in the proof of Theorem 1, we can get a contradiction.

Case 2. Assume that $z'(t)$ is eventually negative. Hence there exists $T_2 \geqslant T_1$ satisfying $z'(t) < 0$ for any $t \geqslant T_2$. It follows from (7) and (28) that for $t \geqslant T_2$,

$$\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^{\alpha} \Big]' - \mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} \int_a^b q(t,\xi) x^{\beta}(g(t,\xi))\,\mathrm{d}\xi = 0. \quad (29)$$

In addition,

$$z'(t) = x'(t) + p'(t)x(\tau(t)) + p(t)x'(\tau(t))\tau'(t), \quad (30)$$

which together with (H_1) and condition (i), yields

$$x'(t) \leqslant 0, \quad t \geqslant T_2. \quad (31)$$

From (31) and (H_2), we deduce

$$\begin{aligned} z(t) &= x(t) + p(t)x(\tau(t)) < x(\tau(t)) + p(t)x(\tau(t)) \\ &= [1+p(t)]x(\tau(t)), \quad t \geqslant T_2. \end{aligned} \quad (32)$$

It follows from (31), (32) and (i) that for any $\xi \in [a,b]$,

$$\begin{aligned} x(g(t,\xi)) &\geqslant x(\tau(t)) > \frac{z(t)}{1+p(t)} = \frac{z(t)(1-p(t))}{1-p^2(t)} \\ &\geqslant (1-p(t))z(t), \quad t \geqslant T_2. \end{aligned}$$

From (H_2), for $t \geqslant T_2$, we have

$$\begin{aligned} \int_a^b q(t,\xi) x^{\beta}(g(t,\xi))\,\mathrm{d}\xi &\geqslant \int_a^b q(t,\xi)[1-p(t)]^{\beta} z^{\beta}(t)\,\mathrm{d}\xi \\ &\geqslant [1-p(t)]^{\beta} z^{\beta}(t) \int_a^b q(t,\xi)\,\mathrm{d}\xi. \end{aligned} \quad (33)$$

From (29) and (33), we get for $t \geqslant T_2$,

$$\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^{\alpha} \Big]' - Q_1(t) z^{\beta}(t) \geqslant 0. \quad (34)$$

Clearly, for $t \geqslant T_2$,

$$w_1(t) := \frac{\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^{\alpha}}{z^{\beta}(t)} > 0. \quad (35)$$

Further, for $t \geqslant T_2$,

$$w_1^{\frac{\alpha+1}{\alpha}}(t) = \frac{\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t) \Big]^{\frac{\alpha+1}{\alpha}} [-z'(t)]^{\alpha+1}}{z^{\frac{\beta(\alpha+1)}{\alpha}}(t)}, \quad (36)$$

$$w_1^{\frac{\beta+1}{\beta}}(t) = \frac{\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t) \Big]^{\frac{\beta+1}{\beta}} [-z'(t)]^{\frac{\alpha(\beta+1)}{\beta}}}{z^{\beta+1}(t)}. \quad (37)$$

It follows from (34), (35) that for any $t \geqslant T_2$,

$$\begin{aligned} w_1'(t) &= \frac{\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^{\alpha} \Big]'}{z^{\beta}(t)} - \frac{\beta\, \mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} z^{\beta-1}(t) z'(t) r(t)(-z'(t))^{\alpha}}{z^{2\beta}(t)} \\ &\geqslant Q_1(t) + \frac{\beta r(t)(-z'(t))^{\alpha+1}\, \mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s}}{z^{\beta+1}(t)}. \end{aligned} \quad (38)$$

If $\alpha \geqslant \beta$, then from (36) and (38) for $t \geqslant T_2$,

$$w_1'(t) \geqslant Q_1(t) + w_1^{\frac{\alpha+1}{\alpha}}(t) \frac{\beta [z(t)]^{\frac{\beta-\alpha}{\alpha}}}{\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t) \Big]^{\frac{1}{\alpha}}}. \quad (39)$$

Note that $z'(t) < 0$ for $t \geqslant T_2$. Therefore

$$[z(t)]^{\frac{\beta-\alpha}{\alpha}} \geqslant [z(T_2)]^{\frac{\beta-\alpha}{\alpha}} = m_3, \quad t \geqslant T_2. \quad (40)$$

Further, $m_3 > 0$. From (39) and (40) for $t \geqslant T_2$,

$$w_1'(t) \geqslant Q_1(t) + w_1^{\frac{\alpha+1}{\alpha}}(t) \frac{\beta m_3}{\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t) \Big]^{\frac{1}{\alpha}}}. \quad (41)$$

