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Concavity and convexity of several maps involving Tracy-Singh products, Khatri-Rao products, and operator-monotone functions of positive operators

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ABSTRACT: We establish concavity and convexity theorems for a number of operator-valued maps involving Tracy-Singh products and Khatri-Rao products of positive operators on a Hilbert space. Operator means serve as useful tools for some convexity results. We also investigate certain maps dealing with positive operator-monotone functions. In this case, the concavity and the convexity of such maps are examined through suitable integral representations of the operator-monotone functions on the unit interval with respect to finite Borel measures.

KEYWORDS: positive operator, Tracy-Singh product, Khatri-Rao product, operator-monotone function, Bochner integration

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INTRODUCTION

This paper focuses on concavity and convexity of certain maps dealing with Tracy-Singh products and Khatri-Rao products of operators. Such operator products are generalizations of famous matrix products in the literature, namely, the Kronecker product, the Hadamard product, the Tracy-Singh product, and the Khatri-Rao product.

Recall that the Kronecker product is defined for two matrices $A = [a_{ij}]$ and B of arbitrary sizes resulting in a block matrix

$$A * B = [a_{ij}B]_{ij}.$$

The Hadamard product is defined for two matrices A and B of the same size

$$A \circ B = [a_{ij}b_{ij}].$$

Concavity and convexity properties of several matrix-valued maps involving Kronecker products and Hadamard products were collected in Refs. 1–3. As a generalization of the Kronecker product, the Tracy-Singh product[4] is defined for partitioned matrices $A = [A_{ij}]$ and $B = [B_{kl}]$ by

$$A \circledast B = [[A_{ij} * B_{kl}]_{kl}]_{ij}.$$

The work of Al-Zhour[5] extends some results of Ando[1] to Tracy-Singh products of positive definite matrices. The Khatri-Rao product[6], as a generalized Hadamard product, for $A = [A_{ij}]$ and $B = [B_{ij}]$ in the same block-matrix form, is defined by

$$A \odot B = [A_{ij} * B_{ij}]_{ij}.$$

In functional analysis aspect, the tensor product of Hilbert space operators can be viewed as an infinite-dimensional extension of the Kronecker product. Mond and Pečarić[7] extended the matrix results of Ando[1] to Hilbert space operators and obtained concavity/convexity theorems associated with positive operator-monotone functions. Ref. 8 extended the notion of tensor product for operators and Tracy-Singh product for matrices to the Tracy-Singh product for Hilbert space operators, and supply its algebraic and order properties. Analytic properties of the Tracy-Singh product were discussed in Ref. 9. Ref. 10 introduced the Khatri-Rao product of Hilbert space operators and gave a relationship between the Khatri-Rao product and the Tracy-Singh product of two operators via isometric selection operators.

In this study, we investigate concavity and convexity of certain maps related to Tracy-Singh products and Khatri-Rao products of oprators. The main tools we use are operator means and suitable integral representations of certain operator-monotone functions. Our results in this paper generalize the results known so far for Tracy-Singh and Khatri-Rao products of matrices and tensor products of operators. Furthermore, we develop new concavity/convexity theorems.

PRELIMINARIES ON TRACY-SINGH AND KHATRI-RAO PRODUCTS

Throughout this paper, let $\mathscr{H}$, $\mathscr{H}'$, $\mathscr{K}$ and $\mathscr{K}'$ be complex Hilbert spaces. When $\mathscr{X}$ and $\mathscr{Y}$ are Hilbert spaces, the symbol $\mathbb{B}(\mathscr{X},\mathscr{Y})$ stands for the algebra of bounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$, and when $\mathscr{X}=\mathscr{Y}$, we write $\mathbb{B}(\mathscr{X})$ instead of $\mathbb{B}(\mathscr{X},\mathscr{X})$. The cone of positive operators on $\mathscr{H}$ is denoted by $\mathbb{B}(\mathscr{H})^+$. For self-adjoint operators A and B on the same space, the situation $A \geqslant B$ means that $A-B$ is positive. Denote the set of all positive invertible operators on $\mathscr{H}$ by $\mathbb{B}(\mathscr{H})^{++}$. If $A \in \mathbb{B}(\mathscr{H})^{++}$, we write $A>0$. The identity operator and the zero operator are denoted by I and 0, respectively.

To define the Tracy-Singh product and the Khatri-Rao product for operators, we decompose

$$\mathscr{H}=\bigoplus_{j=1}^{n}\mathscr{H}_j, \quad \mathscr{H}'=\bigoplus_{i=1}^{m}\mathscr{H}'_i,$$
$$\mathscr{K}=\bigoplus_{l=1}^{q}\mathscr{K}_l, \quad \mathscr{K}'=\bigoplus_{k=1}^{p}\mathscr{K}'_k,$$

where all $\mathscr{H}_j$, $\mathscr{H}'_i$, $\mathscr{K}_l$ and $\mathscr{K}'_k$ are Hilbert spaces. For each j, let $U_j:\mathscr{H}_j\to\mathscr{H}$ be the canonical embedding

$$(0,\dots,0,x_j,0,\dots,0)\mapsto x_j.$$

Similarly, for each l, let $V_l:\mathscr{K}_l\to\mathscr{K}$ be the canonical embedding. For each i and k, let $P_i:\mathscr{H}'\to\mathscr{H}'_i$ and $Q_k:\mathscr{K}'\to\mathscr{K}'_k$ be the orthogonal projections. Each $A\in\mathbb{B}(\mathscr{H},\mathscr{H}')$ and $B\in\mathbb{B}(\mathscr{K},\mathscr{K}')$ can be expressed uniquely as operator matrices

$$A=[A_{ij}]_{i,j=1}^{m,n}, \quad B=[B_{kl}]_{k,l=1}^{p,q},$$

where $A_{ij}=P_iAU_j\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}'_i)$ and $B_{kl}=Q_kBV_l\in\mathbb{B}(\mathscr{K}_l,\mathscr{K}'_k)$ for each i, j, k, l.

