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# Some new bounds related to Fejér-Hermite-Hadamard type inequality and their applications

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**ABSTRACT**: The paper aims to deal with certain integral inequalities which estimate the difference between the left and middle part in the Fejér-Hermite-Hadamard type inequality with new bounds. Also some applications to the estimation of higher moments of continuous random variables are presented.

**KEYWORDS**: invex set, $(\lambda, m)$-MT-preinvex functions, random variable

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## INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping, $a, b \in I$ with $a \neq b$, and $g : [a, b] \to \mathbb{R}$ be nonnegative, integrable, and symmetric with respect to $(a+b)/2$. Then one has

$$f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,\mathrm{d}x \leqslant \int_a^b f(x)g(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b g(x)\,\mathrm{d}x, \quad (1)$$

which is called the Fejér-Hermite-Hadamard inequality.

If we take $g(x) = 1$ in (1), then it becomes the following Hermite-Hadamard inequality

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}. \quad (2)$$

Because of the extensive applications of (1) and (2), some authors extended their studies via mappings of different classes. For example, refer to Refs. 1–4 for convex mappings, to Ref. 5 for strongly convex mappings, to Refs. 6–8 for $s$-convex mappings, to Refs. 9, 10 for $(s, m)$-convex mappings, to Ref. 11 for preinvex convex mappings, to Ref. 12 for $(s, m)$-preinvex convex mappings, to Ref. 13 for harmonically convex mappings, to Ref. 14 for harmonically quasi-convex mappings, to Ref. 15 for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex convex mappings, to Ref. 16 for GA-convex mappings, and to Ref. 17 for Schur convexity. Certain other results associated with (1) and (2) considering fractional integrals can be found in the literature, for instance, in Refs. 18–20 and the references therein.

Consider an invex set $\mathscr{K}$. A set $\mathscr{K} \subseteq \mathbb{R}^n$ is called invex set with respect to the mapping $\eta : \mathscr{K} \times \mathscr{K} \to \mathbb{R}^n$, if $\nu + \lambda\eta(\mu, \nu) \in \mathscr{K}$ holds for all $\mu, \nu \in \mathscr{K}$ and $\lambda \in [0, 1]$. A mapping $h : \mathscr{K} \to \mathbb{R}$ is named preinvex with respect to $\eta$, if the inequality

$$h\big(\nu + \lambda\eta(\mu, \nu)\big) \leqslant (1-\lambda)h(\nu) + \lambda h(\mu) \quad (3)$$

holds for every $\mu, \nu \in \mathscr{K}$ and $\lambda \in [0, 1]$.

In Ref. 21, using mappings which are preinvex in the second sense, Noor proved the following analogous Hadamard's inequality.

**Theorem 1 (Ref. 21)** *Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be an open preinvex function on the interval of real numbers $\mathscr{K}^\circ$ (the interior of $\mathscr{K}$) and $a, b \in \mathscr{K}^\circ$ with $a < a + \eta(b, a)$. Then*

$$f\Big(\frac{2a+\eta(b,a)}{2}\Big) \leqslant \frac{1}{\eta(b,a)}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}. \quad (4)$$

An appealing theme in (1) is the estimation of difference for the right-middle part of this inequality. In Ref. 22, using mappings whose first derivatives in absolute value are $(\alpha, m)$-preinvex, Zhang et al obtained some estimation-type results for the right-middle part of Fejér-Hermite-Hadamard type inequality.

Different from Ref. 22, this paper aims to obtain new estimation-type results for the left-middle part of Fejér-Hermite-Hadamard type inequality through differentiable mappings.

To obtain the principal results, we assume that the considered mapping is generalized $(\lambda, m)$-MT-preinvex. Next, we substitute this hypothesis with the boundedness of the derivative and with a Lipschitz condition for the derivative of the considered mapping to derive integral inequalities with new bounds. Some applications to the estimation of higher moments of continuous random variables are also presented. We end this section by recalling some definitions.

**Definition 1** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $\lambda$-MT-convex function if $f$ is positive and, for all $x, y \in I$, $\lambda \in (0, 1/2]$ and $t \in (0, 1)$, satisfies the following inequality[23]

$$f\big((1-t)x + ty\big) \leqslant \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{(1-\lambda)\sqrt{t}}{2\lambda\sqrt{1-t}} f(y).$$

Clearly, if we put $\lambda = 1/2$ in Definition 1, then $f$ is just an ordinary MT-convex function in Ref. 24.

**Definition 2** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $m$-MT-convex, if $f$ is positive and, for all $x, y \in I$, and $t \in (0, 1)$, with $m \in (0, 1]$, satisfies the following inequality[25]

$$f\big((1-t)x + tmy\big) \leqslant \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{m\sqrt{t}}{2\sqrt{1-t}} f(y).$$

Clearly, if $m = 1$, then Definition 2 reduces to the definition for MT-convex functions.

## NEW DEFINITION AND RESULTS

Let us introduce a new class of functions which will be called $(\lambda, m)$-MT-preinvex functions.

**Definition 3** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ is said to be $(\lambda, m)$-MT-preinvex function, if the following inequality

$$f\big(x + t\eta(my, x)\big) \leqslant \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{m(1-\lambda)\sqrt{t}}{2\lambda\sqrt{1-t}} f(y)$$

holds for all $x, y \in I$, $t \in (0, 1)$, with some fixed $\lambda \in (0, 1/2]$ and $m \in (0, 1]$.

If $t = 1/2$, then the generalized $(\lambda, m)$-MT-preinvex function reduces to

$$f(x + \tfrac{1}{2}\eta(mb, a)) \leqslant \tfrac{1}{2} f(x) + \frac{m(1-\lambda)}{2\lambda} f(y)$$

which is called Jensen-type generalized $(\lambda, m)$-preinvex function.