If $\alpha < \beta$, then, from (37) and (38), for $t \geqslant T_2$,

$$\begin{aligned} w_1'(t) &\geqslant Q_1(t) + w_1^{\frac{\beta+1}{\beta}}(t) \frac{\beta [-z'(t)]^{\frac{\beta-\alpha}{\beta}}}{\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t) \Big]^{\frac{1}{\beta}}} \\ &= Q_1(t) + w_1^{\frac{\beta+1}{\beta}}(t) \frac{\beta \Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^{\alpha} \Big]^{\frac{\beta-\alpha}{\alpha\beta}}}{\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t) \Big]^{\frac{1}{\alpha}}}. \end{aligned} \quad (42)$$

From (34) for $t \geqslant T_2$, $\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^{\alpha} \Big]' > 0$, therefore

$$\begin{aligned} &\Big[\mathrm{e}^{\int_{t_0}^{t} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(t)(-z'(t))^{\alpha} \Big]^{\frac{\beta-\alpha}{\alpha\beta}} \\ &\quad \geqslant \Big[\mathrm{e}^{\int_{t_0}^{T_2} \frac{c(s)}{r(s)}\,\mathrm{d}s} r(T_2)(-z'(T_2))^{\alpha} \Big]^{\frac{\beta-\alpha}{\alpha\beta}} = m_4. \end{aligned} \quad (43)$$

Clearly, $m_4 > 0$. From (42) and (43), for $t \geqslant T_2$,

$$w_1'(t) \geqslant Q_1(t) + w_1^{\frac{\beta+1}{\beta}}(t)\frac{\beta m_4}{\left[e^{\int_{t_0}^{t}\frac{c(s)}{r(s)}ds}r(t)\right]^{\frac{1}{\alpha}}}. \quad (44)$$

It follows from (41) and (44) that there is $M \in (0,1]$ such that for $t \geqslant T_2$,

$$w_1'(t) \geqslant Q_1(t) + w_1^{\frac{\mu+1}{\mu}}(t)\frac{\beta M}{\left[e^{\int_{t_0}^{t}\frac{c(s)}{r(s)}ds}r(t)\right]^{\frac{1}{\alpha}}}. \quad (45)$$

Further, it follows from (H_1) that for $t \geqslant T_2$,

$$\Pi^{\mu}(t)w_1'(t) \geqslant \Pi^{\mu}(t)Q_1(t) + \Pi^{\mu}(t)w_1^{\frac{\mu+1}{\mu}}(t)\frac{\beta M}{\left[e^{\int_{t_0}^{t}\frac{c(s)}{r(s)}ds}r(t)\right]^{\frac{1}{\alpha}}}. \quad (46)$$

Integrating (46) from T_2 to t, we get

$$\int_{T_2}^{t}\Pi^{\mu}(s)w_1'(s)\,ds \geqslant \int_{T_2}^{t}\Pi^{\mu}(s)Q_1(s)\,ds + \int_{T_2}^{t}\left[\Pi^{\mu}(s)w_1^{\frac{\mu+1}{\mu}}(s)\frac{\beta M}{\left[e^{\int_{t_0}^{s}\frac{c(k)}{r(k)}dk}r(s)\right]^{\frac{1}{\alpha}}}\right]ds. \quad (47)$$

Then

$$\Pi^{\mu}(t)w_1(t) - \Pi^{\mu}(T_2)w_1(T_2) \geqslant \int_{T_2}^{t}[\Pi^{\mu}(s)]'w_1(s)\,ds + \int_{T_2}^{t}\Pi^{\mu}(s)Q_1(s)\,ds + \int_{T_2}^{t}\left[\Pi^{\mu}(s)w_1^{\frac{\mu+1}{\mu}}(s)\frac{\beta M}{\left[e^{\int_{t_0}^{s}\frac{c(k)}{r(k)}dk}r(s)\right]^{\frac{1}{\alpha}}}\right]ds.$$

