

Multiple solutions for k -coupled Schrödinger system with variable coefficients

Xin Wang, Lijie Yin, Xiaorui Yue*

College of Information Science and Technology, Hainan University, Haikou, Hainan 570228, China

*Corresponding author, e-mail: yxr@hainu.edu.cn

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ABSTRACT: Consider the k -coupled Schrödinger system with variable coefficients as below which arises in nonlinear optics and other physical problems:

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j(x)u_j^3 + \sum_{i,i \neq j} \beta_{ij}(x)u_i^2 u_j, & x \in \Omega, \\ u_j = 0 \text{ on } \partial\Omega, & j = 1, \dots, k, \end{cases}$$

where Ω is a bounded smooth domain in \mathbb{R}^N , $N \leq 3$, $k \geq 2$; $\lambda_j > -\lambda_1(\Omega)$ for $j = 1, \dots, k$ and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition; $\mu_j(x)$ and $\beta_{ij}(x) = \beta_{ji}(x)$ are positive bounded functions for $i, j = 1, \dots, k$, $i \neq j$. We obtain multiple solutions with some components sign-changing while the others positive, and one positive solution for the above problem.

KEYWORDS: k -coupled Schrödinger system, variable coefficients, multiple mixed states of nodal solutions, the positive solution

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INTRODUCTION

In recent years, there has been extensive mathematical work for the following coupled elliptic system:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & x \in \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain or $\Omega = \mathbb{R}^N$. System (1) arises when we consider the standing wave solutions to the following time-dependent Schrödinger system, which consists of two coupled Gross-Pitaevskii equations:

$$\begin{cases} -i \frac{\partial}{\partial t} \Phi_1 = \Delta \Phi_1 + \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1, \\ x \in \Omega, t > 0, \\ -i \frac{\partial}{\partial t} \Phi_2 = \Delta \Phi_2 + \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2, \\ x \in \Omega, t > 0, \\ \Phi_j = \Phi_j(x, t) \in \mathbb{C}, j = 1, 2, \\ \Phi_j(x, t) = 0, x \in \partial\Omega, t > 0, j = 1, 2, \end{cases} \quad (2)$$

where i is the imaginary unit. It has applications in many physics and nonlinear optics, see Ref. 1. Phys-

ically, the solution Φ_j denotes the j th component of the beam in Kerr-like photorefractive media, see Ref. 2. μ_j is for self-focusing in the j th component, and the coupling constant β is the interaction between the two components of the beam. (2) is also called the Bose-Einstein condensates system since it arises in the Hartree-Fock theory for a double condensate, see Ref. 3 and references therein. To obtain solitary wave solutions of system (2), we set $\Phi_j(x, t) = e^{i\lambda_j t} u_j(x)$ for $j = 1, 2$, then it will be reduced to system (1). The existence of the least energy and other finite energy solutions was studied in Refs. 4–6 and references therein. The existence and the multiplicity of positive and sign-changing solutions were studied in Refs. 7–9 and references therein.

Later, the general k -coupled case attracts more and more interest because of its many more possible properties of solutions:

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j u_j^3 + \sum_{i,i \neq j} \beta_{ij} u_i^2 u_j, & x \in \Omega, \\ u_j = 0 \text{ on } \partial\Omega, & j = 1, \dots, k. \end{cases} \quad (3)$$

Clearly, system (3) is reduced to system (1) when $k = 2$. Recently, Ref. 10 provided the existence of

infinitely many sign-changing solutions to system (1) for each fixed $\beta < 0$. Independently, when $\beta < 0$, similar results for the general k -coupled system were obtained in Refs. 11, 12 by a method different from that of Ref. 10. In Ref. 11, the authors obtained infinitely many mixed states of nodal solutions for system (3) when $\beta_{ij} = \beta_{ji} < 0$, $1 \leq i < j \leq k$. In Ref. 12, the authors obtained a solution $\mathbf{u} = (u_1, \dots, u_k)$ such that some components are sign-changing functions that change sign exactly once in Ω and the others are one-sign functions. Both proofs in Refs. 11, 12 depend heavily on the negative sign of the coupling constants β_{ij} while there are not so many results when $\beta > 0$. Multiple sign-changing solutions to system (1) for $\beta > 0$ being small were obtained in Ref. 13. Later on, more general result was obtained in Ref. 14 for the k -coupled system (3), i.e., they obtained mixed states of nodal solutions with some components positive and the others sign-changing through a method completely different from that in Ref. 13. In fact, the authors of Ref. 14 developed a perturbation method for functionals related to nonlinear elliptic systems to produce a prescribed number of mixed states of nodal solutions and then applied the abstract theorem (Theorem 2.2 in Ref. 14) to system (3).