Let us discuss some special cases of Definition 3. (I) If $m = 1$, then Definition 3 reduces to the definition for $\lambda$-MT-preinvex functions. (II) If $m = 1$ and $\lambda = 1/2$, then Definition 3 reduces to the definition for MT-preinvex functions. (III) If $\lambda = 1/2$ and $\eta(mb, a) = mb - a$ with $m = 1$, then Definition 3 reduces to the definition for MT-convex functions.

Throughout this work, let $\mathcal{K} \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ and $a, b \in \mathcal{K}$, $a < mb \leqslant b$ with $\eta(mb, a) > 0$, and let $f : \mathcal{K} \to \mathbb{R}$ is a differentiable mapping on $\mathcal{K}^\circ$ such that $f' \in L^1([a, a + \eta(mb, a)])$. Assume that $g : [a, a + \eta(mb, a)] \to [0, \infty)$ is integrable and symmetrical about $a + \eta(mb, a)/2$. We need the following lemma for our main theorems.

**Lemma 1** *One has the following identity*

$$\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a + \frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x = \frac{\eta(mb,a)}{2}\bigg\{\int_0^1 \int_a^{\varphi(t)} g(x)\,\mathrm{d}x \times \Big[f'(\psi(t)) - f'(\varphi(t))\Big]\mathrm{d}t\bigg\}, \quad (5)$$

*where $\varphi(t) = a + ((1-t)/2)\eta(mb, a)$ and $\psi(t) = a + ((1+t)/2)\eta(mb, a)$. In particular, we obtain*

$$\bigg|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a + \frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\bigg| \leqslant \frac{\eta(mb,a)}{2}\bigg\{\int_0^1 \int_a^{\varphi(t)} g(x)\,\mathrm{d}x \times \Big[|f'(\varphi(t))| + |f'(\psi(t))|\Big]\mathrm{d}t\bigg\} \quad (6)$$

*and*

$$\bigg|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a + \frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\bigg| \leqslant \frac{\eta^2(mb,a)}{4}\|g\|_\infty \times \int_0^1 (1-t)\Big[|f'(\varphi(t))| + |f'(\psi(t))|\Big]\mathrm{d}t, \quad (7)$$

*where $\|g\|_\infty = \sup_{t\in[a,a+\eta(mb,a)]} g(t)$.*

*Proof*: Since $g(x)$ is symmetrical about $a + \eta(mb, a)/2$, $\int_a^{\varphi(t)} g(x)\,\mathrm{d}x = \int_{\psi(t)}^{a+\eta(mb,a)} g(x)\,\mathrm{d}x$

for all $t\in[0,1]$. By this, we obtain

$$\begin{aligned}
I &= \frac{\eta(mb,a)}{2}\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big[f'(\psi(t))-f'(\varphi(t))\Big]\mathrm{d}t\\
&= \frac{\eta(mb,a)}{2}\int_0^1\Big[\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big]f'(\psi(t))\,\mathrm{d}t\\
&\quad -\frac{\eta(mb,a)}{2}\int_0^1\Big[\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big]f'(\varphi(t))\,\mathrm{d}t\\
&:= I_1-I_2.
\end{aligned}$$

Via integration by parts, we obtain

$$\begin{aligned}
I_1 &= \frac{\eta(mb,a)}{2}\int_0^1\Big[\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big]f'(\psi(t))\,\mathrm{d}t\\
&= \int_0^1\Big[\int_{\psi(t)}^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\Big]\mathrm{d}\big[f(\psi(t))\big]\\
&= \Big[\int_{\psi(t)}^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\Big]f(\psi(t))\Big|_0^1\\
&\quad +\frac{\eta(mb,a)}{2}\int_0^1 f(\psi(t))g(\psi(t))\,\mathrm{d}t\\
&= -f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_{a+\eta(mb,a)/2}^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\\
&\quad +\int_{a+\eta(mb,a)/2}^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)/2} g(x)\,\mathrm{d}x\\
&\quad -\int_a^{a+\eta(mb,a)/2} f(x)g(x)\,\mathrm{d}x.
\end{aligned}$$

From $I=I_1-I_2$, it follows that

$$I=\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x-f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x,$$

which is the required result in (5). Using Minkowski's inequality, we obtain (6) and (7). □

**Remark 1** If we take $g(x)=1$, $x\in[a,a+\eta(mb,a)]$ with $m=1$ in Lemma 1, then the identity (5) reduces to

$$\begin{aligned}
&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x-f\Big(a+\frac{\eta(b,a)}{2}\Big)=\\
&\quad\frac{\eta(b,a)}{4}\int_0^1(1-t)\Big[f'(\psi(t))-f'(\varphi(t))\Big]\mathrm{d}t. \qquad (8)
\end{aligned}$$

**Remark 2** If $\eta(mb,a)=mb-a$ with $m\in(0,1]$ in Lemma 1, then the identity (5) can be written as

$$\begin{aligned}
&\int_a^{mb} f(x)g(x)\,\mathrm{d}x-f\Big(\frac{a+mb}{2}\Big)\int_a^{mb} g(x)\,\mathrm{d}x\\
&\quad=\frac{mb-a}{2}\Big\{\int_0^1\Big[\int_a^{\mathscr{L}(t)} g(x)\,\mathrm{d}x\Big]\\
&\qquad\times\Big[f'\big(\mathscr{U}(t)\big)-f'\big(\mathscr{L}(t)\big)\Big]\mathrm{d}t\Big\}, \qquad (9)
\end{aligned}$$

where

$$\mathscr{L}(t)=\frac{1+t}{2}a+m\frac{1-t}{2}b=ta+(1-t)\frac{a+mb}{2} \qquad (10)$$

and

$$\mathscr{U}(t)=\frac{1-t}{2}a+m\frac{1+t}{2}b=tmb+(1-t)\frac{a+mb}{2}. \qquad (11)$$

The identity (9) with $m=1$ is proved by Liu[26].