Definition 1 Let $A=[A_{ij}]_{i,j=1}^{m,n}\in\mathbb{B}(\mathscr{H},\mathscr{H}')$ and $B=[B_{kl}]_{k,l=1}^{p,q}\in\mathbb{B}(\mathscr{K},\mathscr{K}')$. We define the *Tracy-Singh product* of A and B to be the bounded linear operator

$$A\boxtimes B=[[A_{ij}\otimes B_{kl}]_{kl}]_{ij},$$
$$A\boxtimes B:\bigoplus_{j=1}^{n}\bigoplus_{l=1}^{q}\mathscr{H}_j\otimes\mathscr{K}_l\to\bigoplus_{i=1}^{m}\bigoplus_{k=1}^{p}\mathscr{H}'_i\otimes\mathscr{K}'_k.$$

When $m=p$ and $n=q$, we define the *Khatri-Rao product* of A and B to be the bounded linear operator

$$A\boxdot B=[A_{ij}\otimes B_{ij}]_{ij}:\bigoplus_{i=1}^{n}\mathscr{H}_i\otimes\mathscr{K}_i\to\bigoplus_{j=1}^{m}\mathscr{H}'_j\otimes\mathscr{K}'_j.$$

Lemma 1 (Refs. 8, 9) *Let A, B, C, D be compatible operators. Then*
(i) *The map $(A,B)\mapsto A\boxtimes B$ is bilinear and jointly continuous.*
(ii) $(A\boxtimes B)(C\boxtimes D)=(AC)\boxtimes(BD)$.
(iii) *If A and B are invertible, then $(A\boxtimes B)^{-1}=A^{-1}\boxtimes B^{-1}$.*
(iv) *If A and B are positive, then $(A\boxtimes B)^{\alpha}=A^{\alpha}\boxtimes B^{\alpha}$ for any $\alpha>0$.*
(v) *If $A\geqslant C\geqslant 0$ and $B\geqslant D\geqslant 0$, then $A\boxtimes B\geqslant C\boxtimes D\geqslant 0$.*
(vi) *If $A>0$ and $B>0$, then $A\boxtimes B>0$.*

Lemma 2 (Ref. 9) *Let $A\in\mathbb{B}(\mathscr{H})$.*
(i) *If f is an analytic function on a region containing the spectra of A and $I\boxtimes A$, then $f(I\boxtimes A)=I\boxtimes f(A)$.*
(ii) *If f is an analytic function on a region containing the spectra of A and $A\boxtimes I$, then $f(A\boxtimes I)=f(A)\boxtimes I$.*

Lemma 3 (Ref. 10) *Let $A\in\mathbb{B}(\mathscr{H})$ and $B\in\mathbb{B}(\mathscr{K})$. If $A\geqslant 0$ and $B\geqslant 0$, then $A\boxdot B\geqslant 0$.*

Lemma 4 (Ref. 10) *There are isometries Z_1 and Z_2 such that*

$$A\boxdot B=Z_1^*(A\boxtimes B)Z_2 \tag{1}$$

for all $A\in\mathbb{B}(\mathscr{H},\mathscr{H}')$ and $B\in\mathbb{B}(\mathscr{K},\mathscr{K}')$. For the case $\mathscr{H}=\mathscr{H}'$ and $\mathscr{K}=\mathscr{K}'$, we have $Z_1=Z_2:=Z$.

Lemma 5 *The Khatri-Rao product of operators is jointly continuous.*

Proof: It follows from (1) and the continuity of the Tracy-Singh product (Lemma 1). □

For each $i=1,\dots,k$, let $\mathscr{H}_i$ and $\mathscr{H}'_i$ be Hilbert spaces and decompose

$$\mathscr{H}_i=\bigoplus_{r=1}^{n_i}\mathscr{H}_{i,r}, \quad \mathscr{H}'_i=\bigoplus_{s=1}^{m_i}\mathscr{H}'_{i,s},$$

where all $\mathscr{H}_{i,r}$ and $\mathscr{H}'_{i,s}$ are Hilbert spaces. For a finite number of operator matrices $A_i\in\mathbb{B}(\mathscr{H}_i,\mathscr{H}'_i)$ for $i=1,\dots,k$, we use the following notations,

$$\mathop{\boxtimes}_{i=1}^{k}A_i=((A_1\boxtimes A_2)\boxtimes\cdots\boxtimes A_{k-1})\boxtimes A_k,$$
$$\mathop{\boxdot}_{i=1}^{k}A_i=((A_1\boxdot A_2)\boxdot\cdots\boxdot A_{k-1})\boxdot A_k.$$

Lemma 6 *There are isometries Z_1 and Z_2*

$$\mathop{\boxdot}_{i=1}^{k} A_i = Z_1^* \Big(\mathop{\boxtimes}_{i=1}^{k} A_i \Big) Z_2 \tag{2}$$

for any $A_i \in \mathbb{B}(\mathscr{H}_i, \mathscr{H}_i')$, $i = 1, \dots, k$. If $\mathscr{H}_i$ and $\mathscr{H}_i'$ are the same space for all i, the $Z_1 = Z_2 := Z$.