In this paper, we consider the existence of multiple sign-changing and positive solutions for the following k -coupled equations with variable coefficients:

$$\left. \begin{aligned} -\Delta u_j + \lambda_j u_j &= \mu_j(x) u_j^3 + \sum_{i, i \neq j} \beta_{ij}(x) u_i^2 u_j, \quad x \in \Omega, \\ u_j &= 0 \text{ on } \partial\Omega, \quad j = 1, \dots, k, \end{aligned} \right\} (\mathcal{P})$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \leq 3$; $\mu_j(x) > 0$ and $\beta_{ij}(x) = \beta_{ji}(x) > 0$ for $i, j = 1, \dots, k, i \neq j$ are functions of x . As we know, system (\mathcal{P}) will be reduced to (3) when the coupling coefficients are constants. Thus we will discuss a more general problem which may suit more precisely to physical models in the real world. Firstly, we will show the existence of infinitely many solutions for the problem when all of the coupling coefficients $\beta_{ij} \equiv 0$. Then we use the perturbation idea as in Ref. 14 to show that similar results of multiple sign-changing and positive solutions still hold for the more general case with variable coefficients and $\lambda_1(\Omega)$ being the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition. Our main results can be summarized as follows.

Theorem 1 Assume that $\lambda_j > -\lambda_1(\Omega)$, $\mu_j(x)$ and $\beta_{ij}(x)$, $1 \leq i < j \leq k$, are positive bounded functions. Then for any $m \in \mathbb{N}$, there is $\beta_m > 0$ such that whenever $\beta_{\max} := \max_{x \in \Omega, 1 \leq i < j \leq k} \beta_{ij}(x) < \beta_m$, the system (\mathcal{P}) has at least m distinct solutions with each component changing sign.

Theorem 2 Under the same conditions of Theorem 1. Given $0 < l < k$. Then for any $m \in \mathbb{N}$, there is $\hat{\beta}_m > 0$ such that whenever $\beta_{\max} < \hat{\beta}_m$, system (\mathcal{P}) has at least m distinct solutions with l components sign-changing and the others positive.

Theorem 3 Under the same conditions of Theorem 1, the system (\mathcal{P}) has at least one positive solution for sufficiently small β_{\max} .

Remark 1 Since Ω is a bounded domain, we can just take the variable coefficients to be positive continuous functions which clearly satisfy the assumptions in our theorems.

PRELIMINARIES

In this section, we construct some useful preliminaries for the proof of our main results. Firstly, we will introduce some notation and the appropriate working space. We assume that all integrations below are taken over Ω if not stated otherwise. Let \mathbf{H} be the k -time product space $H_0^1(\Omega) \times \dots \times H_0^1(\Omega)$. For $\mathbf{u} \in \mathbf{H}$, we write $\mathbf{u} = (u_1, \dots, u_k)$. It is easy to see that we can define the equivalent norm in $H_0^1(\Omega)$, since $\lambda_j > -\lambda_1(\Omega)$ and the Poincaré inequality, by

$$\|u\|_j = \left(\int |\nabla u|^2 + \lambda_j u^2 \right)^{1/2}, \quad j = 1, \dots, k, \quad (4)$$

and then the product space \mathbf{H} is endowed with the norm:

$$\|\mathbf{u}\| = \left(\sum_{j=1}^k \|u_j\|_j^2 \right)^{1/2}.$$

We also write the L^p norm as $|u|_p := (\int |u|^p)^{1/p}$ for convenience. It is well known that solutions for system (\mathcal{P}) are critical points of the functional

$$I(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}\|^2 - \frac{1}{4} \int \sum_j \mu_j(x) u_j^4 - \frac{1}{4} \int \sum_{i, j, i \neq j} \beta_{ij}(x) u_i^2 u_j^2.$$

Our main idea is to regard the coupled terms as perturbations for the system, when $\beta_{ij}(x) \equiv 0$ for $i, j = 1, \dots, k, i \neq j$, similar to that in Ref. 14, i.e., consider the system (\mathcal{P}_0) where equations have no connections with each other:

$$\left. \begin{aligned} -\Delta u_j + \lambda_j u_j &= \mu_j(x) u_j^3, \quad x \in \Omega, \\ u_j &= 0 \text{ on } \partial\Omega, \quad j = 1, \dots, k. \end{aligned} \right\} (\mathcal{P}_0)$$

The corresponding energy functional is

$$I_0(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}\|^2 - \frac{1}{4} \int \sum_j \mu_j(x) u_j^4.$$

We need the following existence results about sign-changing and positive solutions for (\mathcal{P}_0) .