Our first main result is as follows.

**Theorem 2** *If $|f'|^q$ for $q\geqslant 1$ is $(\lambda,m)$-MT-preinvex on $[a,a+\eta(mb,a)]$, then the following inequality holds:*

$$\begin{aligned}
&\Big|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x-f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\Big|\\
&\leqslant\frac{\eta^2(mb,a)}{4}\Big(\tfrac{1}{2}\Big)^{1-1/q}\|g\|_\infty\\
&\quad\times\Big\{\Big(\frac{\pi}{8}|f'(a)|^q+\frac{m(1-\lambda)(3\pi-8)}{8\lambda}|f'(b)|^q\Big)^{1/q}\\
&\qquad+\Big(\frac{3\pi-8}{8}|f'(a)|^q+\frac{m(1-\lambda)\pi}{8\lambda}|f'(b)|^q\Big)^{1/q}\Big\}. \qquad (12)
\end{aligned}$$

*Proof*: Firstly, we suppose that $q=1$. Since $|f'|$ is $(\lambda,m)-MT$-preinvex, then for any $t\in(0,1)$,

$$\begin{aligned}
&\Big|f'\Big(a+\frac{1-t}{2}\eta(mb,a)\Big)\Big|\\
&\leqslant\frac{\sqrt{1+t}}{2\sqrt{1-t}}|f'(a)|+\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{1+t}}|f'(b)| \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
&\Big|f'\Big(a+\frac{1+t}{2}\eta(mb,a)\Big)\Big|\\
&\leqslant\frac{\sqrt{1-t}}{2\sqrt{1+t}}|f'(a)|+\frac{m(1-\lambda)\sqrt{1+t}}{2\lambda\sqrt{1-t}}|f'(b)|. \qquad (14)
\end{aligned}$$

Hence from (7) in Lemma 1, using (13) with (14) we have

$$\left|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\right|$$
$$\leqslant \frac{\eta^2(mb,a)}{4}\|g\|_\infty\bigg\{\int_0^1(1-t)\Big[\frac{\sqrt{1+t}}{2\sqrt{1-t}}|f'(a)| + \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{1+t}}|f'(b)|\Big]\mathrm{d}t$$
$$+\int_0^1(1-t)\Big[\frac{\sqrt{1-t}}{2\sqrt{1+t}}|f'(a)| + \frac{m(1-\lambda)\sqrt{1+t}}{2\lambda\sqrt{1-t}}|f'(b)|\Big]\mathrm{d}t\bigg\}$$
$$\leqslant \frac{\eta^2(mb,a)}{4}\|g\|_\infty\Big\{\frac{\pi-2}{2}|f'(a)| + \frac{m(1-\lambda)(\pi-2)}{2\lambda}|f'(b)|\Big\},$$

which completes the proof for this case.

Secondly, we suppose that $q>1$. Using (7) in Lemma 1 and the power mean inequality, we obtain

$$\left|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\right|$$
$$\leqslant \frac{\eta^2(mb,a)}{4}\|g\|_\infty\int_0^1(1-t)^{1-1/q}(1-t)^{1/q}\times\Big[\big|f'(\varphi(t))\big|+\big|f'(\psi(t))\big|\Big]\mathrm{d}t$$
$$\leqslant \frac{\eta^2(mb,a)}{4}\|g\|_\infty\Big(\int_0^1(1-t)\,\mathrm{d}t\Big)^{1-1/q}\times\bigg\{\Big[\int_0^1(1-t)\big|f'(\varphi(t))\big|^q\,\mathrm{d}t\Big]^{1/q} + \Big[\int_0^1(1-t)\big|f'(\psi(t))\big|^q\,\mathrm{d}t\Big]^{1/q}\bigg\}. \quad (15)$$

Since $|f'|^q$ is $(\lambda,m)$-MT-preinvex on $[a,a+\eta(mb,a)]$, we know that for every $t\in(0,1)$

$$\Big|f'\Big(a+\frac{1-t}{2}\eta(mb,a)\Big)\Big|^q \leqslant \frac{\sqrt{1+t}}{2\sqrt{1-t}}|f'(a)|^q + \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{1+t}}|f'(b)|^q$$

and

$$\Big|f'\Big(a+\frac{1+t}{2}\eta(mb,a)\Big)\Big|^q \leqslant \frac{\sqrt{1-t}}{2\sqrt{1+t}}|f'(a)|^q + \frac{m(1-\lambda)\sqrt{1+t}}{2\lambda\sqrt{1-t}}|f'(b)|^q.$$

So we obtain

$$\left|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\right|$$
$$\leqslant \frac{\eta^2(mb,a)}{4}\big(\tfrac{1}{2}\big)^{1-1/q}\|g\|_\infty \times\bigg\{\Big[\int_0^1(1-t)\Big[\frac{\sqrt{1+t}}{2\sqrt{1-t}}|f'(a)|^q + \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{1+t}}|f'(b)|^q\Big]\mathrm{d}t\Big]^{1/q}$$
$$+\Big[\int_0^1(1-t)\Big[\frac{\sqrt{1-t}}{2\sqrt{1+t}}|f'(a)|^q + \frac{m(1-\lambda)\sqrt{1+t}}{2\lambda\sqrt{1-t}}|f'(b)|^q\Big]\mathrm{d}t\Big]^{1/q}\bigg\}$$
$$= \frac{\eta^2(mb,a)}{4}\big(\tfrac{1}{2}\big)^{1-1/q}\|g\|_\infty \times\bigg\{\Big(\frac{\pi}{8}|f'(a)|^q + \frac{m(1-\lambda)(3\pi-8)}{8\lambda}|f'(b)|^q\Big)^{1/q} + \Big(\frac{3\pi-8}{8}|f'(a)|^q + \frac{m(1-\lambda)\pi}{8\lambda}|f'(b)|^q\Big)^{1/q}\bigg\},$$

which completes the proof. □

**Corollary 1** *Consider Theorem 2. (i) If $q=1$ and $g(x)=1$, then we have*

$$\left|\frac{1}{\eta(mb,a)}\int_a^{a+\eta(mb,a)} f(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\right| \leqslant \frac{\eta(mb,a)}{4}\Big[\frac{\pi-2}{2}|f'(a)| + \frac{m(1-\lambda)(\pi-2)}{2\lambda}|f'(b)|\Big]. \quad (16)$$