Proof: We proceed by induction on k. If $k = 2$, the property (2) is true by Lemma 4. Suppose that there exist isometries R_1 and R_2 such that

$$\mathop{\boxdot}_{i=1}^{k-1} A_i = R_1^* \Big(\mathop{\boxtimes}_{i=1}^{k-1} A_i \Big) R_2.$$

By Lemma 4, there are isometries S_1, S_2 such that

$$\Big(\mathop{\boxdot}_{i=1}^{k-1} A_i \Big) \boxdot A_k = S_1^* [(\mathop{\boxdot}_{i=1}^{k-1} A_i) \boxtimes A_k] S_2.$$

Then

$$\begin{aligned} \mathop{\boxdot}_{i=1}^{k} A_i &= \Big(\mathop{\boxdot}_{i=1}^{k-1} A_i \Big) \boxdot A_k \\ &= S_1^* [(\mathop{\boxdot}_{i=1}^{k-1} A_i) \boxtimes A_k] S_2 \\ &= S_1^* [R_1^* (\mathop{\boxtimes}_{i=1}^{k-1} A_i) R_2 \boxtimes A_k] S_2 \\ &= S_1^* (R_1^* \boxtimes I) \Big[\Big(\mathop{\boxtimes}_{i=1}^{k-1} A_i \Big) \boxtimes A_k \Big] (R_2 \boxtimes I) S_2 \\ &= [(R_1 \boxtimes I) S_1]^* \Big(\mathop{\boxtimes}_{i=1}^{k} A_i \Big) (R_2 \boxtimes I) S_2. \end{aligned}$$

Set $Z_1 = (R_1 \boxtimes I) S_1$ and $Z_2 = (R_2 \boxtimes I) S_2$. Then Z_1 and Z_2 are isometries. When $\mathscr{H}_i = \mathscr{H}_i'$ for all $i = 1, \dots, k$, we have $Z_1 = Z_2$ from the construction. □

CONCAVITY AND CONVEXITY

In this section, we provide concavity and convexity theorems related to Tracy-Singh products of operators. First of all, recall the following terminologies.

Definition 2 A function $f : (0, \infty) \to (0, \infty)$ is said to be operator-monotone if $f[A] \geqslant f[B]$ whenever $A \geqslant B > 0$. Here, $f[A]$ is the (continuous) functional calculus of f defined on the spectrum of A.

Definition 3 Let $\mathscr{H}_1, \dots, \mathscr{H}_k, \mathscr{K}$ be Hilbert spaces. For each $i = 1, \dots, k$, let E_i be a convex subset of $\mathbb{B}(\mathscr{H}_i)$. A function $\phi : E_1 \times \cdots \times E_k \to \mathbb{B}(\mathscr{K})$ is said to be concave if

$$\begin{aligned} &\phi((1-t)A_1 + tB_1, \dots, (1-t)A_k + tB_k) \\ &\qquad \leqslant (1-t)\phi(A_1, \dots, A_k) + t\phi(B_1, \dots, B_k) \end{aligned}$$

for any $A_i, B_i \in E_i$ $(i = 1, \dots, k)$ and $t \in (0,1)$. A function ϕ is convex if $-\phi$ is concave. A map between two convex sets is said to be affine if it preserves convex combinations.

Recall that, for each $t \in (0,1)$, the t-weighted harmonic mean and the t-weighted geometric mean of $A, B \in \mathbb{B}(\mathscr{H})^{++}$ is defined respectively by

$$\begin{aligned} A \,!_t\, B &= [(1-t)A^{-1} + tB^{-1}]^{-1}, \\ A \#_t B &= A^{1/2} \big(A^{-1/2} B A^{-1/2} \big)^t A^{1/2}. \end{aligned}$$

For arbitrary $A, B \in \mathbb{B}(\mathscr{H})^{+}$, we define the t-weighted geometric mean of A and B to be

$$A \#_t B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \#_t (B + \varepsilon I),$$

where the limit is taken in the strong-operator topology.

Lemma 7 (Ref. 11) *For each $t \in [0,1]$, the map $(A,B) \mapsto A \,!_t\, B$ is concave on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{H})^{++}$.*

The next lemma gives an integral representation of operator-monotone functions on $(0, \infty)$ in terms of Borel measures on $[0,1]$.

Lemma 8 (Ref. 12) *Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. Then there is a finite Borel measure μ on $[0,1]$ such that*

$$f(x) = \int_0^1 1 \,!_t\, x \,\mathrm{d}\mu(t), \quad x > 0. \tag{3}$$

Theorem 1 *Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. If $\phi_1 : \mathbb{B}(\mathscr{H})^{++} \to \mathbb{B}(\mathscr{H}')^{++}$ and $\phi_2 : \mathbb{B}(\mathscr{K})^{++} \to \mathbb{B}(\mathscr{K}')^{++}$ are concave maps, then the maps*

$$(A,B) \mapsto f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)), \tag{4}$$

$$(A,B) \mapsto f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (\phi_1(A) \boxtimes I) \tag{5}$$

are concave on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Proof: Let $A \in \mathbb{B}(\mathscr{H})^{++}$ and $B \in \mathbb{B}(\mathscr{K})^{++}$. Then $\phi_1(A) > 0$ and $\phi_2(B) > 0$. Lemma 1 implies that $f[\phi_1(A) \boxtimes \phi_2(B)^{-1}]$ and $f[\phi_1(A)^{-1} \boxtimes \phi_2(B)]$ are well-defined operators. By Lemma 8, there is a finite Borel measure μ on $[0,1]$ such that (3) holds. Using Bochner integration, we have

$$\begin{aligned} &f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)) \\ &\quad = \int_0^1 \{ (I \boxtimes I) \,!_t\, (\phi_1(A) \boxtimes \phi_2(B)^{-1}) \} (I \boxtimes \phi_2(B)) \,\mathrm{d}\mu(t). \end{aligned}$$