Theorem 4 For $j = 1, \dots, k$, assume that $\lambda_j > -\lambda_1(\Omega)$, $\mu_j(x) > 0$ is a bounded function. Then each equation in (\mathcal{P}_0) has infinitely many sign-changing solutions.

Proof: This result can be obtained by Zou’s theorem (see Theorem 5.7 in Ref. 15) with a little modification, so we will just give a brief prove here. Given $j \in \{1, \dots, k\}$, let $f_j(x, u_j) = \mu_j(x) u_j^3$. Since $\mu_j(x) > 0$ is bounded, there is $C > 0$ such that $0 < \mu_j(x) \leq C$ for all $x \in \Omega$. Then it is easy to check that the following three hypotheses hold:

- (i) $f_j : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function with subcritical growth: $|f_j(x, u_j)| \leq C|u_j|^3$ and $f_j(x, u_j)u_j = \mu_j(x)u_j^4 \geq 0$ for all $u_j \in \mathbb{R}$, $x \in \bar{\Omega}$, and $f_j(x, u_j) = o(|u_j|)$ as $|u_j| \rightarrow 0$ uniformly for $x \in \bar{\Omega}$;

- (ii) for all $x \in \Omega$, $F_j(x, u_j) := \int_0^{u_j} f_j(x, v) dv$,

$$0 < F_j(x, u_j) = \frac{1}{4} \mu_j(x) u_j^4 \leq \frac{1}{3} f_j(x, u_j) u_j;$$

- (iii) $f_j(x, -u_j) = -f_j(x, u_j)$.

Since $\lambda_j > -\lambda_1(\Omega)$ and then we use the equivalent norm $\|u_j\|_j$ as defined in (4) instead of that in Ref. 15 where is the case $\lambda_j = 0$. Thus the proof in Ref. 15 can still stand for problem (\mathcal{P}_0) and infinitely many sign-changing solutions can be obtained. \square

Similarly, we can prove the following result as shown in Ref. 16 with a little modification like that in the proof of Theorem 4 above, so the details will be omitted.

Theorem 5 Under the same conditions in Theorem 4, each equation in (\mathcal{P}_0) has at least one positive solution.

To describe the sign of each component, we make use of the closed convex positive cone $P := \{u \in H_0^1(\Omega) : u \geq 0, \text{ a.e.}\}$ and denote d as the metric in $H_0^1(\Omega)$. For $j = 1, \dots, k$, set $P_j := \{u \in \mathbf{H} : u_j \in P\}$. For $\delta > 0$ we define open cones in \mathbf{H} by

$$\pm P_j(\delta) := \{u \in \mathbf{H} : d(u_j, \pm P) < \delta\}. \quad (5)$$

Then for $0 \leq l \leq k$, we construct the complement of the unions in \mathbf{H} :

$$S_l := \mathbf{H} \setminus \left\{ \left[\bigcup_{j=1}^l (P_j(\delta) \cup (-P_j(\delta))) \right] \cup \left[\bigcup_{j=l+1}^k (P_j(\delta) \cap (-P_j(\delta))) \right] \right\}. \quad (6)$$

For $l = k$ or $l = 0$, we mean that S_l only consists of the first part or the second part of the union in (6), respectively. Let the functional $J \in C^1(\mathbf{H}, \mathbb{R})$ and the gradient $J'(x) = x - K_J(x)$. We need an important property for the operator K_J which means that the above open neighbourhoods defined in (5) are invariant under the operator K_J , that is:

$P(J)$ For any sufficiently small $\delta_0 > 0$, it holds that $K_J(\pm P_j(\delta_0)) \subset \pm P_j(\delta)$ for some $\delta \in (0, \delta_0)$ and $j = 1, \dots, k$.

It is easy to see that if J satisfies condition $(P(J))$ then any critical point $\mathbf{u} \in S_l$ implies its first l components being sign-changing and the others being nontrivial.