From Lemmas 1 and 2, for $t \geqslant \max\{T, T_2\}$, we have

$$\begin{aligned} L - \Pi^{\mu}(T_2)w_1(T_2) &\geqslant \int_{T_2}^{t}\Pi^{\mu}(s)Q_1(s)\,ds \\ &\quad - \int_{T_2}^{t}\mu\Pi^{\mu-1}(s)\left[e^{\int_{t_0}^{s}\frac{c(k)}{r(k)}dk}r(s)\right]^{-\frac{1}{\alpha}}w_1(s)\,ds \\ &\quad - \int_{T_2}^{t}\Pi^{\mu}(s)\beta M\left[e^{\int_{t_0}^{s}\frac{c(k)}{r(k)}dk}r(s)\right]^{-\frac{1}{\alpha}}w_1^{\frac{\mu+1}{\mu}}(s)\,ds \\ &= \int_{T_2}^{t}\Pi^{\mu}(s)Q_1(s)\,ds - \int_{T_2}^{t}\Pi^{\mu-1}(s)\left[e^{\int_{t_0}^{s}\frac{c(k)}{r(k)}dk}r(s)\right]^{-\frac{1}{\alpha}} \\ &\quad \times\left[\mu w_1(s) - \Pi(s)\beta M w_1^{\frac{\mu+1}{\mu}}(s)\right]ds \\ &\geqslant \int_{T_2}^{t}\Pi^{\mu}(s)Q_1(s)\,ds \\ &\quad - \int_{T_2}^{t}\frac{\mu^{2\mu+1}}{(\beta M)^{\mu}\Pi(s)(\mu+1)^{\mu+1}\left[e^{\int_{t_0}^{s}\frac{c(k)}{r(k)}dk}r(s)\right]^{\frac{1}{\alpha}}}\,ds. \end{aligned} \quad (48)$$

Letting $t \to \infty$ in (48), it follows from condition (ii) that $L \geqslant \infty$. This is a contradiction. Hence, (7) is oscillatory. This completes the proof. □

Remark 1 If $q(t,\xi) = \frac{1}{b-a}q(t)$, $g(t,\xi) = \sigma(t)$, $c(t) \equiv 0$, then (6) reduces to (2). Further, Theorem 1 reduces to Theorem 1 in Ref. 22. In addition, Theorem 2.1 in Refs. 20, 21 requires the condition $\alpha \geqslant \beta$; Theorem 3.1 in Ref. 21 requires the condition $\beta \geqslant \alpha$. Hence, Theorem 1 improves and generalizes these results.

Remark 2 If $q(t,\xi) = \frac{1}{b-a}q(t)$, $g(t,\xi) = \sigma(t)$, then (6) reduces to (4). Theorems 1 and 4 in Ref. 24 require the condition $\alpha \geqslant \beta$ and $\beta \geqslant \alpha$, respectively. Hence, Theorem 1 improves these theorems.

EXAMPLES

Example 1 Consider the following equation

$$\left[t^{\frac{\alpha}{2}}|z'(t)|^{\alpha-1}z'(t)\right]' + \frac{\alpha}{2}t^{\frac{\alpha}{2}-1}|z'(t)|^{\alpha-1}z'(t) + \int_{\frac{1}{2}}^{1}t^{\frac{\alpha}{2}}\xi|x(t\xi)|^{\beta-1}x(t\xi)\,d\xi = 0, \quad (49)$$

where $t \in [1,\infty)$, $z(t) = x(t) + x(t-1)/2$, $\alpha, \beta > 0$, $r(t) = t^{\frac{\alpha}{2}}$, $c(t) = \frac{\alpha}{2}t^{\frac{\alpha}{2}-1}$, $p(t) = 1/2$, $q(t,\xi) = t^{\frac{\alpha}{2}}\xi$, $g(t,\xi) = t\xi$, and $\tau(t) = t-1$.

It is easy to see that

$$\lim_{t\to\infty} g(t,\xi) = \infty,\ 0 \leqslant g(t,\xi) \leqslant t,\ g_t(t,\xi) = \xi > 0,$$
$$g_{\xi}(t,\xi) = t > 0, \quad Q(t) = 3(\tfrac{1}{2})^{\beta+3}t^{\alpha}.$$

Further,

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty}\int_1^t \Big[r(s)\,\mathrm{e}^{\int_{t_0}^{s}\frac{c(k)}{r(k)}\,\mathrm{d}k}\Big]^{-\frac{1}{\alpha}}\,\mathrm{d}s = \lim_{t\to\infty}\int_1^t \frac{1}{s}\,\mathrm{d}s = \infty.$$

So (H_3) holds. Choose $\rho(t)=t$. For any $m\in(0,1]$, we have

$$\int_1^\infty \Big[\rho(t)Q(t) - \frac{r(t)[\rho'(t)]^{\lambda+1}\,\mathrm{e}^{\int_{t_0}^{t}\frac{c(s)}{r(s)}\,\mathrm{d}s}}{(\lambda+1)^{\lambda+1}(mg_t(t,a)\rho(t))^{\lambda}}\Big]\mathrm{d}t = \int_1^\infty \Big[3(\tfrac{1}{2})^{\beta+3}t^{\alpha+1} - \frac{2^\lambda}{m^\lambda(\lambda+1)^{\lambda+1}}t^{\alpha-\lambda}\Big]\mathrm{d}t = \Big[(\tfrac{1}{2})^{\beta+3}\frac{3t^{\alpha+2}}{\alpha+2} - \frac{2^\lambda t^{\alpha-\lambda+1}}{m^\lambda(\alpha-\lambda+1)(\lambda+1)^{\lambda+1}}\Big]_1^\infty = \infty.$$

Therefore, (H_4) holds. Hence, the conditions of Theorem 1 are all satisfied. Thus, (49) is oscillatory.