*(ii) If $q=1$, $\lambda=\frac{1}{2}$, and $\eta(mb,a)=mb-a$ with $m=1$, then we have*

$$\left|\int_a^b f(x)g(x)\,\mathrm{d}x - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,\mathrm{d}x\right| \leqslant \frac{(b-a)^2(\pi-2)}{8}\|g\|_\infty\big[|f'(a)|+|f'(b)|\big]. \quad (17)$$

**Corollary 2** *In Theorem 2, if $q=1$ and $\eta(mb,a)=mb-a$, then we have*

$$\left|\int_a^{mb} f(x)g(x)\,\mathrm{d}x - f\Big(\frac{a+mb}{2}\Big)\int_a^{mb} g(x)\,\mathrm{d}x\right| \leqslant \frac{(mb-a)^2}{4}\|g\|_\infty\bigg\{\frac{\pi}{16}\big[|f'(a)|+|f'(mb)|\big] + \frac{3m(1-\lambda)\pi}{8\lambda}\Big|f'\Big(\frac{a+mb}{2m}\Big)\Big|\bigg\}. \quad (18)$$

*Proof*: By (7) in Lemma 1 with the second equality in (10) and (11), and using the $(\lambda, m)$-MT-convexity of $|f'|$ on $[a, mb]$ with $a < mb$, we have

$$\begin{aligned}
&\Big|\int_a^{mb} f(x)g(x)\,\mathrm{d}x - f\Big(\frac{a+mb}{2}\Big)\int_a^{mb} g(x)\,\mathrm{d}x\Big| \\
&\leqslant \frac{(mb-a)^2}{4}\|g\|_\infty\bigg\{\int_0^1 (1-t)\Big[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)| \\
&\quad + \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\Big|f'\Big(\frac{a+mb}{2m}\Big)\Big|\Big]\mathrm{d}t \\
&\quad + \int_0^1 (1-t)\Big[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(mb)| \\
&\quad + \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\Big|f'\Big(\frac{a+mb}{2m}\Big)\Big|\Big]\mathrm{d}t\bigg\} \\
&= \frac{(mb-a)^2}{4}\|g\|_\infty\bigg\{\frac{1}{2}\big(|f'(a)|+|f'(mb)|\big) \\
&\quad \times \int_0^1 (1-t)^{1/2}t^{1/2}\,\mathrm{d}t + \frac{m(1-\lambda)}{\lambda} \\
&\quad \times \Big|f'\Big(\frac{a+mb}{2m}\Big)\Big|\int_0^1 (1-t)^{3/2}t^{-1/2}\,\mathrm{d}t\bigg\}. \qquad (19)
\end{aligned}$$

From (19) we obtain (18), since

$$\int_0^1 (1-t)^{1/2}t^{1/2}\,\mathrm{d}t = \beta(\tfrac{3}{2},\tfrac{3}{2}) = \frac{\pi}{8},$$
$$\int_0^1 (1-t)^{3/2}t^{-1/2}\,\mathrm{d}t = \beta(\tfrac{1}{2},\tfrac{5}{2}) = \frac{3\pi}{8}.$$

Here,

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

□

**Remark 3** In Corollary 2, if $g(x)=1$, $\lambda=\frac{1}{2}$ with $m=1$, then we have the following inequality for MT-convex functions

$$\begin{aligned}
&\Big|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\Big(\frac{a+b}{2}\Big)\Big| \\
&\leqslant \frac{(b-a)}{4}\bigg\{\frac{\pi}{16}\big[|f'(a)|+|f'(b)|\big]+\frac{3\pi}{8}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|\bigg\}.
\end{aligned}$$

The following result holds for $(\lambda, m)$-MT-preinvexity of $|f'|^q$.

**Theorem 3** *For $q > 1$ with $p^{-1}+q^{-1}=1$, if $|f'|^q$ is $(\lambda, m)$-MT-preinvex on $[a, a+\eta(mb,a)]$, then the following inequality holds:*

$$\begin{aligned}
&\Big|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\Big| \\
&\leqslant \frac{\eta^2(mb,a)}{4}\|g\|_\infty\Big(\frac{1}{p+1}\Big)^{1/p} \\
&\times\bigg\{\Big[\frac{\pi+2}{4}|f'(a)|^q + \frac{m(1-\lambda)(\pi-2)}{4\lambda}|f'(b)|^q\Big]^{1/q} \\
&+\Big[\frac{\pi-2}{4}|f'(a)|^q + \frac{m(1-\lambda)(\pi+2)}{4\lambda}|f'(b)|^q\Big]^{1/q}\bigg\}. \qquad (20)
\end{aligned}$$