Taking consideration of Theorems 2.1 and 2.2 in Ref. 14, we can obtain the result about the connection between one problem and the corresponding perturbed problem as below.

Theorem 6 Assume that the functional J has a critical point $\mathbf{u} \in S_l$ for $0 \leq l \leq k$ with $J(\mathbf{u}) = c \in \mathbb{R}$. Then for every $\varepsilon > 0$, there is $\rho > 0$ such that for every C^1 functional $J_\rho : \mathbf{H} \rightarrow \mathbb{R}$ which satisfies $\sup_{u \in \mathbf{H}} |J(u) - J_\rho(u)| < \rho$ and $(P(J_\rho))$, we have that J_ρ has a critical value in $[c - \varepsilon, c + \varepsilon]$ with at least one critical point also in S_l .

By Theorem 4 and Theorem 5, the functional I_0 has infinitely many critical points in S_l for $0 < l \leq k$ with critical values

$$0 < c_{01} < c_{02} < \dots < c_{0n} < \dots \rightarrow \infty. \quad (7)$$

And at least one positive critical point in S_0 . To connect the problem (\mathcal{P}) with (\mathcal{P}_0) , we need two auxiliary problems which satisfy perturbation assumptions in Theorem 6 and lead back to system (\mathcal{P}) . Consider

$$\left. \begin{aligned} -\Delta u_j + \lambda_j u_j &= \mu_j(x) u_j^3 + \sum_{i, i \neq j} \rho_{ij}(x) g_{ij}(u_i, u_j) u_j, \quad x \in \Omega, \\ u_j &= 0 \text{ on } \partial\Omega, \quad j = 1, \dots, k, \end{aligned} \right\} (\mathcal{P}_1)$$

where $\rho_{ij}(x) > 0$ is bounded, $g_{ij} \in C(\mathbb{R} \times \mathbb{R}, [0, 1])$, for $i, j = 1, \dots, k$. Then we have a uniformly estimate by elliptic theory referring to Ref. 17.

Lemma 1 For a given number $m > 0$, assume $\mu_j(x) > 0$ is bounded for $j = 1, \dots, k$. Then for any \mathbf{u} in the system (\mathcal{P}_1) with $\|\mathbf{u}\| \leq 2(c_{0(2m+1)} + 2)^{1/2}$, there is a constant $M^* > 0$ depending only on m and $|\Omega|$ while independent from ρ_{ij} and g_{ij} such that

$$|\mathbf{u}| \leq M^* \text{ uniformly on } x \in \Omega.$$

Proof: For $i, j = 1, \dots, k$ and a given number $n > 2$, since $\mu_j(x), \rho_{ij}(x), g_{ij} > 0$ are bounded, it holds that

$$\lim_{|u_j| \rightarrow \infty} \frac{\mu_j(x)u_j^3 + \sum_{i,i \neq j} \rho_{ij}(x)g_{ij}(u_i, u_j)u_j - \lambda_j u_j}{|u_j|^{2^*-1}} = 0. \quad (8)$$

Here 2^* is the Sobolev critical number defined by $2^* = 2N/(N - 2)$ where N is the dimension. Multiply the equation of the component u_j in (\mathcal{P}_1) by $|u_j|^{2n}u_j$, respectively, and integrate over Ω by parts, considering (8), we have that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} & \frac{2n+1}{(n+1)^2} \int |\nabla(|u_j|^{n+1})|^2 \\ &= \int [\mu_j(x)u_j^3 + \sum_{i,i \neq j} \rho_{ij}(x)g_{ij}(u_i, u_j)u_j - \lambda_j u_j] |u_j|^{2n}u_j \\ &\leq \int (\varepsilon |u_j|^{2^*+2n} + C_\varepsilon) \\ &\leq \varepsilon \left[\int |u_j|^{2^*} \right]^{2/N} \left[\int |u_j|^{2N(n+1)/(N-2)} \right]^{(N-2)/N} + C_\varepsilon |\Omega| \\ &\leq \varepsilon C(|\Omega|) \|u_j\|_j^{4/(N-2)} \left[\int |\nabla(|u_j|^{n+1})|^2 \right]^{(N-2)/N} + C_\varepsilon |\Omega|, \end{aligned}$$

