RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2019.45.285

ScienceAsia 45 (2019): **285–291**

Weaving of K-g-frames in Hilbert spaces

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Received 10 Jan 2019
Accepted 12 May 2019

ABSTRACT: In this study, we mainly discuss the weaving of K-g-frames in Hilbert spaces. Note that the concept of weaving was recently proposed by Bemrose et al to solve a question in distributed signal processing. We give a sufficient condition such that the left sequence can still be K-woven in $R(K)$ by deleting some elements from a K-woven pair of K-g-frames. We then give three different types of perturbation conditions such that, under them, the K-g-frames $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ or the types $\{\Lambda_j T_1^* : j \in J\}$ and $\{\Gamma_j T_2^* : j \in J\}$ can be K-woven in $R(K)$, where T_1, T_2 are surjective operators on $\mathscr{U}$.

KEYWORDS: G-frame, K-g-frame, weaving, perturbation

MSC2010: 42C15

INTRODUCTION

Sun[1] proposed a more general type of frame called g-frame to deal with all existing frames at that time as a united object. Given J being a countable index set; $\mathscr{U}$, $\mathscr{V}_j$, $j \in J$, being Hilbert spaces; and Λ_j being a bounded linear operator from $\mathscr{U}$ to $\mathscr{V}_j$, recall that $\{\Lambda_j : j \in J\}$ is called a g-frame for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ if there exist $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{j\in J}\|\Lambda_j f\|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathscr{U}.$$

Here, A and B are the lower and upper frame bounds, respectively, for the g-frame $\{\Lambda_j : j \in J\}$. Xiao et al[2] further generalized g-frames[3–5] and K-frames[6–8] and introduced another notion, namely, K-g-frames. For more information on g-frames and K-g-frames, see Refs. 9–11 and the references therein.

Note that the notion of weaving was recently proposed by Bemrose et al[12–14] to simulate a question in distributed signal processing. Since then weaving as a research hotspot has been studied by many scholars. We refer the readers to check Refs. 15–18 for more information on the weaving of g-frames or fusion frames and Ref. 19 for information on the weaving of K-frames.

In this study, we will mainly discuss the erasures and perturbations of weaving for K-g-frames in Hilbert spaces. We give a sufficient condition such that the left sequence can still be K-woven in $R(K)$ by deleting some elements from a K-woven pair of K-g-frames. We then give three different types of perturbation conditions such that, under them, the K-g-frames $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ or the types $\{\Lambda_j T_1^* : j \in J\}$ and $\{\Gamma_j T_2^* : j \in J\}$ can be K-woven in $R(K)$, where T_1 and T_2 are surjective operators on $\mathscr{U}$.

Throughout this study, we will adopt the following notations: $\mathscr{H}$ is a separable Hilbert space; $I_{\mathscr{H}}$ is the identity operator for $\mathscr{H}$; $L(X,Y)$ is the collection of all bounded linear operators from X to Y, where X, Y are Banach spaces, and if $X = Y$, then $L(X,Y)$ is denoted by $L(X)$; the range and the kernel of $K \in L(\mathscr{H})$ are denoted by $R(K)$ and $N(K)$, respectively; finally, the pseudo-inverse of $K \in L(\mathscr{H})$ is denoted by $K^\dagger$.

PRELIMINARIES OF K-G-FRAMES

In this section, we recall the definitions and some basic properties of K-g-frames and weaving, and apply the woven principle to K-g-frames.

Definition 1 [Ref. 2] A sequence $\{\Lambda_j \in L(\mathscr{U},\mathscr{V}_j) : j \in J\}$ is called a K-g-frame for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ if there exist $A, B > 0$ such that

$$A\|K^* f\|^2 \leqslant \sum_{j\in J}\|\Lambda_j f\|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathscr{U}. \quad (1)$$

We call A and B the lower frame bound and the upper frame bound, respectively, for the K-g-frame

$\{\Lambda_j : j \in J\}$. We call $\{\Lambda_j : j \in J\}$ the g-Bessel sequence if only the right-hand side of (1) holds.

If $A\|K^* f\|^2 = \sum_{j\in J}\|\Lambda_j f\|^2$, $\forall f \in \mathscr{U}$, we call $\{\Lambda_j : j \in J\}$ a tight K-g-frame; furthermore, if $A = 1$, $\{\Lambda_j : j \in J\}$ is called a Parseval K-g-frame.

Observe that, from (1), $\{\Lambda_j : j \in J\}$ is an $I_{\mathscr{U}}$-g-frame for $\mathscr{U}$ if and only if $\{\Lambda_j : j \in J\}$ is a g-frame for $\mathscr{U}$.

The concept of weaving of frames was introduced by Bemrose et al[12] to simulate a question in distributed signal processing. We now recall it as follows.

Definition 2 [Ref. 12] Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be frames for $\mathscr{H}$. If for any partition $\{\sigma_j\}_{j=1}^2$ of I there exist A, $B > 0$ such that $\{f_i\}_{i\in\sigma_1} \cup \{g_i\}_{i\in\sigma_2}$ is a frame for $\mathscr{H}$ with the frame bounds A and B, then we consider that $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are woven in $\mathscr{H}$ with the frame bounds A and B, and $\{f_i\}_{i\in\sigma_1} \cup \{g_i\}_{i\in\sigma_2}$ is called a weaving.