*Proof*: By (7) in Lemma 1 and Hölder's inequality,

$$\begin{aligned}
&\Big|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\Big| \\
&\quad\leqslant \frac{\eta^2(mb,a)}{4}\|g\|_\infty \\
&\quad\quad\times\int_0^1 (1-t)\big[|f'(\varphi(t))|+|f'(\psi(t))|\big]\,\mathrm{d}t \\
&\quad\leqslant \frac{\eta^2(mb,a)}{4}\|g\|_\infty\Big[\int_0^1 (1-t)^p\,\mathrm{d}t\Big]^{1/p} \\
&\times\bigg\{\Big[\int_0^1 \big|f'(\varphi(t))\big|^q\,\mathrm{d}t\Big]^{1/q}+\Big[\int_0^1 \big|f'(\psi(t))\big|^q\,\mathrm{d}t\Big]^{1/q}\bigg\}. \qquad (21)
\end{aligned}$$

From (21) we deduce the required inequality in (20), since $|f'|^q$ is $(\lambda, m)$-MT-preinvex on $[a, a+\eta(mb,a)]$, we have

$$\int_0^1 |f'(\varphi(t))|^q\,\mathrm{d}t \leqslant \frac{\pi+2}{4}|f'(a)|^q + \frac{m(1-\lambda)(\pi-2)}{4\lambda}|f'(b)|^q$$

and

$$\int_0^1 |f'(\psi(t))|^q\,\mathrm{d}t \leqslant \frac{\pi-2}{4}|f'(a)|^q + \frac{m(1-\lambda)(\pi+2)}{4\lambda}|f'(b)|^q.$$

□

**Corollary 3** *If we take $g(x)=1$, $\lambda=1/2$, and $m=1$ in Theorem 3, then we obtain*

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,\mathrm{d}x-f\Big(a+\frac{\eta(b,a)}{2}\Big)\right|\\&\quad\leqslant\frac{\eta(b,a)}{4}\Big(\frac{1}{p+1}\Big)^{1/p}\\&\qquad\times\Big\{\Big[\frac{\pi+2}{4}|f'(a)|^q+\frac{\pi-2}{4}|f'(b)|^q\Big]^{1/q}\\&\qquad\quad+\Big[\frac{\pi-2}{4}|f'(a)|^q+\frac{\pi+2}{4}|f'(b)|^q\Big]^{1/q}\Big\}\\&\quad\leqslant\frac{\eta(b,a)}{4}\Big(\frac{1}{p+1}\Big)^{1/p}\\&\qquad\times\Big[\frac{(\pi+2)^{1/q}+(\pi-2)^{1/q}}{4^{1/q}}\Big]\big(|f'(a)|+|f'(b)|\big).\end{aligned}$$

Here, $0<1/q<1$ for $q>1$. To prove the second inequality above, we use the fact $\sum_{i=1}^n(u_i+v_i)^\gamma\leqslant\sum_{i=1}^n u_i^\gamma+\sum_{i=1}^n v_i^\gamma$, for $0\leqslant\gamma<1$, $u_1,\dots,u_\mathrm{n}\geqslant 0$, and $v_1,\dots,v_\mathrm{n}\geqslant 0$.

In the next theorem, we will use the hypergeometric function: for $c>b>0$ and $|z|<1$,

$$\begin{aligned}&{}_2F_1(a,b;c;z)\\&\quad=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t.\end{aligned}$$

**Theorem 4** *Suppose that all conditions of Theorem 3 are satisfied. Then we have*

$$\begin{aligned}&\left|\int_a^{a+\eta(mb,a)}f(x)g(x)\,\mathrm{d}x-f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)}g(x)\,\mathrm{d}x\right|\\&\leqslant\frac{\eta^2(mb,a)}{4}\Big(\frac{q-1}{2q-p-1}\Big)^{(q-1)/q}\|g\|_\infty\\&\quad\times\Big\{\Big[\tfrac{1}{2}\beta\big(1,p+\tfrac{1}{2}\big){}_2F_1(-\tfrac{1}{2},1;p+\tfrac{3}{2};-1)|f'(a)|^q\\&\quad+\frac{m(1-\lambda)}{2\lambda}\beta\big(1,p+\tfrac{3}{2}\big){}_2F_1(\tfrac{1}{2},1;p+\tfrac{5}{2};-1)|f'(b)|^q\Big]^{1/q}\\&\quad+\Big[\tfrac{1}{2}\beta\big(1,p+\tfrac{3}{2}\big){}_2F_1(\tfrac{1}{2},1;p+\tfrac{5}{2};-1)|f'(a)|^q\\&+\frac{m(1-\lambda)}{2\lambda}\beta\big(1,p+\tfrac{1}{2}\big){}_2F_1(-\tfrac{1}{2},1;p+\tfrac{3}{2};-1)|f'(b)|^q\Big]^{1/q}\Big\}.\end{aligned}\tag{22}$$

*Proof*: By (7) in Lemma 1 and Hölder's inequality for $p>1$, we obtain

$$\begin{aligned}&\left|\int_a^{a+\eta(mb,a)}f(x)g(x)\,\mathrm{d}x-f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)}g(x)\,\mathrm{d}x\right|\\&\leqslant\frac{\eta^2(mb,a)}{4}\|g\|_\infty\Big[\int_0^1(1-t)^{(q-p)/(q-1)}\,\mathrm{d}t\Big]^{(q-1)/q}\\&\quad\times\Big\{\Big[\int_0^1(1-t)^p|f'(\varphi(t))|^q\,\mathrm{d}t\Big]^{1/q}\\&\qquad+\Big[\int_0^1(1-t)^p|f'(\psi(t))|^q\,\mathrm{d}t\Big]^{1/q}\Big\}\\&\leqslant\frac{\eta^2(mb,a)}{4}\Big(\frac{q-1}{2q-p-1}\Big)^{(q-1)/q}\|g\|_\infty\\&\quad\times\Big\{\Big[\int_0^1(1-t)^p|f'(\varphi(t))|^q\,\mathrm{d}t\Big]^{1/q}\\&\qquad+\Big[\int_0^1(1-t)^p|f'(\psi(t))|^q\,\mathrm{d}t\Big]^{1/q}\Big\}.\end{aligned}\tag{23}$$