where $C(|\Omega|)$ is a positive constant by Sobolev imbedding theorem, and it implies that $|u_j|^{n+1}$ is bounded in $H_0^1(\Omega)$ because of $\|\mathbf{u}\| \leq 2(c_{0(2m+1)} + 2)^{1/2}$ and the arbitrariness of ε . Then $|u_j|^{n+1}$ is also bounded in $L^{2^*}(\Omega)$. Returning to (8), we again obtain that $(\mu_j(x)u_j^3 + \sum_{i,i \neq j} \rho_{ij}(x)g_{ij}(u_i, u_j)u_j - \lambda_j u_j)$ is bounded in $L^{2N(n+1)/(N+2)}(\Omega)$. Thus by the standard elliptic theory¹⁷, we have that u_j is bounded in $W^{2,2N(n+1)/(N+2)}(\Omega)$. Choose n so large that $W^{2,2N(n+1)/(N+2)}(\Omega)$ is imbedded into $C(\bar{\Omega})$ continuously, and then we obtain that $|\mathbf{u}|$ is bounded uniformly on x . \square

We define a cutoff function $\varphi \in C^\infty(\mathbb{R}, [0, 1])$,

$$\varphi(s) = \begin{cases} 1, & |s| \leq M^* + 1, \\ 0, & |s| \geq M^* + 2. \end{cases}$$

Then we consider another auxiliary system,

$$\left. \begin{aligned} -\Delta u_j + \lambda_j u_j &= \mu_j(x)u_j^3 + 2 \sum_{i,i \neq j} \beta_{ij}(x) \times \\ & \left[\int_0^{u_i} \varphi(s) s \, ds \right] \varphi(u_j) u_j, \quad x \in \Omega, \\ u_j &= 0 \text{ on } \partial\Omega, \quad j = 1, \dots, k. \end{aligned} \right\} (\mathcal{P}_2)$$

Its energy functional is

$$\begin{aligned} I_\varphi(\mathbf{u}) &= \frac{1}{2} \|\mathbf{u}\|^2 - \frac{1}{4} \int \sum_j \mu_j(x) u_j^4 \\ & \quad - \int \sum_{i,j,i \neq j} \beta_{ij}(x) \psi(u_i, u_j), \end{aligned} \quad (9)$$

where

$$\psi(u_i, u_j) = \int_0^{u_i} \varphi(s) s \, ds \int_0^{u_j} \varphi(s) s \, ds. \quad (10)$$

Set $\beta_{\max} := \max_{x \in \Omega, 1 \leq i < j \leq k} \beta_{ij}(x)$, we can verify that I_φ satisfies the assumption in Theorem 6.

Lemma 2 Under the same assumptions in Theorem 4, let $\beta_{ij}(x) > 0$ and β_{\max} be sufficiently small. Then the functional I_φ satisfies $(P(I_\varphi))$.

Proof: By direct calculation, for $\mathbf{u}, \mathbf{v} \in \mathbf{H}$, set the operator $K_\varphi(\mathbf{u}) := I'_\varphi(\mathbf{u}) - \mathbf{u}$, we have

$$\begin{aligned} \langle K_\varphi(\mathbf{u}), \mathbf{v} \rangle &= \int \left[\sum_j \mu_j(x) u_j^3 v_j \right. \\ & \quad \left. + 2 \sum_{i,j,i \neq j} \beta_{ij}(x) \left(\int_0^{u_i} \varphi(s) s \, ds \right) \varphi(u_j) u_j v_j \right]. \end{aligned}$$

For $j = 1, \dots, k, 2 \leq p \leq 2^*$, by Sobolev inequalities, there exists a constant $C_p > 0$ depending only on $|\Omega|$ and p such that

$$\begin{aligned} |u_j^\pm|_p &= \min_{w \in \mp P} |u_j - w|_p \\ &\leq C_p \min_{w \in \mp P} \|u_j - w\|_j = C_p d(u_j, \mp P) \\ &\leq C_p \|u_j^\pm\|_j, \end{aligned} \quad (11)$$

where $u_j^\pm := \max\{\pm u_j, 0\}$. By the definition of the function φ , we know that $\int_0^{u_i} \varphi(s) s \, ds$ is bounded independently from $\beta_{ij}(x)$, so we can choose β_{\max} sufficiently small such that

$$\left| 2\beta_{ij}(x) \int_0^{u_i} \varphi(s) s \, ds \right| \leq \beta_{\max}^{1/2}, \quad \forall x \in \Omega. \quad (12)$$