For each $t \in [0,1]$, by Lemma 1 we obtain

$$\begin{aligned}
&\{(I \boxtimes I)\,!_t(\phi_1(A) \boxtimes \phi_2(B)^{-1})\} \cdot (I \boxtimes \phi_2(B)) \\
&\quad = \left[(1-t)(I \boxtimes I) + t(\phi_1(A) \boxtimes \phi_2(B)^{-1})^{-1}\right]^{-1} \\
&\qquad \cdot (I \boxtimes \phi_2(B)) \\
&\quad = \Big[(I \boxtimes \phi_2(B)^{-1}) \\
&\qquad \cdot \left\{(1-t)I \boxtimes I + t\phi_1(A)^{-1} \boxtimes \phi_2(B)\right\}\Big]^{-1} \\
&\quad = \left[(1-t)(I \boxtimes \phi_2(B))^{-1} + t(\phi_1(A) \boxtimes I)^{-1}\right]^{-1} \\
&\quad = (I \boxtimes \phi_2(B))\,!_t(\phi_1(A) \boxtimes I).
\end{aligned}$$

Since the weighted harmonic mean is concave (Lemma 7), so is the map

$$(A,B) \mapsto \{(I \boxtimes I)\,!_t(\phi_1(A) \boxtimes \phi_2(B)^{-1})\} \cdot (I \boxtimes \phi_2(B)).$$

Thus the map (4) is concave. Similarly, the map (5) is concave. □

Remark 1 Since $\phi_1(A) \boxtimes \phi_2(B)^{-1}$ commutes with $I \boxtimes \phi_2(B)$, we have

$$\begin{aligned}
&f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)) \\
&\qquad = (I \boxtimes \phi_2(B)) \cdot f[\phi_1(A) \boxtimes \phi_2(B)^{-1}].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&f[\phi_1(A)^{-1} \boxtimes \phi_2(A)] \cdot (\phi_1(A) \boxtimes I) \\
&\qquad = (\phi_1(A) \boxtimes I) \cdot f[\phi_1(A)^{-1} \boxtimes \phi_2(B)].
\end{aligned}$$

Example 1 Recall that the function $t \mapsto t^p$ is operator-monotone for any $0 \leqslant p \leqslant 1$. Given two concave maps $\phi_1 : \mathbb{B}(\mathscr{H})^{++} \to \mathbb{B}(\mathscr{H}')^{++}$ and $\phi_2 : \mathbb{B}(\mathscr{K})^{++} \to \mathbb{B}(\mathscr{K}')^{++}$, by Theorem 1 the maps

$$\begin{aligned}
(A,B) &\mapsto [\phi_1(A) \boxtimes \phi_2(B)^{-1}]^p \cdot (I \boxtimes \phi_2(B)), \\
(A,B) &\mapsto [\phi_1(A)^{-1} \boxtimes \phi_2(B)]^p \cdot (\phi_1(A) \boxtimes I)
\end{aligned}$$

are concave on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Corollary 1 *Let $f : (0,\infty) \to (0,\infty)$ be operator-monotone. If $\phi_1 : \mathbb{B}(\mathscr{H})^{++} \to \mathbb{B}(\mathscr{H}')^{++}$ and $\phi_2 : \mathbb{B}(\mathscr{K})^{++} \to \mathbb{B}(\mathscr{K}')^{++}$ are concave maps, then the maps*

$$(A,B) \mapsto f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (I \boxtimes \phi_2(B)^{-1}), \tag{6}$$

$$(A,B) \mapsto f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (\phi_1(A)^{-1} \boxtimes I) \tag{7}$$

are convex on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Proof: Note that the function $g(x) := f(x^{-1})^{-1}$ is operator-monotone. By Lemma 1, we have

$$\begin{aligned}
&f[\phi_1(A)^{-1} \boxtimes \phi_2(B)](I \boxtimes \phi_2(B)^{-1}) \\
&\qquad = g[\phi_1(A)^{-1} \boxtimes \phi_2(B)]^{-1}(I \boxtimes \phi_2(B)^{-1}).
\end{aligned}$$

Theorem 1 implies the concavity of the map

$$\begin{aligned}
(A,B) &\mapsto g[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)) \\
&\quad = \{(I \boxtimes \phi_2(B)^{-1}) \cdot f[\phi_1(A)^{-1} \boxtimes \phi_2(B)]\}^{-1} \\
&\quad = \{f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (I \boxtimes \phi_2(B)^{-1})\}^{-1}.
\end{aligned}$$

Thus the map (6) is convex. Similarly, the map (7) is convex. □

Theorem 2 *Let $f : (0,\infty) \to (0,\infty)$ be an operator-monotone function. If $\phi_1 : \mathbb{B}(\mathscr{H})^{++} \to \mathbb{B}(\mathscr{H}')^{++}$ is a concave map and $\phi_2 : \mathbb{B}(\mathscr{K})^{++} \to \mathbb{B}(\mathscr{K}')^{++}$ is an affine map, then the maps*

$$(A,B) \mapsto f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (I \boxtimes \phi_2(B)), \tag{8}$$

$$(A,B) \mapsto f[\phi_2(B) \boxtimes \phi_1(A)^{-1}] \cdot (\phi_2(B) \boxtimes I) \tag{9}$$

are convex on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Proof: By Lemma 8, there is a finite Borel measure μ on $[0,1]$ such that (3) holds. Then

$$\begin{aligned}
&f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (I \boxtimes \phi_2(B)) \\
&\quad = \int_0^1 \{(I \boxtimes I)\,!_t(\phi_1(A)^{-1} \boxtimes \phi_2(B))\}(I \boxtimes \phi_2(B))\, \mathrm{d}\mu(t).
\end{aligned}$$