Example 2 Consider the following equation

$$\big[t^\alpha|z'(t)|^{\alpha-1}z'(t)\big]' + \alpha t^{\alpha-1}|z'(t)|^{\alpha-1}z'(t) + \int_{\frac{1}{2}}^{1} t^\mu\xi|x(t\xi)|^{\beta-1}x(t\xi)\,\mathrm{d}\xi = 0, \quad (50)$$

where $t\in[1,\infty)$, $z(t)=x(t)+x(t-1)/2$, $\alpha,\beta>0$, $\mu=\max\{\alpha,\beta\}$, $r(t)=t^\alpha$, $p(t)=1/2$, $c(t)=\alpha t^{\alpha-1}$, $q(t,\xi)=t^\mu\xi$, $g(t,\xi)=t\xi$, and $\tau(t)=t-1$.

It is easy to see that

$$\lim_{t\to\infty} g(t,\xi)=\infty,\ 0\leqslant g(t,\xi)\leqslant t,\ g_t(t,\xi)=\xi>0, \quad g_\xi(t,\xi)=t>0, \quad Q(t)=3(\frac{1}{2})^{\beta+3}t^{\alpha+\mu}.$$

Further,

$$\Pi(t) = \int_t^\infty \Big[r(s)\,\mathrm{e}^{\int_{t_0}^{s}\frac{c(k)}{r(k)}\,\mathrm{d}k}\Big]^{-\frac{1}{\alpha}}\,\mathrm{d}s = \int_t^\infty [s^\alpha s^\alpha]^{-\frac{1}{\alpha}}\,\mathrm{d}s = \int_t^\infty s^{-2}\,\mathrm{d}s = \frac{1}{t} < \infty.$$

So (H_5) holds. Choose $\rho(t)=t$. For any $m\in(0,1]$, we have

$$\int_1^\infty \Big[\rho(t)Q(t) - \frac{r(t)[\rho'(t)]^{\lambda+1}\,\mathrm{e}^{\int_{t_0}^{t}\frac{c(s)}{r(s)}\,\mathrm{d}s}}{(\lambda+1)^{\lambda+1}(m\rho(t)g_t(t,a))^{\lambda}}\Big]\mathrm{d}t = \int_1^\infty \Big[3(\tfrac{1}{2})^{\beta+3}t^{\alpha+\mu+1} - \frac{2^\lambda}{m^\lambda(\lambda+1)^{\lambda+1}}t^{2\alpha-\lambda}\Big]\mathrm{d}t \geqslant \int_1^\infty \Big[3(\tfrac{1}{2})^{\beta+1}t^{2\alpha+1} - \frac{2^\lambda}{m^\lambda(\lambda+1)^{\lambda+1}}t^{2\alpha-\lambda}\Big]\mathrm{d}t = \Big[(\tfrac{1}{2})^{\beta+1}\frac{3t^{2\alpha+2}}{2\alpha+2} - \frac{2^\lambda t^{2\alpha-\lambda+1}}{(2\alpha-\lambda+1)m^\lambda(\lambda+1)^{\lambda+1}}\Big]_1^\infty = \infty.$$

So (H_4) holds. In addition, for any $M\in(0,1]$,

$$\int_1^\infty \Pi^\mu(t)Q_1(t)\,\mathrm{d}t - \int_1^\infty \frac{\mu^{2\mu+1}}{(\mu+1)^{\mu+1}(\beta M)^\mu\Pi(t)\Big[r(t)\,\mathrm{e}^{\int_{t_0}^{t}\frac{c(s)}{r(s)}\,\mathrm{d}s}\Big]^{\frac{1}{\alpha}}}\,\mathrm{d}t = \int_1^\infty \Big[3(\tfrac{1}{2})^{\beta+1}t^\alpha - \frac{\mu^{2\mu+1}}{(\mu+1)^{\mu+1}(M\beta)^\mu}t^{-1}\Big]\mathrm{d}t = \Big[3(\tfrac{1}{2})^{\beta+1}\frac{1}{\alpha+1}t^{\alpha+1} - \frac{\mu^{2\mu+1}}{(\mu+1)^{\mu+1}(M\beta)^\mu}\ln t\Big]_1^\infty = \infty.$$

Therefore the conditions of Theorem 2 hold. Hence, (50) is oscillatory.

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