Then set $\mathbf{w} = K_\varphi(\mathbf{u})$, by Hölder and Sobolev inequalities, we have

$$\begin{aligned} \|w_j^\pm\|_j^2 &= \langle w_j, w_j^\pm \rangle_j \\ &= \int \mu_j(x) u_j^3 w_j^\pm + 2 \int \sum_{i,j,i \neq j} \beta_{ij}(x) \left[\int_0^{u_i} \varphi(s) s \, ds \right] \varphi(u_j) u_j w_j^\pm \\ &\leq \int \mu_j(x) (u_j^\pm)^3 w_j^\pm + 2 \int \sum_{i,j,i \neq j} \beta_{ij}(x) \left[\int_0^{u_i} \varphi(s) s \, ds \right] u_j^\pm w_j^\pm \\ &\leq C |u_j^\pm|_4^3 \|w_j^\pm\|_4 + \beta_{\max}^{1/2} |u_j^\pm|_2 \|w_j^\pm\|_2 \\ &\leq C C_4 |u_j^\pm|_4^3 \|w_j^\pm\|_j + C_2 \beta_{\max}^{1/2} |u_j^\pm|_2 \|w_j^\pm\|_j. \end{aligned}$$

Hence, by (11), for $\mathbf{u} \in \pm P_j(\delta)$, and β_{\max} sufficiently small, we obtain

$$d(w_j, \mp P) \leq \|w_j^\pm\|_j < C C_4 \delta^3 + C_2 \beta_{\max}^{1/2} \delta \leq \frac{\delta}{2}$$

for sufficiently small $\delta > 0$. That is, for any small enough $\delta > 0$, it holds that $K_\varphi(\pm P_j(\delta)) \subset \pm P_j(\frac{1}{2}\delta)$, which completes the proof. \square

The next estimate for the norm of critical points of I_φ ensures that the auxiliary problem (\mathcal{P}_2) is able to go back to the original problem (\mathcal{P}) which we are concerned with.

Lemma 3 Assume the same conditions in Lemma 2, then for any critical points \mathbf{u} of I_φ with $I_\varphi(\mathbf{u}) \leq C$ for some constant $C > 0$, there exists a constant $C^* > 0$ depending only on C such that $\|\mathbf{u}\| \leq C^*$.

Proof: The proof is standard when β_{\max} is small. Since $I'_\varphi(\mathbf{u})\mathbf{u} = 0$, by direct calculation, we have

$$\begin{aligned} I_\varphi(\mathbf{u}) &= I_\varphi(\mathbf{u}) - \frac{1}{4} I'_\varphi(\mathbf{u})\mathbf{u} \\ &= \frac{1}{4} \|\mathbf{u}\|^2 - \int \sum_{i,j,i \neq j} \beta_{ij}(x) \psi(u_i, u_j) \\ &\quad + \frac{1}{2} \int \sum_{i,j,i \neq j} \beta_{ij}(x) \left(\int_0^{u_i} \varphi(s) s \, ds \right) \varphi(u_j) u_j^2 \\ &:= \frac{1}{4} \|\mathbf{u}\|^2 + I_{\varphi\beta 1} + I_{\varphi\beta 2}. \end{aligned}$$

By the definitions of ψ and φ , $\psi(u_i, u_j)$ and $\varphi(u_j) u_j^2$ are also bounded independently from $\beta_{ij}(x)$. Thus we can choose β_{\max} small enough such that

$$|I_{\varphi\beta 1}| \leq 1, \quad |I_{\varphi\beta 2}| \leq 1. \tag{13}$$

It follows that $\|\mathbf{u}\| \leq 2(C + 2)^{1/2}$. \square

PROOF OF Theorem 1

In this section, we apply Theorem 6 in the set S_k to I_φ and I_0 to obtain multiple solutions with each component changing sign for the perturbed problem, then we show that these critical points of I_φ are indeed those of the original problem (\mathcal{P}) . Let $m \in \mathbb{N}$ be a given number, and we consider the case of $l = k$.

Proof: We show that I_φ has m sign-changing critical points. By the definition of ψ in (10), for $\rho > 0$, we can choose β_{\max} small enough such that

$$|I_\varphi(\mathbf{u}) - I_0(\mathbf{u})| = \left| \int \sum_{i,j,i \neq j} \beta_{ij}(x) \psi(u_i, u_j) \right| \leq \rho$$

for all $\mathbf{u} \in \mathbf{H}$. Thus by Lemma 2 and applying Theorem 6 for $l = k$, we obtain that for ρ small enough, I_φ has m critical values, say $c_{\varphi 1} < c_{\varphi 2} < \dots < c_{\varphi m}$. In particular, it can be supposed that $c_{\varphi m} < c_{0(2m+1)}$. Each critical value corresponds to at least one critical point, and we denote them by $\mathbf{u}_{\varphi 1}, \dots, \mathbf{u}_{\varphi m}$.