In this study, we will apply the woven principle to K-g-frames.

Definition 3 Let $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ be K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$. If for any partition $\{\sigma_j\}_{j=1}^2$ of J there exist A, $B > 0$ such that $\{\Lambda_j\}_{i\in\sigma_1} \cup \{\Gamma_j\}_{i\in\sigma_2}$ is a K-g-frame for $\mathscr{U}$ with the frame bounds A and B, then we consider that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-woven in $\mathscr{U}$ with the frame bounds A and B, and each $\{\Lambda_j\}_{i\in\sigma_1} \cup \{\Gamma_j\}_{i\in\sigma_2}$ is called a weaving.

Note that, if $K = I_{\mathscr{H}}$, then K-frames are the classical frames for $\mathscr{H}$ and, at the same time, the weaving of the K-frames is the weaving of the classical frames.

Assume that $\{\Lambda_j \in L(\mathscr{U}, \mathscr{V}_j) : j \in J\}$ is a g-Bessel sequence in $\mathscr{U}$; then, the synthesis operator T, analysis operator U, and frame operator S of $\{\Lambda_j : j \in J\}$ are defined as follows:

$$T : l^2(\{\mathscr{V}_j\}_{j\in J}) \to \mathscr{U}, \quad T(\{g_j\}_{j\in J}) = \sum_{j\in J}\Lambda_j^* g_j, \tag{2}$$

$$U : \mathscr{U} \to l^2(\{\mathscr{V}_j\}_{j\in J}), \quad Uf = \{\Lambda_j f\}_{j\in J}, \tag{3}$$

$$S : \mathscr{U} \to \mathscr{U}, \quad Sf = \sum_{j\in J}\Lambda_j^*\Lambda_j f, \tag{4}$$

where $l^2(\{\mathscr{V}_j\}_{j\in J})$ is a Hilbert space defined as

$$l^2(\{\mathscr{V}_j\}_{j\in J}) = \Big\{\{g_j\}_{j\in J} : g_j \in \mathscr{V}_j, j \in J, \sum_{j\in J}\|g_j\|^2 < \infty\Big\}$$

with inner product $\langle\{f_j\}_{j\in J}, \{g_j\}_{j\in J}\rangle = \sum_{j\in J}\langle f_j, g_j\rangle$. It is easy to check that $U = T^*$ and $S = TU$.

Lemma 1 (Ref. 20) *Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are two Hilbert spaces and that $Q \in L(\mathscr{H}_1, \mathscr{H}_2)$ is an operator with a closed range. Then, there exists a unique bounded operator $Q^\dagger : \mathscr{H}_2 \to \mathscr{H}_1$, called the pseudo-inverse operator of Q, satisfying*

$$\begin{aligned} &N(Q^\dagger) = R(Q)^\perp, \quad R(Q^\dagger) = N(Q)^\perp, \\ &QQ^\dagger = P_{R(Q)}, \quad Q^\dagger Q = P_{R(Q^\dagger)}. \end{aligned} \tag{5}$$

If Q is bounded and invertible, then $Q^\dagger = Q^{-1}$.

In this study, we always assume that $K \in L(\mathscr{U})$ is an operator with a closed range. By Lemma 1, there exists a pseudo-inverse operator $K^\dagger$ such that $KK^\dagger = P_{R(K)}$. It follows that $I_{R(K)} = I^*_{R(K)} = (K^\dagger)^*K^*$. Hence, for any $f \in R(K)$,

$$\|f\| = \|(K^\dagger)^*K^*f\| \leqslant \|(K^\dagger)^*\|\|K^*f\| = \|K^\dagger\|\|K^*f\|. \tag{6}$$

ERASURES OF THE WEAVING OF K-G-FRAMES

In this section, we provide a sufficient condition such that the left sequence can still be K-woven in $R(K)$ by deleting some elements from a K-woven pair of K-g-frames.

Theorem 1 *Let $K \in L(\mathscr{U})$ be an operator with a closed range. Suppose that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ and are K-woven in $\mathscr{U}$ with the universal bounds A and B. If there exist $\sigma \subset J$ and $\alpha, \beta, \gamma \geqslant 0$ satisfying $A > \beta + (\alpha B + \gamma)\|K^\dagger\|^2$ such that*

$$\sum_{j\in\sigma}\|\Lambda_j f\|^2 \leqslant \alpha\sum_{j\in\sigma}\|\Gamma_j f\|^2 + \beta\|K^*f\|^2 + \gamma\|f\|^2 \tag{7}$$

for all $f \in \mathscr{U}$, then $\{\Lambda_j : j \in J\backslash\sigma\}$ and $\{\Gamma_j : j \in J\backslash\sigma\}$ are K-woven in $R(K)$ with the universal bounds $A - \beta - (\alpha B + \gamma)\|K^\dagger\|^2$ and B.