For each $t \in [0,1]$, it follows from Lemma 1 that

$$\begin{aligned}
&\{(I \boxtimes I)\,!_t(\phi_1(A)^{-1} \boxtimes \phi_2(B))\} \\
&= \left[(1-t)(I \boxtimes I) + t(\phi_1(A)^{-1} \boxtimes \phi_2(B))^{-1}\right]^{-1} \\
&= \left[(1-t)(I \boxtimes I) + t(\phi_1(A) \boxtimes \phi_2(B)^{-1})\right]^{-1} \\
&= (I \boxtimes \phi_2(B))\left[(1-t)(I \boxtimes \phi_2(B)) + t(\phi_1(A) \boxtimes I)\right]^{-1}.
\end{aligned}$$

The concavity of the map $(A,B) \mapsto (1-t)(I \boxtimes \phi_2(B)) + t(\phi_1(A) \boxtimes I)$ and the affinity of the map $(A,B) \mapsto I \boxtimes \phi_2(B)$ together yield the convexity of the map

$$\begin{aligned}
&(A,B) \mapsto \\
&(I \boxtimes \phi_2(B))\{(1-t)I \boxtimes \phi_2(B) + t\phi_1(A) \boxtimes I\}^{-1}(I \boxtimes \phi_2(B)) \\
&\qquad = \{(I \boxtimes I)\,!_t(\phi_1(A)^{-1} \boxtimes \phi_2(B))\}(I \boxtimes \phi_2(B)).
\end{aligned}$$

Hence the map (8) is convex. Similarly, the map (9) is convex. □

Corollary 2 *The maps*

$$(A,B) \mapsto I \boxtimes (B\log[B]) - \log[A] \boxtimes B, \tag{10}$$

$$(A,B) \mapsto (A\log[A]) \boxtimes I - A \boxtimes \log[B] \tag{11}$$

are convex on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{H})^{++}$.

Proof: Using Lemmas 1 and 2, we obtain

$$
\begin{aligned}
I \boxtimes (B\log[B]) - \log[A] \boxtimes B
&= \{I \boxtimes \log[B] - \log[A] \boxtimes I\} \cdot (I \boxtimes B)\\
&= \{\log[I \boxtimes B] - \log[A \boxtimes I]\} \cdot (I \boxtimes B)\\
&= \log[(I \boxtimes B)(A \boxtimes I)^{-1}] \cdot (I \boxtimes B)\\
&= (I \boxtimes B) \cdot \log[A^{-1} \boxtimes B].
\end{aligned}
$$

Since $\log x$ is operator-monotone, by Theorem 2 we obtain that the map

$$(A,B) \mapsto \log[A^{-1} \boxtimes B] \cdot (I \boxtimes B)$$

is convex. Hence the map (10) is convex. Similarly, the map (11) is convex. □

Example 2 Let $\phi_1 : \mathbb{B}(\mathscr{H})^{++} \to \mathbb{B}(\mathscr{H}')^{++}$ be a concave map and $\phi_2 : \mathbb{B}(\mathscr{K})^{++} \to \mathbb{B}(\mathscr{K}')^{++}$ an affine map. For any $0 \leqslant p \leqslant 1$, we have by Theorem 2 that the maps

$$
\begin{aligned}
(A,B) &\mapsto [\phi_1(A)^{-1} \boxtimes \phi_2(B)]^p \cdot (I \boxtimes \phi_2(B)),\\
(A,B) &\mapsto [\phi_2(B) \boxtimes \phi_1(A)^{-1}]^p \cdot (\phi_2(B) \boxtimes I)
\end{aligned}
$$

are convex on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

We mention that the maps (5), (7), (9) and (11) are extensions of results discussed in Ref. 7.

CONCAVITY THEOREMS FOR TRACY-SINGH AND KHATRI-RAO PRODUCTS

In this section, we present concavity theorems for Tracy-Singh products of operators. Concavity theorems for Khatri-Rao products of operators are established by using the concavity theorems for Tracy-Singh products and the connection between the Khatri-Rao and Tracy-Singh products.

The next result generalizes Corollary 6.2 of Ref. 1 to the case of Tracy-Singh product of operators.

Theorem 3 *Let $0 \leqslant p_i \leqslant 1$, $i = 1,\dots,k$, be such that $\sum_{i=1}^k p_i \leqslant 1$. Then the map*

$$(A_1,\dots,A_k) \mapsto \mathop{\boxtimes}_{i=1}^{k} A_i^{p_i} \tag{12}$$

is concave on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: We proceed by induction on k. Clearly, the map $A_1 \mapsto A_1^{p_1}$ is concave. Suppose the assertion is generally true for the case $k-1$. If $p_k = 0$, then the map becomes

$$(A_1,\dots,A_k) \mapsto ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{k-1}) \boxtimes I,$$

which is concave. If $p_k = 1$, then $p_i = 0$ for all $i = 1,\dots,k-1$ and the map is clearly concave. Now suppose $0 < p_k < 1$. By the induction assumption, the map

$$\phi(A_1,\dots,A_{k-1}) = \mathop{\boxtimes}_{i=1}^{k-1} A_i^{p_i/(1-p_k)}$$

is concave. By applying Theorem 1 with $f(x) = x^{p_k}$, the map

$$
\begin{aligned}
&(A_1,\dots,A_k) \mapsto\\
&\quad f(\phi(A_1,\dots,A_{k-1})^{-1} \boxtimes A_k) \cdot (\phi(A_1,\dots,A_{k-1}) \boxtimes I)
\end{aligned}
$$

is concave. We obtain the concavity of the map (12),

$$
\begin{aligned}
&f(\phi(A_1,\dots,A_{k-1})^{-1} \boxtimes A_k)(\phi(A_1,\dots,A_{k-1}) \boxtimes I)\\
&\quad = (\phi(A_1,\dots,A_{k-1})^{-p_k} \boxtimes A_k^{p_k})(\phi(A_1,\dots,A_{k-1}) \boxtimes I)\\
&\quad = \phi(A_1,\dots,A_{k-1})^{1-p_k} \boxtimes A_k^{p_k} = \mathop{\boxtimes}_{i=1}^{k} A_i^{p_i}.
\end{aligned}
$$

□

A special case of Theorem 3 is when $k = 2$.