Now we prove that for $n = 1, \dots, m$, $\mathbf{u}_{\varphi n}$ is a critical point of I . Since $I_\varphi(\mathbf{u}_{\varphi n}) < c_{0(2m+1)}$, we have $\|\mathbf{u}_{\varphi n}\| \leq 2(c_{0(2m+1)} + 2)^{1/2}$ by Lemma 3. For $i, j = 1, \dots, k, i \neq j$, take

$$\begin{aligned} g_{ij}(u_i, u_j) &= \beta_{\max}^{1/2} \int_0^{u_i} \varphi(s) s \, ds \varphi(u_j), \\ \rho_{ij}(x) &= \frac{\beta_{ij}(x)}{\beta_{\max}^{1/2}}, \end{aligned}$$

then by Lemma 1, we obtain $\|\mathbf{u}_{\varphi n}\| \leq M^*$, thus by the definition of φ , the system (\mathcal{P}_2) returns to (\mathcal{P}) . Consequently, we obtain m solutions of the system (\mathcal{P}) with each component changing sign. \square

PROOF OF Theorem 2

In this section, we apply Theorem 6 for the case of $0 < l < k$ to obtain multiple solutions with the first l components changing sign and the others positive. The proof is similar to that of Theorem 1, but we need to make a little modification by constructing another auxiliary problem which is a cutoff of u_j for $j = l + 1, \dots, k$. Also, let $m \in \mathbb{N}$ be a given number and we take use of S_l with $0 < l < k$.

Proof: Consider another problem with a little change from problem (\mathcal{P}) :

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j(x) \hat{u}_j^3 + 2 \sum_{i \neq j} \beta_{ij}(x) \hat{u}_i^2 \hat{u}_j, & x \in \Omega, \\ u_j = 0 \text{ on } \partial\Omega, & j = 1, \dots, k, \end{cases} \tag{\mathcal{P}_3}$$

where

$$\hat{u}_j = \begin{cases} u_j, & j = 1, \dots, l, \\ u_j^+, & j = l + 1, \dots, k. \end{cases}$$

Follow the proof of Theorem 1 above, that is, construct the auxiliary problem similar to (\mathcal{P}_2) corresponding to (\mathcal{P}_3) and then apply Theorem 6 in the set S_l . Thus when β_{\max} is small enough, we obtain m solutions for the system (\mathcal{P}_3) , denoted by $\mathbf{u}_n := (u_{n1}, \dots, u_{nk})$ for $n = 1, \dots, m$, where u_{nj} changes sign for $j = 1, \dots, l$.

Finally, for $j = l + 1, \dots, k$, $n = 1, \dots, m$, we multiply the equation for u_{nj} in the system (\mathcal{P}_3) by u_{nj}^- , and then it is easy to see that $\|u_{nj}^-\|_j = 0$, that is, u_{nj} is positive. Thus the critical points we obtain for (\mathcal{P}_3) are those for the system (\mathcal{P}) which are mixed states of nodal solutions. \square

PROOF OF Theorem 3

Proof: We apply Theorem 6 in the set S_0 to obtain a critical point for I and the technical skill to ensure the positivity of each component is similar to the construction of problem (\mathcal{P}_3) . Note that we just obtain one positive critical point for I_0 by Theorem 5 and so as for I . In fact, we first consider the problem:

$$\left. \begin{aligned} -\Delta u_j + \lambda_j u_j &= \mu_j(x)(u_j^+)^3 \\ &+ 2 \sum_{i \neq j} \beta_{ij}(x)(u_i^+)^2 u_j^+, \quad x \in \Omega, \\ u_j &= 0 \text{ on } \partial\Omega, \quad j = 1, \dots, k. \end{aligned} \right\} (\mathcal{P}_4)$$

Since I_0 has a critical point in S_0 , we can also obtain a critical point \mathbf{u} for the problem (\mathcal{P}_4) when β_{\max} is small enough, and then it can be shown that \mathbf{u} is positive, which consequently is a critical point of the original functional I with each component positive. \square

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