Proof: For any partition $\{\sigma_j\}_{j=1}^2$ of $J\backslash\sigma$, and any $f \in R(K) \subset \mathscr{U}$, we have

$$\begin{aligned} &\sum_{j\in\sigma_1}\|\Lambda_j f\|^2 + \sum_{j\in\sigma_2}\|\Gamma_j f\|^2 \\ &\quad = \Bigg(\sum_{j\in J\backslash\sigma_2}\|\Lambda_j f\|^2 + \sum_{j\in\sigma_2}\|\Gamma_j f\|^2\Bigg) - \sum_{j\in\sigma}\|\Lambda_j f\|^2 \\ &\quad \geqslant A\|K^*f\|^2 - \Bigg(\alpha\sum_{j\in\sigma}\|\Gamma_j f\|^2 + \beta\|K^*f\|^2 + \gamma\|f\|^2\Bigg) \\ &\quad \geqslant (A-\beta)\|K^*f\|^2 - (\alpha B + \gamma)\|f\|^2 \\ &\quad \geqslant (A-\beta)\|K^*f\|^2 - (\alpha B + \gamma)\|K^\dagger\|^2\|K^*f\|^2 \\ &\quad = [A - \beta - (\alpha B + \gamma)\|K^\dagger\|^2]\|K^*f\|^2, \end{aligned} \tag{8}$$

where the first inequality is deduced by (7) and that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-woven with the universal bounds A and B, and the third inequality is deduced by (6). On the other hand, we have

$$\sum_{j\in\sigma_1}\|\Lambda_j f\|^2+\sum_{j\in\sigma_2}\|\Gamma_j f\|^2 \leqslant \sum_{j\in\sigma_1\cup\sigma}\|\Lambda_j f\|^2+\sum_{j\in\sigma_2}\|\Gamma_j f\|^2 \leqslant B\|f\|^2.$$

We can also know that $\{\Lambda_j : j\in J\backslash\sigma\}$ and $\{\Gamma_j : j\in J\backslash\sigma\}$ are K-g-frames for $R(K)$ if we take $\sigma_1 = J\backslash\sigma$ and $\varnothing$ from (8). Hence $\{\Lambda_j : j\in J\backslash\sigma\}$ and $\{\Gamma_j : j\in J\backslash\sigma\}$ are K-woven in $R(K)$. □

We can easily obtain the following result if we take $K = I_{\mathscr{U}}$ in Theorem 1.

Corollary 1 *Suppose that $\{\Lambda_j : j\in J\}$ and $\{\Gamma_j : j\in J\}$ are g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j\in J\}$ and are woven in $\mathscr{U}$ with the universal frame bounds A and B. If there exist $\sigma\subset J$ and $\alpha\beta\geqslant 0$ satisfying $A>\alpha B+\beta$ such that*

$$\sum_{j\in\sigma}\|\Lambda_j f\|^2 \leqslant \alpha\sum_{j\in\sigma}\|\Gamma_j f\|^2+\beta\|f\|^2, \quad \forall f\in\mathscr{U}, \tag{9}$$

then $\{\Lambda_j : j\in J\backslash\sigma\}$ and $\{\Gamma_j : j\in J\backslash\sigma\}$ are woven in $\mathscr{U}$ with the universal frame bounds $A-\alpha B-\beta$ and B.

PERTURBATIONS OF THE WEAVING OF K-G-FRAMES

In this section, we mainly discuss the perturbation stabilities of the weaving of $\{\Lambda_j : j\in J\}$ and $\{\Gamma_j : j\in J\}$ under different types of perturbation conditions.

Given that $\{\Lambda_j : j\in J\}$ and $\{\Gamma_j : j\in J\}$ are K-g-frames for $\mathscr{U}$, we show that, under condition (10), $\{\Lambda_j T_1^* : j\in J\}$ and $\{\Gamma_j T_2^* : j\in J\}$ are K-woven in $R(K)$ for some surjective operators T_1, T_2 on $\mathscr{U}$.

Theorem 2 *Let $K\in L(\mathscr{U})$ be an operator with a closed range. Suppose that $\{\Lambda_j : j\in J\}$ and $\{\Gamma_j : j\in J\}$ are K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j\in J\}$ with the frame bounds A_1, B_1 and A_2, B_2, respectively. Suppose that $T_1, T_2\in L(\mathscr{U})$ are surjective on $\mathscr{U}$ and satisfy $T_iK = KT_i$, $i = 1, 2$. If there exist $\alpha,\beta,\gamma\geqslant 0$ satisfying $\frac{\sqrt{A_1}}{\|T_1^\dagger\|}-\frac{\gamma}{\|T_2^\dagger\|} > (\sqrt{B_1}\|T_2-T_1\|+\alpha\sqrt{B_1}\|T_2\|+\beta\sqrt{B_2}\|T_2\|)\|K^\dagger\|$ such that, $\forall f\in\mathscr{U}$,*