From (23) we deduce the desired inequality (22), since $|f'|^q$ is $(\lambda,m)$-MT-preinvex on $[a,a+\eta(mb,a)]$, for any $t\in(0,1)$, we have

$$\begin{aligned}&\int_0^1(1-t)^p|f'(\varphi(t))|^q\,\mathrm{d}t\\&\quad\leqslant\int_0^1(1-t)^p\Big[\frac{\sqrt{1+t}}{2\sqrt{1-t}}|f'(a)|^q\\&\qquad+\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{1+t}}|f'(b)|^q\Big]\mathrm{d}t\\&\quad=\tfrac{1}{2}\beta(1,p+\tfrac{1}{2}){}_2F_1(-\tfrac{1}{2},1;p+\tfrac{3}{2};-1)|f'(a)|^q\\&\quad+\frac{m(1-\lambda)}{2\lambda}\beta(1,p+\tfrac{3}{2}){}_2F_1(\tfrac{1}{2},1;p+\tfrac{5}{2};-1)|f'(b)|^q\end{aligned}$$

and

$$\begin{aligned}&\int_0^1(1-t)^p\left|f'(\psi(t))\right|^q\,\mathrm{d}t\\&\qquad\leqslant\tfrac{1}{2}\beta(1,p+\tfrac{3}{2}){}_2F_1(\tfrac{1}{2},1;p+\tfrac{5}{2};-1)|f'(a)|^q\\&+\frac{m(1-\lambda)}{2\lambda}\beta(1,p+\tfrac{1}{2}){}_2F_1(-\tfrac{1}{2},1;p+\tfrac{3}{2};-1)|f'(b)|^q.\end{aligned}$$

This ends the proof. □

By using (6) in Lemma 1, we are in a position to present the following results.

**Theorem 5** *If $|f'|^q$ for $q\geqslant 1$ is $(\lambda,m)$-MT-preinvex on $[a,a+\eta(mb,a)]$, then the following holds:*

$$\left|\int_a^{a+\eta(mb,a)} f(x)g(x)\,dx - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,dx\right| \leqslant 2^{-1/q}\eta(mb,a)\Big[|f'(a)|^q + \frac{m(1-\lambda)}{\lambda}|f'(b)|^q\Big]^{1/q} \times\Big[\int_0^1\int_a^{\varphi(t)} g(x)\,dx\,dt\Big]^{1-1/q}\Big[\int_0^1 \frac{1}{\sqrt{1-t^2}}\int_a^{\varphi(t)} g(x)\,dx\,dt\Big]^{1/q}. \tag{24}$$

*Proof*: By (6) in Lemma 1 and the power mean inequality, we have

$$\left|\int_a^{a+\eta(mb,a)} f(x)g(x)\,dx - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,dx\right| \leqslant \frac{\eta(mb,a)}{2}\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)dt\Big]^{1-1/q} \times\Big\{\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)\big|f'(\varphi(t))\big|^q\,dt\Big]^{1/q} +\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)\big|f'(\psi(t))\big|^q\,dt\Big]^{1/q}\Big\}. \tag{25}$$

From the inequality $(u^\gamma+v^\gamma)\leqslant 2^{1-\gamma}(u+v)^\gamma$, for $u,v>0$, $\gamma\leqslant 1$, and the $(\lambda,m)$-MT-preinvexity of $|f'|^q$ on $[a,a+\eta(mb,a)]$, we obtain

$$\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)\big|f'(\varphi(t))\big|^q\,dt\Big]^{1/q} +\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)\big|f'(\psi(t))\big|^q\,dt\Big]^{1/q}$$
$$\leqslant 2^{1-1/q}\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)\times\Big(\big|f'(\varphi(t))\big|^q+\big|f'(\psi(t))\big|^q\Big)dt\Big]^{1/q}$$
$$\leqslant 2^{1-1/q}\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)\Big(\frac{\sqrt{1+t}}{2\sqrt{1-t}}|f'(a)|^q + \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{1+t}}|f'(b)|^q + \frac{\sqrt{1-t}}{2\sqrt{1+t}}|f'(a)|^q + \frac{m(1-\lambda)\sqrt{1+t}}{2\lambda\sqrt{1-t}}|f'(b)|^q\Big)dt\Big]^{1/q}$$
$$= 2^{1-1/q}\Big[|f'(a)|^q + \frac{m(1-\lambda)}{\lambda}|f'(b)|^q\Big]^{1/q} \times\Big[\int_0^1\frac{1}{\sqrt{1-t^2}}\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)dt\Big]^{1/q}. \tag{26}$$

The inequality (24) follows from (25) and (26), and Theorem 5 is proved. □

**Corollary 4** *In Theorem 5, if $q=1$ and $g(x)=1$, then we have*

$$\left|\frac{1}{\eta(mb,a)}\int_a^{a+\eta(mb,a)} f(x)\,dx - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\right| \leqslant \frac{\eta(mb,a)(\pi-2)}{8}\Big[|f'(a)| + \frac{m(1-\lambda)}{\lambda}|f'(b)|\Big].$$

**Theorem 6** *If $|f'|^q$ for $q>1$ is $(\lambda,m)$-MT-preinvex on $[a,a+\eta(mb,a)]$, then the following inequality holds:*

$$\left|\int_a^{a+\eta(mb,a)} f(x)g(x)\,dx - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,dx\right| \leqslant \frac{\eta(mb,a)}{2}\Big\{\Big[\frac{\pi+2}{4}|f'(a)|^q + \frac{m(1-\lambda)(\pi-2)}{4\lambda}|f'(b)|^q\Big]^{1/q} + \Big[\frac{\pi-2}{4}|f'(a)|^q + \frac{m(1-\lambda)(\pi+2)}{4\lambda}|f'(b)|^q\Big]^{1/q}\Big\} \times\Big\{\int_0^1\Big[\int_a^{\varphi(t)} g(x)\,dx\Big]^{q/(q-1)}dt\Big\}^{1-1/q}. \tag{27}$$