Corollary 3 *For each $r \in (0,1)$, the map*

$$(A,B) \mapsto A^{1-r} \boxtimes B^r \tag{13}$$

is concave on $\mathbb{B}(\mathscr{H})^{+} \times \mathbb{B}(\mathscr{K})^{+}$.

Proof: Theorem 3 implies that the map (13) is concave on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$. Since the Tracy-Singh product is jointly continuous (Lemma 1), this map is also concave on $\mathbb{B}(\mathscr{H})^{+} \times \mathbb{B}(\mathscr{K})^{+}$. □

Next, we develop concavity theorems for Khatri-Rao products of operators.

Theorem 4 *Let $0 \leqslant p_i \leqslant 1$, $i = 1,\dots,k$, be such that $\sum_{i=1}^k p_i \leqslant 1$. Then the map*

$$(A_1,\dots,A_k) \mapsto \mathop{\boxdot}_{i=1}^{k} A_i^{p_i} \tag{14}$$

is concave on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: From Lemma 6, the map $X \mapsto Z^*XZ$, taking the Tracy-Singh product $\boxtimes_{i=1}^k A_i$ into the Khatri-Rao product $\boxdot_{i=1}^k A_i$, is linear and preserves positivity. Recall that the composition between a linear map and a concave map results in a concave map. Since the map $(A_1,\dots,A_k) \mapsto \boxtimes_{i=1}^k A_i^{p_i}$ is concave by Theorem 3, we have the concavity of the map is concave. We obtain the concavity of the map from (12), since

$$(A_1,\dots,A_k) \mapsto Z^*(\mathop{\boxtimes}_{i=1}^{k} A_i^{p_i})Z = \mathop{\boxdot}_{i=1}^{k} A_i^{p_i}.$$

□

Corollary 4 *For each $r \in (0,1)$, the map*

$$(A,B) \mapsto A^{1-r} \boxdot B^{r},$$

is concave on $\mathbb{B}(\mathscr{H})^{+} \times \mathbb{B}(\mathscr{K})^{+}$.

Proof: It follows from Theorem 4 when $k = 2$ together with the continuity of the Khatri-Rao product, Lemma 5. □

CONVEXITY THEOREMS FOR TRACY-SINGH AND KHATRI-RAO PRODUCTS

In this section, we establish convexity theorems for Tracy-Singh products and Khatri-Rao products of operators. Weighted arithmetic/geometric/harmonic means of operators serve as useful tools.

Lemma 9 (Ref. 13) *Let $A_i, B_i \in \mathbb{B}(\mathscr{H})^{+}$, $1 \leqslant i \leqslant k$. Then*

$$\Big(\mathop{\boxtimes}_{i=1}^{k} A_i\Big) \#_t \Big(\mathop{\boxtimes}_{i=1}^{k} B_i\Big) = \mathop{\boxtimes}_{i=1}^{k} \big(A_i \#_t B_i\big).$$

Theorem 5 *Let ϕ_i, $i = 1, \dots, k$, be a concave map from $B(\mathscr{H}_i)^{++}$ to $B(\mathscr{H}_i')^{++}$. Then the map*

$$(A_1, \dots, A_k) \mapsto \mathop{\boxtimes}_{i=1}^{k} \phi_i(A_i)^{-1} \tag{15}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: Let $t \in [0,1]$. By continuity, we may assume that A_i and B_i are positive invertible operators. Applying Lemmas 1 and 9 and the arithmetic-geometric means inequality for operators, we have

$$\begin{aligned}
&\mathop{\boxtimes}_{i=1}^{k} \phi_i((1-t)A_i + tB_i)^{-1} \\
&\quad \leqslant \mathop{\boxtimes}_{i=1}^{k} ((1-t)\phi_i(A_i) + t\phi_i(B_i))^{-1} \\
&\quad \leqslant \mathop{\boxtimes}_{i=1}^{k} (\phi_i(A_i) \#_t \phi_i(B_i))^{-1} \\
&\quad = \mathop{\boxtimes}_{i=1}^{k} \phi_i(A_i)^{-1} \#_t \mathop{\boxtimes}_{i=1}^{k} \phi_i(B_i)^{-1} \\
&\quad \leqslant (1-t) \mathop{\boxtimes}_{i=1}^{k} \phi_i(A_i)^{-1} + t \mathop{\boxtimes}_{i=1}^{k} \phi_i(B_i)^{-1}.
\end{aligned}$$

Hence the map (15) is convex. □

Corollary 5 *Let $0 < p_i \leqslant 1$, $i = 1, \dots, k$. Then the map*

$$(A_1, \dots, A_k) \mapsto \mathop{\boxtimes}_{i=1}^{k} A_i^{-p_i}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proposition 1 *Let $0 \leqslant p_i \leqslant 1$, $i = 1, \dots, k$, and $1 \leqslant q \leqslant 2$ be such that $\sum_{i=1}^{k} p_i \leqslant q-1$. Then the map*

$$(A_1, \dots, A_{k+1}) \mapsto \Big(\mathop{\boxtimes}_{i=1}^{k} A_i^{-p_i}\Big) \boxtimes A_{k+1}^{q}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_{k+1})^{++}$.