$$\Big(\sum_{j\in J}\|(\Gamma_j-\Lambda_j)f\|^2\Big)^{1/2} \leqslant \alpha\Big(\sum_{j\in J}\|\Lambda_j f\|^2\Big)^{1/2} +\beta\Big(\sum_{j\in J}\|\Gamma_j f\|^2\Big)^{1/2}+\gamma\|K^* f\|, \tag{10}$$

then $\{\Lambda_j T_1^* : j\in J\}$ and $\{\Gamma_j T_2^* : j\in J\}$ are K-woven in $R(K)$ with the universal frame bounds $\Big(\frac{\sqrt{A_1}}{\|T_1^\dagger\|}-\frac{\gamma}{\|T_2^\dagger\|}-\big(\sqrt{B_1}\|T_2-T_1\|+\alpha\sqrt{B_1}\|T_2\|+\beta\sqrt{B_2}\|T_2\|\big)\|K^\dagger\|\Big)^2$ and $B_1\|T_1\|^2+B_2\|T_2\|^2$.

Proof: Since $T_1\in L(\mathscr{U})$ is surjective on $\mathscr{U}$, similar to (6), we obtain, for any $f\in\mathscr{U}$,

$$\|f\| = \|(T_1^\dagger)^* T_1^* f\| \leqslant \|T_1^\dagger\|\|T_1^* f\|. \tag{11}$$

Hence we obtain

$$\|T_1^* f\| \geqslant \frac{1}{\|T_1^\dagger\|}\|f\|, \quad \forall f\in\mathscr{U}. \tag{12}$$

We can now show that $\{\Lambda_j T_1^* : j\in J\}$ and $\{\Gamma_j T_2^* : j\in J\}$ are K-g-frames for $\mathscr{U}$. Since $\{\Lambda_j : j\in J\}$ is a K-g-frame for $\mathscr{U}$, therefore (1) holds. For any $f\in\mathscr{U}$, we obtain

$$\begin{aligned}\frac{A_1}{\|T_1^\dagger\|^2}\|K^* f\|^2 &\leqslant A_1\|T_1^* K^* f\|^2 = A_1\|K^* T_1^* f\|^2\\ &\leqslant \sum_{j\in J}\|\Lambda_j T_1^* f\|^2\\ &\leqslant B_1\|T_1^* f\|^2 \leqslant B_1\|T_1\|^2\|f\|^2,\end{aligned} \tag{13}$$

where the first inequality is deduced by (12). Hence $\{\Lambda_j T_1^* : j\in J\}$ is a K-g-frame for $\mathscr{U}$. Similarly, we can show that $\{\Gamma_j T_2^* : j\in J\}$ is a K-g-frame for $\mathscr{U}$.

Next, we show that $\{\Lambda_j T_1^* : j\in J\}$ and $\{\Gamma_j T_2^* : j\in J\}$ are woven in $R(K)$. For any partition $\{\sigma_j\}_{j=1}^2$ of J, and any $f\in\mathscr{U}$, we have

$$\sum_{j\in\sigma_1}\|\Lambda_j T_1^* f\|^2+\sum_{j\in\sigma_2}\|\Gamma_j T_2^* f\|^2 \leqslant \big(B_1\|T_1\|^2+B_2\|T_2\|^2\big)\|f\|^2.$$

Let $x = \{\Lambda_j T_1^* f\}_{j\in\sigma_1}\cup\{\Lambda_j T_1^* f\}_{j\in\sigma_2}$ and $y = \{0\}_{j\in\sigma_1}\cup\{\Lambda_j(T_2-T_1)^* f+(\Gamma_j-\Lambda_j)T_2^* f\}_{j\in\sigma_2}$. Then $x, y\in l^2(\{\mathscr{V}_j\}_{j\in J})$ since $\{\Lambda_j T_1^* : j\in J\}$ and $\{\Lambda_j(T_2-T_1)^* : j\in J\}\bigcup\{(\Lambda_j-\Gamma_j)T_2^* : j\in J\}$ can be proved to be g-Bessel sequences in $\mathscr{U}$. Furthermore, there is

$$x+y = \{\Lambda_j T_1^* f\}_{j\in\sigma_1}\bigcup\{\Gamma_j T_2^* f\}_{j\in\sigma_2}\in l^2(\{\mathscr{V}_j\}_{j\in J}).$$