*Proof*: Noting that $|f'|^q$ is $(\lambda,m)$-MT-preinvex on $[a,a+\eta(mb,a)]$, by (6) in Lemma 1, and using the Hölder inequality, we have

$$\left|\int_a^{a+\eta(mb,a)} f(x)g(x)\,dx - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,dx\right| \leqslant \frac{\eta(mb,a)}{2}\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)^{q/(q-1)}dt\Big]^{1-1/q} \times\Big[\Big(\int_0^1\big|f'(\varphi(t))\big|^q\,dt\Big)^{1/q}+\Big(\int_0^1\big|f'(\psi(t))\big|^q\,dt\Big)^{1/q}\Big]$$
$$\leqslant \frac{\eta(mb,a)}{2}\Big[\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,dx\Big)^{q/(q-1)}dt\Big]^{1-1/q} \times\Big\{\Big[\int_0^1\Big(\frac{\sqrt{1+t}}{2\sqrt{1-t}}|f'(a)|^q + \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{1+t}}|f'(b)|^q\Big)dt\Big]^{1/q} + \Big[\int_0^1\Big(\frac{\sqrt{1-t}}{2\sqrt{1+t}}|f'(a)|^q + \frac{m(1-\lambda)\sqrt{1+t}}{2\lambda\sqrt{1-t}}|f'(b)|^q\Big)dt\Big]^{1/q}\Big\}$$

$$= \frac{\eta(mb,a)}{2}\Big\{\Big[\frac{\pi+2}{4}|f'(a)|^q + \frac{m(1-\lambda)(\pi-2)}{4\lambda}|f'(b)|^q\Big]^{1/q} + \Big[\frac{\pi-2}{4}|f'(a)|^q + \frac{m(1-\lambda)(\pi+2)}{4\lambda}|f'(b)|^q\Big]^{1/q}\Big\} \times \Big\{\int_0^1\Big(\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big)^{q/(q-1)}\mathrm{d}t\Big\}^{1-1/q}.$$

The inequality (27) follows. □

## FURTHER ESTIMATION RESULTS

To obtain new estimation-type results, we deal with the boundedness and the Lipschitzian condition of $f'$, respectively.

**Theorem 7** *Assume that there exist constants $r < R$ such that $-\infty < r \leqslant f'(x) \leqslant R < \infty$ for all $x \in [a, a+\eta(mb,a)]$. Then the following inequality holds:*

$$\Big|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x\Big| \leqslant \frac{(R-r)\eta^2(mb,a)}{8}\|g\|_\infty. \quad (28)$$

*Proof*: From Lemma 1, we have

$$J = \int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x$$
$$= \frac{\eta(mb,a)}{2}\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\,\big[f'(\psi(t)) - f'(\varphi(t))\big]\,\mathrm{d}t$$
$$= \frac{\eta(mb,a)}{2}\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\,\Big[f'(\psi(t)) - \frac{r+R}{2}\Big]\,\mathrm{d}t - \frac{\eta(mb,a)}{2}\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\,\Big[f'(\varphi(t)) - \frac{r+R}{2}\Big]\,\mathrm{d}t.$$

Since the inequality $r \leqslant f'(x) \leqslant R$, we have

$$r - \frac{r+R}{2} \leqslant f'\Big(a+\frac{1-t}{2}\eta(mb,a)\Big) - \frac{r+R}{2} \leqslant R - \frac{r+R}{2},$$

which implies that

$$\Big|f'\Big(a+\frac{1-t}{2}\eta(mb,a)\Big) - \frac{r+R}{2}\Big| \leqslant \frac{R-r}{2}.$$

Similarly, we have

$$\Big|f'\Big(a+\frac{1+t}{2}\eta(mb,a)\Big) - \frac{r+R}{2}\Big| \leqslant \frac{R-r}{2}.$$

Hence

$$|J| \leqslant \frac{\eta(mb,a)}{2}\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big|f'(\psi(t)) - \frac{r+R}{2}\Big|\,\mathrm{d}t + \frac{\eta(mb,a)}{2}\int_0^1\Big|\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big|\Big|f'(\varphi(t)) - \frac{r+R}{2}\Big|\,\mathrm{d}t$$
$$\leqslant \frac{(R-r)\eta(mb,a)}{2}\int_0^1\Big|\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big|\,\mathrm{d}t$$
$$\leqslant \frac{(R-r)\eta^2(mb,a)}{8}\|g\|_\infty.$$

This ends the proof. □

**Corollary 5** *In Theorem 7, if we take $g(x) = 1$, then we have*

$$\Big|\frac{1}{\eta(mb,a)}\int_a^{a+\eta(mb,a)} f(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\Big| \leqslant \frac{(R-r)\eta(mb,a)}{8}. \quad (29)$$

Specially, taking $\eta(mb,a) = mb - a$ with $m = 1$, we obtain

$$\Big|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\Big(\frac{a+b}{2}\Big)\Big| \leqslant \frac{(R-r)(b-a)}{8}.$$

**Theorem 8** *Assume that $f'$ satisfies Lipschitz condition on $\mathscr{K}$ for some $L > 0$, then the following inequality holds:*

$$\Big|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x - \frac{\eta(mb,a)}{2}\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\,\big[f'(a+\eta(mb,a)) - f'(a)\big]\,\mathrm{d}t\Big| \leqslant \frac{\eta^3(mb,a)}{12}L\|g\|_\infty. \quad (30)$$