Proof: By Theorem 3, the map

$$(A_1, \dots, A_{k+1}) \mapsto \Big(\mathop{\boxtimes}_{i=1}^{k} A_i^{p_i}\Big) \boxtimes A_{k+1}^{2-q}$$

is concave on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_{k+1})^{++}$. Clearly, the map

$$(A_1, \dots, A_{k+1}) \mapsto \Big(\mathop{\boxtimes}_{i=1}^{k} I\Big) \boxtimes A_{k+1}$$

is affine. It follows from Lemma 1 that the map

$$\begin{aligned}
&(A_1, \dots, A_{k+1}) \mapsto \\
&\Big[\Big(\mathop{\boxtimes}_{i=1}^{k} I\Big) \boxtimes A_{k+1}\Big]\Big[\Big(\mathop{\boxtimes}_{i=1}^{k} A_i^{p_i}\Big) \boxtimes A_{k+1}^{2-q}\Big]^{-1}\Big(\mathop{\boxtimes}_{i=1}^{k} I\Big) \boxtimes A_{k+1} \\
&\qquad = \Big(\mathop{\boxtimes}_{i=1}^{k} A_i^{-p_i}\Big) \boxtimes A_{k+1}^{q}
\end{aligned}$$

is convex. □

Theorem 6 *For each $r \in (0,1)$, the maps*

$$(A,B) \mapsto A^{-r} \boxtimes B^{1+r}, \tag{16}$$

$$(A,B) \mapsto A^{1+r} \boxtimes B^{-r} \tag{17}$$

are convex on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Proof: The convexity of the map (16) follows from Proposition 1. By continuity, we may assume that A and B are invertible. Lemma 1 implies that

$$A^{1+r} \boxtimes B^{-r} = (A^{r} \boxtimes B^{-r})(A \boxtimes I) = (A \boxtimes B^{-1})^{r}(A \boxtimes I).$$

It follows from Lemmas 1 and 8 that

$$\begin{aligned}
A^{1+r} \boxtimes B^{-r} &= \int_0^1 ((I \boxtimes I) \,!_t\, (A \boxtimes B^{-1}))\, \mathrm{d}\mu(t)\, (A \boxtimes I) \\
&= \int_0^1 \Big[(1-t)(I \boxtimes I) + t(A \boxtimes B^{-1})^{-1}\Big]^{-1} (A \boxtimes I)\, \mathrm{d}\mu(t) \\
&= \int_0^1 \Big[(1-t)(I \boxtimes I) + t(A^{-1} \boxtimes B)\Big]^{-1} (A \boxtimes I)\, \mathrm{d}\mu(t) \\
&= \int_0^1 (A \boxtimes I)\,[(1-t)(A \boxtimes I) + t(I \boxtimes B)]^{-1} (A \boxtimes I)\, \mathrm{d}\mu(t).
\end{aligned}$$

Since the map $A \mapsto A^{-1}$ is convex and the map $(A,B) \mapsto (1-t)(A \boxtimes I) + t(I \boxtimes B)$ is affine, the map

$$(A,B) \mapsto (I \boxtimes B)\{(1-t)(A \boxtimes I) + t(I \boxtimes B)\}^{-1}(I \boxtimes B)$$

is convex. Thus the map $(A,B) \mapsto A^{1+r} \boxtimes B^{-r}$ is convex. □

Proposition 2 *Let ϕ_i, $i=1,\dots,k$, be concave maps from $B(\mathscr{H}_i)^{++}$ to $B(\mathscr{H}_i')^{++}$. Then the map*

$$(A_1,\dots,A_k) \mapsto \mathop{\boxdot}_{i=1}^{k} \phi_i(A_i)^{-1}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: It follows from Lemma 6 and Theorem 5. □

Corollary 6 *Let $0 < p_i \leqslant 1$ for each $i=1,\dots,k$. Then the map*

$$(A_1,\dots,A_k) \mapsto \mathop{\boxdot}_{i=1}^{k} A_i^{-p_i}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: It follows from Proposition 2 by putting $\phi_i(A_i) = A_i^{p_i}$ for each i. □

Proposition 3 *For each $r \in (0,1)$, the maps*

$$(A,B) \mapsto A^{-r} \boxdot B^{1+r},$$
$$(A,B) \mapsto A^{1+r} \boxdot B^{-r}$$

are convex on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Proof: It follows from Lemma 4 and Theorem 6. □

Recall that the Moore-Penrose inverse of an operator $T \in \mathbb{B}(\mathscr{H},\mathscr{H}')$ is the operator $T^\dagger \in \mathbb{B}(\mathscr{H}',\mathscr{H})$ satisfying the conditions $TT^\dagger T = T$, $T^\dagger T T^\dagger = T$, $(TT^\dagger)^* = TT^\dagger$, and $(T^\dagger T)^* = T^\dagger T$. It is well known that $T^\dagger$ exists if and only if the range of T is closed[14].

Lemma 10 (Ref. 15) *Let*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{bmatrix} \in \mathbb{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$$

be a self-adjoint operator. Suppose that T_{11} has a closed range. Then $T \geqslant 0$ if and only if $T_{11} \geqslant 0$, $T_{12} = T_{11}T_{11}^\dagger T_{12}$, and $T_{22} \geqslant T_{12}^* T_{11}^\dagger T_{12}$.

Recall that for any interval J, a continuous function $f: J \to \mathbb{R}$ is convex if and only if $f(x+h) + f(x-h) - 2f(x) \geqslant 0$ for all $x \in J$ and $h > 0$ such that $x \pm h \in J$.