Hence for any $f \in R(K)$, we obtain

$$
\begin{aligned}
&\Big(\sum_{j\in\sigma_1}\|\Lambda_j T_1^* f\|^2+\sum_{j\in\sigma_2}\|\Gamma_j T_2^* f\|^2\Big)^{1/2}\\
&\quad=\|x+y\|\geqslant\|x\|-\|y\|\\
&\quad=\Big(\sum_{j\in J}\|\Lambda_j T_1^* f\|^2\Big)^{1/2}\\
&\qquad-\Big(\sum_{j\in\sigma_2}\|\Lambda_j(T_2-T_1)^* f+(\Gamma_j-\Lambda_j)T_2^* f\|^2\Big)^{1/2}\\
&\quad\geqslant\sqrt{A_1}\|K^*T_1^* f\|-\Big(\sum_{j\in\sigma_2}\|\Lambda_j(T_2-T_1)^* f\|^2\Big)^{1/2}\\
&\qquad-\Big(\sum_{j\in\sigma_2}\|(\Gamma_j-\Lambda_j)T_2^* f\|^2\Big)^{1/2}\\
&\quad\geqslant\sqrt{A_1}\|K^*T_1^* f\|-\Big(\sum_{j\in J}\|\Lambda_j(T_2-T_1)^* f\|^2\Big)^{1/2}\\
&\qquad-\Big(\sum_{j\in J}\|(\Gamma_j-\Lambda_j)T_2^* f\|^2\Big)^{1/2}\\
&\quad\geqslant\sqrt{A_1}\|T_1^*K^* f\|-\sqrt{B_1}\|(T_2-T_1)^* f\|-\gamma\|K^*T_2^* f\|\\
&\qquad-\alpha\Big(\sum_{j\in J}\|\Lambda_j T_2^* f\|^2\Big)^{1/2}-\beta\Big(\sum_{j\in J}\|\Gamma_j T_2^* f\|^2\Big)^{1/2}\\
&\quad\geqslant\sqrt{A_1}\|T_1^*K^* f\|-\sqrt{B_1}\|T_2-T_1\|\|f\|-\gamma\|T_2^*K^* f\|\\
&\qquad-\alpha\sqrt{B_1}\|T_2^* f\|-\beta\sqrt{B_2}\|T_2^* f\|\\
&\quad\geqslant\Big(\frac{\sqrt{A_1}}{\|T_1^\dagger\|}-\frac{\gamma}{\|T_2^\dagger\|}\Big)\|K^* f\|-\Big(\sqrt{B_1}\|T_2-T_1\|\\
&\qquad+\alpha\sqrt{B_1}\|T_2\|+\beta\sqrt{B_2}\|T_2\|\Big)\|f\|\\
&\quad\geqslant\Big(\frac{\sqrt{A_1}}{\|T_1^\dagger\|}-\frac{\gamma}{\|T_2^\dagger\|}\Big)\|K^* f\|-\Big(\sqrt{B_1}\|T_2-T_1\|\\
&\qquad+\alpha\sqrt{B_1}\|T_2\|+\beta\sqrt{B_2}\|T_2\|\Big)\|K^\dagger\|\|K^* f\|\\
&\quad=\Big(\frac{\sqrt{A_1}}{\|T_1^\dagger\|}-\frac{\gamma}{\|T_2^\dagger\|}-\Big(\sqrt{B_1}\|T_2-T_1\|\\
&\qquad+\alpha\sqrt{B_1}\|T_2\|+\beta\sqrt{B_2}\|T_2\|\Big)\|K^\dagger\|\Big)\|K^* f\|,
\end{aligned}
$$

where the second inequality is obtained by Minkowski's inequality, the sixth inequality by (12), and the seventh inequality by (6). Hence $\{\Lambda_j T_1^* : j \in J\}$ and $\{\Gamma_j T_2^* : j \in J\}$ are K-woven in $R(K)$. □

If $T_1 = T_2$ in Theorem 2, we can easily have the following corollary.

Corollary 2 *Let $K \in L(\mathscr{U})$ be an operator with a closed range. Suppose that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ with the frame bounds A_1, B_1 and A_2, B_2, respectively. Suppose that $T \in L(\mathscr{U})$ is surjective on $\mathscr{U}$ and satisfies $TK = KT$. If there exist $\alpha, \beta, \gamma \geqslant 0$ satisfying $\sqrt{A_1}-\gamma > (\alpha\sqrt{B_1}+\beta\sqrt{B_2})\|T\|\|T^\dagger\|\|K^\dagger\|$ such that, $\forall f \in \mathscr{U}$,*

$$
\Big(\sum_{j\in J}\|(\Gamma_j-\Lambda_j)f\|^2\Big)^{1/2}\leqslant\alpha\Big(\sum_{j\in J}\|\Lambda_j f\|^2\Big)^{1/2}+\beta\Big(\sum_{j\in J}\|\Gamma_j f\|^2\Big)^{1/2}+\gamma\|K^* f\|,
$$

then $\{\Lambda_j T^* : j \in J\}$ and $\{\Gamma_j T^* : j \in J\}$ are K-woven in $R(K)$ with the universal frame bounds $\big((\sqrt{A_1}-\gamma)/\|T^\dagger\|-\big(\alpha\sqrt{B_1}+\beta\sqrt{B_2}\big)\|T\|\|K^\dagger\|\big)^2$ and $(B_1+B_2)\|T\|^2$.

If $\alpha = \beta = \gamma = 0$ in Theorem 2, then from (10) we can deduce that $\Lambda_j = \Gamma_j$, $\forall j \in J$, and a result follows from Theorem 2.