*Proof*: From Lemma 1, we have

$$\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x$$
$$= \frac{\eta(mb,a)}{2}\Big\{\int_0^1\Big[\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big]\Big[f'(\psi(t)) - f'(a+\eta(mb,a)) + f'(a+\eta(mb,a))\Big]\,\mathrm{d}t\Big\}$$
$$-\frac{\eta(mb,a)}{2}\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\,\big[f'(\varphi(t)) - f'(a) + f'(a)\big]\,\mathrm{d}t$$

$$= \frac{\eta(mb,a)}{2}\Big\{\int_0^1\Big[\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\Big]\Big[(f'(\psi(t)) - f'(a+\eta(mb,a))) - (f'(\varphi(t)) - f'(a))\Big]\mathrm{d}t\Big\} + \frac{\eta(mb,a)}{2}\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\,\big[f'\big(a+\eta(mb,a)\big) - f'(a)\big]\,\mathrm{d}t.$$

Since $f'$ satisfies Lipschitz condition for some $L > 0$, we have

$$\Big|f'\Big(a+\frac{1+t}{2}\eta(mb,a)\Big) - f'\big(a+\eta(mb,a)\big)\Big| \leqslant L\Big|\Big(a+\frac{1+t}{2}\eta(mb,a)\Big) - \big(a+\eta(mb,a)\big)\Big| = L\frac{1-t}{2}\eta(mb,a)$$

and

$$\Big|f'\Big(a+\frac{1-t}{2}\eta(mb,a)\Big) - f'(a)\Big| \leqslant L\Big|\Big(a+\frac{1-t}{2}\eta(mb,a)\Big) - a\Big| = L\frac{1-t}{2}\eta(mb,a).$$

Hence

$$\Big|\int_a^{a+\eta(mb,a)} f(x)g(x)\,\mathrm{d}x - f\Big(a+\frac{\eta(mb,a)}{2}\Big)\int_a^{a+\eta(mb,a)} g(x)\,\mathrm{d}x - \frac{\eta(mb,a)}{2}\big[f'(a+\eta(mb,a)) - f'(a)\big]\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leqslant \frac{\eta^2(mb,a)}{2}L\int_0^1\int_a^{\varphi(t)} g(x)\,\mathrm{d}x\,(1-t)\,\mathrm{d}t \leqslant \frac{\eta^3(mb,a)}{12}L\|g\|_\infty.$$

This ends the proof. □

**Corollary 6** *If we take $g(x) = 1$ and $\eta(mb,a) = mb - a$ with $m = 1$ in Theorem 8, then we obtain*

$$\Big|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\Big(\frac{a+b}{2}\Big) - \frac{b-a}{8}\big[f'(b) - f'(a)\big]\Big| \leqslant \frac{(b-a)^2}{12}L.$$

## APPLICATIONS FOR RANDOM VARIABLES

Suppose that for $0 < a < b$, $g : [a,b] \to [0,\infty)$ is a continuous probability density function related to a continuous random variable $X$ which is symmetric about $(a+b)/2$. Also, for $\tau \in \mathbb{R}$, suppose that the $\tau$-moment

$$E_\tau(X) = \int_a^b x^\tau g(x)\,\mathrm{d}x \tag{31}$$

is finite.

From the fact that $g$ is symmetric and $\int_a^b g(x)\,\mathrm{d}x = 1$, we have

$$E(X) = \int_a^b x g(x)\,\mathrm{d}x = \frac{a+b}{2}, \tag{32}$$

since

$$\int_a^b x g(x)\,\mathrm{d}x = \int_a^b (b+a-x)g(b+a-x)\,\mathrm{d}x = \int_a^b (b+a-x)g(x)\,\mathrm{d}x.$$

Based on the above-mentioned derivations, we obtain the following estimations of the $\tau$-moment.

**Proposition 1** *For $\tau \geqslant 2$, we have the following inequality*

$$\Big|E_\tau(X) - \big(E(X)\big)^\tau\Big| \leqslant \frac{\tau(b-a)^2(\pi-2)}{8}\|g\|_\infty\big(a^{\tau-1} + b^{\tau-1}\big). \tag{33}$$

*Proof*: Let $f(x) = x^\tau$. For $\tau \geqslant 2$, the function $|f'(x)| = \tau x^{\tau-1}$ is an MT-convex function. Utilizing the identities (31) and (32), we immediately obtain inequality (33) from (17) in Corollary 1. □

**Proposition 2** *If $m = 1$ and $\lambda = 1/2$ in Corollary 2, then we have*

$$\Big|E_\tau(X) - \big(E(X)\big)^\tau\Big| \leqslant \frac{\tau(b-a)^2}{4}\|g\|_\infty \times \Big[\frac{\pi}{16}\big(a^{\tau-1} + b^{\tau-1}\big) + \frac{3\pi}{8}\Big(\frac{a+b}{2}\Big)^{\tau-1}\Big].$$

*Proof*: The proof is analogous to that of Proposition 1 but replace (17) in Corollary 1 with (18) in Corollary 2. □

**Proposition 3** *Assume that $\eta(mb,a) = mb - a$ with $m = 1$ in Theorem 7, and notice that if we consider $f(x) = x^\tau$ for $\tau \in \mathbb{R}$ together with $x \in [a,b]$. Then $r = \tau a^{\tau-1} \leqslant f'(x) \leqslant \tau b^{\tau-1} = R$, and we have*

$$\Big|E_\tau(X) - \big(E(X)\big)^\tau\Big| \leqslant \frac{\tau(b^{\tau-1} - a^{\tau-1})(b-a)^2}{8}\|g\|_\infty.$$

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