Theorem 7 *Let $A \in \mathbb{B}(\mathscr{H})^+$ and $B \in \mathbb{B}(\mathscr{K})^+$ have closed ranges. Then the operator-valued function*

$$\begin{aligned} \phi &: [-1,1] \to \mathbb{B}\Big(\bigoplus_{i=1}^{n} \mathscr{H}_i \otimes \mathscr{K}_i\Big), \\ \phi(t) &= A^{1+t} \boxdot B^{1-t} + A^{1-t} \boxdot B^{1+t} \end{aligned} \tag{18}$$

is convex on $[-1,1]$, decreasing on $[-1,0]$, increasing on $[0,1]$, attains minimality at $t=0$, and attains maximality at $t=-1$, 1.

Proof: Let $s \in [-1,1]$ and $t > 0$ be such that $s \pm t \in [-1,1]$. Consider the operator matrices

$$T_1 = \begin{bmatrix} A^{1+s+t} & A^{1+s} \\ A^{1+s} & A^{1+s-t} \end{bmatrix}, \quad T_2 = \begin{bmatrix} A^{1-s-t} & A^{1-s} \\ A^{1-s} & A^{1-s+t} \end{bmatrix},$$
$$T_3 = \begin{bmatrix} B^{1+s+t} & B^{1+s} \\ B^{1+s} & B^{1+s-t} \end{bmatrix}, \quad T_4 = \begin{bmatrix} B^{1-s-t} & B^{1-s} \\ B^{1-s} & B^{1-s+t} \end{bmatrix}.$$

Note that

$$A^{1+s} = (AA^\dagger A)^{1+s+t}A^{-t} = A^{1+s+t}(A^{1+s+t})^\dagger A^{1+s},$$
$$A^{1+s-t} = A^{-t}(AA^\dagger A)^{1+s+t}A^{-t} = A^{1+s}(A^{1+s+t})^\dagger A^{1+s}.$$

We have by Lemma 10 that T_i is positive for all $i = 1,2,3,4$. By the monotonicity of Khatri-Rao product, Lemma 3, we have that the operator $X \equiv T_1 \boxdot T_4 + T_2 \boxdot T_3$ is

$$\begin{bmatrix} A^{1+s+t} \boxdot B^{1-s-t} + A^{1-s-t} \boxdot B^{1+s+t} & A^{1+s} \boxdot B^{1-s} + A^{1-s} \boxdot B^{1+s} \\ A^{1+s} \boxdot B^{1-s} + A^{1-s} \boxdot B^{1+s} & A^{1+s-t} \boxdot B^{1-s+t} + A^{1-s+t} \boxdot B^{1+s-t} \end{bmatrix},$$

which is positive. Similarly, the operator Y,

$$\begin{bmatrix} A^{1+s-t} \boxdot B^{1-s+t} + A^{1-s+t} \boxdot B^{1+s-t} & A^{1+s} \boxdot B^{1-s} + A^{1-s} \boxdot B^{1+s} \\ A^{1+s} \boxdot B^{1-s} + A^{1-s} \boxdot B^{1+s} & A^{1+s+t} \boxdot B^{1-s-t} + A^{1-s+t} \boxdot B^{1+s+t} \end{bmatrix}$$

is also positive. It follows that

$$\begin{aligned} 0 \leqslant X + Y &= \begin{bmatrix} \phi(s+t)+\phi(s-t) & 2\phi(s) \\ 2\phi(s) & \phi(s+t)+\phi(s-t) \end{bmatrix} \\ &= U \begin{bmatrix} \phi(s+t)+\phi(s-t)+2\phi(s) & 0 \\ 0 & \phi(s+t)+\phi(s-t)-2\phi(s) \end{bmatrix} U^*, \end{aligned}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}.$$

Again, Lemma 10 guarantees that

$$\phi(s+t) + \phi(s-t) \geqslant 2\phi(s).$$

This means that ϕ is convex. The fact that $\phi(t) = \phi(-t)$ for all $t \in [-1,1]$ and the convexity of ϕ implies that ϕ has the minimal value at 0. Hence ϕ is decreasing on $[-1,0]$ and increasing on $[0,1]$. □

Corollary 7 *Let $A \in \mathbb{B}(\mathscr{H})^+$ and $B \in \mathbb{B}(\mathscr{K})^+$ have closed ranges. Then the parameterization*

$$\psi : [0,1] \to \mathbb{B}\Big(\bigoplus_{i=1}^{n} \mathscr{H}_i \otimes \mathscr{K}_i\Big),$$
$$\psi(t) = A^t \boxdot B^{1-t} + A^{1-t} \boxdot B^t$$

is convex on $[0,1]$, decreasing on $[0,1/2]$, increasing on $[1/2,1]$, attains minimality at $t = 1/2$, and attains maximality at $t = 0$, 1.

Proof: Let $f : [0,1] \to [-1,1]$ be defined by $f(t) = 2t-1$. Then $\psi = \phi \circ f$ where ϕ is given by (18). Now, the desired results follow from Theorem 7 by using $f([0,1]) = [-1,1]$, $f([0,1/2]) = [-1,0]$, $f([1/2,0]) = [0,1]$, and $f(1/2) = 0$. □

As a consequence, we obtain an operator version of the arithmetic-geometric mean inequality as follows.

Corollary 8 *Let $A \in \mathbb{B}(\mathscr{H})^+$ and $B \in \mathbb{B}(\mathscr{K})^+$ have closed ranges. For any $t \in [1/2,1]$, we have*

$$2(A^{1/2} \boxdot B^{1/2}) \leqslant A^t \boxdot B^{1-t} + A^{1-t} \boxdot B^t \leqslant A \boxplus B,$$

where $\boxplus$ denotes the Khatri-Rao sum[16] defined by $A \boxplus B = A \boxdot I + I \boxdot B$.

We mention that Theorem 5, Corollary 5, and Proposition 1 generalize the matrix results involving Tracy-Singh products provided in Ref. 5.

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