Corollary 3 *Let $K \in L(\mathscr{U})$ be an operator with a closed range. Suppose that $\{\Lambda_j : j \in J\}$ is a K-g-frame for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ with the frame bounds A and B. Suppose that $T_1, T_2 \in L(\mathscr{U})$ are surjective on $\mathscr{U}$ and satisfy $T_iK = KT_i$, $i = 1, 2$. If $\sqrt{A}/\|T_1^\dagger\| > \sqrt{B}\|T_2 - T_1\|\|K^\dagger\|$, then $\{\Lambda_j T_1^* : j \in J\}$ and $\{\Lambda_j T_2^* : j \in J\}$ are K-woven in $R(K)$ with the frame bounds $\big(\sqrt{A}/\|T_1^\dagger\|-\sqrt{B}\|T_2-T_1\|\|K^\dagger\|\big)^2$ and $B(\|T_1\|^2+\|T_2\|^2)$.*

In Ref. 12, the authors reported that, in general, applying two different operators to woven frames can give frames that are not woven (see Example 2 in Ref. 12). Corollary 3 also provides us with a sufficient condition for applying different operators (T_1, T_2) to $\{\Lambda_j : j \in J\}$ such that $\{\Lambda_j T_1^* : j \in J\}$ and $\{\Lambda_j T_2^* : j \in J\}$ are woven in $\mathscr{U}$ (here, $K = I_{\mathscr{U}}$ and, clearly, $\{\Lambda_j : j \in J\}$ is woven with itself).

Note also that Corollary 3 is a generalization of Proposition 6.2 in Ref. 12. In fact, let $K = T_1 = I_{\mathscr{U}}$; then $\{\Lambda_j : j \in J\}$ is a g-frame for $\mathscr{U}$ and $\sqrt{A}/\|T_1^\dagger\| > \sqrt{B}\|T_2 - T_1\|\|K^\dagger\|$ in Corollary 3 can be rewritten as $\|I_{\mathscr{U}} - T_2\|^2 < A/B$. Then, from Corollary 3, we can obtain a g-frame version of Proposition 6.2 in Ref. 12.

Corollary 4 *Suppose that $\{\Lambda_j : j \in J\}$ is a g-frame for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$, with the frame bounds A and B. Suppose that T is surjective on $\mathscr{U}$. If $\|I_{\mathscr{U}} - T\|^2 < A/B$, then $\{\Lambda_j : j \in J\}$ and $\{\Lambda_j T^* : j \in J\}$ are woven in $\mathscr{U}$ with the frame bounds $\big(\sqrt{A}-\sqrt{B}\|T-I_{\mathscr{U}}\|\big)^2$ and $B(1+\|T\|^2)$.*

Furthermore, if we let $\Lambda_j f = \langle f, f_i\rangle$, $\mathscr{V}_j = \mathbb{C}$, $j \in J$. Then $\{\Lambda_j : j \in J\}$ is a g-frame for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ if and only if $\{f_j\}_{j\in J}$ is a frame for $\mathscr{U}$. Hence from Corollary 4, we can obtain Proposition 6.2 in Ref. 12.

The next theorem tells us that, under condition (14), $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-woven in $R(K)$, where $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are assumed to be K-g-frames for $\mathscr{U}$.

Theorem 3 *Let $K \in L(\mathscr{U})$ be an operator with a closed range. Suppose that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ with the frame bounds A_1, B_1 and A_2, B_2, respectively. If there exist $\alpha, \beta, \gamma \in [0, \infty)$ satisfying $A_1 > (\sqrt{B_1} + \sqrt{B_2})(\alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma)\|K^\dagger\|^2$ such that, for any $\{g_j\}_{j\in J} \in l^2(\{\mathscr{V}_j\}_{j\in J})$,*

$$\Big\|\sum_{j\in J}(\Lambda_j^* - \Gamma_j^*)g_j\Big\| \leqslant \alpha\Big\|\sum_{j\in J}\Lambda_j^* g_j\Big\| + \beta\Big\|\sum_{j\in J}\Gamma_j^* g_j\Big\| + \gamma\left\|\{g_j\}_{j\in J}\right\|, \quad (14)$$

then $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-woven in $R(K)$ with the universal frame bounds $A_1 - \big(\sqrt{B_1} + \sqrt{B_2}\big)\big(\alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma\big)\|K^\dagger\|^2$ and $B_1 + B_2$.

To prove Theorem 3, we need to give a lemma as follows.

Lemma 2 *Let $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ be K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$, with the Bessel bounds B_1 and B_2 and the synthesis operators T_1 and T_2, respectively. If there exist α, β, $\gamma > 0$ such that (14) holds, then, for any subset $\sigma \subset J$, we have*

$$\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| \leqslant \big(\sqrt{B_1} + \sqrt{B_2}\big)\big(\alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma\big)\|f\|. \quad (15)$$

Proof: For any $\{g_j\}_{j\in J} \in l^2(\{\mathscr{V}_j\}_{j\in J})$, from (14), we obtain

$$\begin{aligned}
\|(T_1 - T_2)(\{g_j\}_{j\in J})\| &= \Big\|\sum_{j\in J}(\Lambda_j^* - \Gamma_j^*)g_j\Big\| \\
&\leqslant \alpha\Big\|\sum_{j\in J}\Lambda_j^* g_j\Big\| + \beta\Big\|\sum_{j\in J}\Gamma_j^* g_j\Big\| + \gamma\|\{g_j\}_{j\in J}\| \\
&= \alpha\|T_1(\{g_j\}_{j\in J})\| + \beta\|T_2(\{g_j\}_{j\in J})\| + \gamma\|\{g_j\}_{j\in J}\| \\
&\leqslant \alpha\|T_1\|\|\{g_j\}_{j\in J}\| + \beta\|T_2\|\|\{g_j\}_{j\in J}\| + \gamma\|\{g_j\}_{j\in J}\| \\
&\leqslant \big(\alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma\big)\|\{g_j\}_{j\in J}\|.
\end{aligned}$$

It follows that

$$\|T_1 - T_2\| \leqslant \alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma \quad (16)$$

since $\{g_j\}_{j\in J} \in l^2(\{\mathscr{V}_j\}_{j\in J})$ is arbitrary.

Denote $\Phi_j = \Lambda_j - \Gamma_j$, $j \in J$, and the synthesis operator of $\{\Phi_j\}_{j\in J}$ by T_3. It is trivial to show that $\{\Phi_j\}_{j\in J}$ is a g-Bessel sequence and $T_3 = T_1 - T_2$. For any $\sigma \subset I$, $f \in \mathscr{H}$, we have

$$\begin{aligned}
&\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| \\
&= \Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma}\Gamma_j^*\Lambda_j f + \sum_{j\in\sigma}\Gamma_j^*\Lambda_j f - \sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| \\
&= \Big\|\sum_{j\in\sigma}(\Lambda_j - \Gamma_j)^*\Lambda_j f + \sum_{j\in\sigma}\Gamma_j^*(\Lambda_j - \Gamma_j)f\Big\| \\
&\leqslant \Big\|\sum_{j\in\sigma}\Phi_j^*\Lambda_j f\Big\| + \Big\|\sum_{j\in\sigma}\Gamma_j^*\Phi_j f\Big\| \\
&\leqslant \|T_3\|\|T_1\|\|f\| + \|T_2\|\|T_3\|\|f\| \\
&= (\|T_1\| + \|T_2\|)\|T_1 - T_2\|\|f\| \\
&\leqslant \big(\sqrt{B_1} + \sqrt{B_2}\big)\big(\alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma\big)\|f\|,
\end{aligned}$$

where the last inequality is obtained by (16). Hence (15) holds. □

Proof Theorem 3: For any $f \in R(K)$ and any partition $\{\sigma_j\}_{j=1}^2$ of J, we have

$$\begin{aligned}
&\sum_{j\in\sigma_1}\|\Lambda_j f\|^2 + \sum_{j\in\sigma_2}\|\Gamma_j f\|^2 \\
&= \sum_{j\in J}\|\Lambda_j f\|^2 + \sum_{j\in\sigma_2}\|\Gamma_j f\|^2 - \sum_{j\in\sigma_2}\|\Lambda_j f\|^2 \\
&\geqslant A_1\|K^* f\|^2 - \Big\langle \sum_{j\in\sigma_2}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma_2}\Gamma_j^*\Gamma_j f,\ f\Big\rangle \\
&\geqslant A_1\|K^* f\|^2 - \Big\|\sum_{j\in\sigma_2}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma_2}\Gamma_j^*\Gamma_j f\Big\|\|f\| \\
&\geqslant A_1\|K^* f\|^2 - \big(\sqrt{B_1} + \sqrt{B_2}\big)\big(\alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma\big)\|f\|^2 \\
&\geqslant \Big[A_1 - \big(\sqrt{B_1} + \sqrt{B_2}\big)\big(\alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma\big)\|K^\dagger\|^2\Big]\|K^* f\|^2,
\end{aligned}$$

where the third and fourth inequalities are, respectively, deduced by (15) and (6). The upper bound for every weaving is trivial. Hence $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-woven in $R(K)$. □

In case $K = I_{\mathscr{U}}$ in Theorem 3, we can easily obtain a result as follows.

Corollary 5 *Suppose that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$, with the frame bounds A_1, B_1 and A_2, B_2, respectively. If there exist $\alpha, \beta, \gamma \in [0, \infty)$ satisfying $A_1 > (\sqrt{B_1} + \sqrt{B_2})(\alpha\sqrt{B_1} + \beta\sqrt{B_2} + \gamma)$ such that, for any $\{g_j\}_{j\in J} \in$*

$l^2(\{\mathscr{V}_j\}_{j\in J})$,

$$\Big\|\sum_{j\in J}(\Lambda_j^* - \Gamma_j^*)g_j\Big\| \leqslant \alpha\Big\|\sum_{j\in J}\Lambda_j^* g_j\Big\| + \beta\Big\|\sum_{j\in J}\Gamma_j^* g_j\Big\| + \gamma\|\{g_j\}_{j\in J}\|,$$

then $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are woven in $\mathscr{U}$ with the universal frame bounds $A_1 - \big(\sqrt{B_1}+\sqrt{B_2}\big)\big(\alpha\sqrt{B_1}+\beta\sqrt{B_2}+\gamma\big)$ and B_1+B_2.

Furthermore, if $\alpha = \beta = 0$ in Corollary 5, we can obtain a g-frame version of Theorem 6.1 in Ref. 12.

Next we provide the third type of perturbation condition (17) such that, under it, $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ can be K-woven in $R(K)$.

Theorem 4 *Let $K \in L(\mathscr{U})$ be an operator with a closed range. Suppose that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ with the frame bounds A_1, B_1 and A_2, B_2, respectively. If there exist $\alpha, \beta, \gamma \in [0, \infty)$ satisfying $A_1 > (\alpha B_1 + \beta B_2 + \gamma)\|K^\dagger\|^2$ such that, for any $\{g_j\}_{j\in J} \in l^2(\{\mathscr{V}_j\}_{j\in J})$,*

$$\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| \leqslant \alpha\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f\Big\| + \beta\Big\|\sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| + \gamma\|f\|, \quad (17)$$

then $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-woven in $R(K)$ with the universal frame bounds $A_1 - (\alpha B_1 + \beta B_2 + \gamma)\|K^\dagger\|^2$ and $B_1 + B_2$.

Proof: Since $\{\Lambda_j : j \in J\}$ is a K-g-frame for $\mathscr{U}$ with the frame bounds A_1 and B_1, for any $\sigma \subset J$, we have

$$\begin{aligned}\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f\Big\| &= \sup_{g\in\mathscr{U},\|g\|=1}\Big|\Big\langle\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f,\ g\Big\rangle\Big| \\ &= \sup_{g\in\mathscr{U},\|g\|=1}\Big|\sum_{j\in\sigma}\langle\Lambda_j f,\ \Lambda_j g\rangle\Big| \\ &\leqslant \sup_{g\in\mathscr{U},\|g\|=1}\sum_{j\in\sigma}|\langle\Lambda_j f, \Lambda_j g\rangle| \\ &\leqslant \sup_{g\in\mathscr{U},\|g\|=1}\Big(\sum_{j\in J}\|\Lambda_j f\|^2\Big)^{1/2}\Big(\sum_{j\in J}\|\Lambda_j g\|^2\Big)^{1/2} \\ &\leqslant \sup_{g\in\mathscr{U},\|g\|=1}\sqrt{B_1}\|f\|\sqrt{B_1}\|g\| = B_1\|f\|. \quad (18)\end{aligned}$$

Similarly, we can obtain $\|\sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\| \leqslant B_2\|f\|$. Hence if we combine (17) and (18), it follows that

$$\begin{aligned}&\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| \\ &\quad\leqslant \alpha\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f\Big\| + \beta\Big\|\sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| + \gamma\|f\| \\ &\quad\leqslant (\alpha B_1 + \beta B_2 + \gamma)\|f\|. \quad (19)\end{aligned}$$

For any $f \in R(K)$, and any partition $\{\sigma_j\}_{j=1}^2$ of J, by the same method of Theorem 3, we obtain

$$\begin{aligned}&\sum_{j\in\sigma_1}\|\Lambda_j f\|^2 + \sum_{j\in\sigma_2}\|\Gamma_j f\|^2 \\ &\quad\geqslant A_1\|K^* f\|^2 - \Big\|\sum_{j\in\sigma_2}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma_2}\Gamma_j^*\Gamma_j f\Big\|\|f\| \\ &\quad\geqslant A_1\|K^* f\|^2 - (\alpha B_1 + \beta B_2 + \gamma)\|f\|^2 \\ &\quad\geqslant A_1\|K^* f\|^2 - (\alpha B_1 + \beta B_2 + \gamma)\|K^\dagger\|^2\|K^* f\|^2 \\ &\quad= \Big(A_1 - (\alpha B_1 + \beta B_2 + \gamma)\|K^\dagger\|^2\Big)\|K^* f\|^2,\end{aligned}$$

where the second inequality is deduced by (19). Hence $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are K-woven in $R(K)$. □

If $K = I_{\mathscr{U}}$, a result follows immediately from Theorem 4.

Corollary 6 *Suppose that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_j : j \in J\}$ with the frame bounds A_1, B_1 and A_2, B_2, respectively. If there exist $\alpha, \beta, \gamma \in [0, \infty)$ satisfying $A_1 > \alpha B_1 + \beta B_2 + \gamma$ such that, for any $\{g_j\}_{j\in J} \in l^2(\{\mathscr{V}_j\}_{j\in J})$,*

$$\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f - \sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| \leqslant \alpha\Big\|\sum_{j\in\sigma}\Lambda_j^*\Lambda_j f\Big\| + \beta\Big\|\sum_{j\in\sigma}\Gamma_j^*\Gamma_j f\Big\| + \gamma\|f\|,$$

then $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are woven in $\mathscr{U}$ with the universal frame bounds $A_1 - (\alpha B_1 + \beta B_2 + \gamma)$ and $B_1 + B_2$.

Acknowledgements: This work is partly supported by the Natural Science Foundation of Fujian Province, China (grant No. 2016J01014), and by the projects of Xiamen University of Technology (grant No. G2017005).

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