Concavity and convexity of several maps involving Tracy-Singh products, Khatri-Rao products, and operator-monotone functions of positive operators

Arnon Ploymukda, Pattrawut Chansangiam

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520 Thailand

∗Corresponding author, e-mail: pattrawut.ch@kmitl.ac.th

Received 12 Nov 2018
Accepted 27 Mar 2019

ABSTRACT: We establish concavity and convexity theorems for a number of operator-valued maps involving Tracy-Singh products and Khatri-Rao products of positive operators on a Hilbert space. Operator means serve as useful tools for some convexity results. We also investigate certain maps dealing with positive operator-monotone functions. In this case, the concavity and the convexity of such maps are examined through suitable integral representations of the operator-monotone functions on the unit intervals with respect to finite Borel measures.

KEYWORDS: positive operator, Tracy-Singh product, Khatri-Rao product, operator-monotone function, Bochner integration

MSC2010: 46G10 47A05 47A63 47A64 47A80

INTRODUCTION

This paper focuses on concavity and convexity of certain maps dealing with Tracy-Singh products and Khatri-Rao products of operators. Such operator products are generalizations of famous matrix products in the literature, namely, the Kronecker product, the Hadamard product, the Tracy-Singh product, and the Khatri-Rao product.

Recall that the Kronecker product is defined for two matrices \( A = [a_{ij}] \) and \( B = [b_{ij}] \) of arbitrary sizes resulting in a block matrix

\[
A \otimes B = [a_{ij}B_{ij}].
\]

The Hadamard product is defined for two matrices \( A \) and \( B \) of the same size

\[
A \odot B = [a_{ij}b_{ij}].
\]

Concavity and convexity properties of several matrix-valued maps involving Kronecker products and Hadamard products were collected in Refs. 1–3. As a generalization of the Kronecker product, the Tracy-Singh product\(^4\) is defined for partitioned matrices \( A = [A_{ij}] \) and \( B = [B_{ij}] \) by

\[
A \star B = [[A_{ij}B_{kl}]],
\]

The work of Al-Zhour\(^5\) extends some results of Ando\(^1\) to Tracy-Singh products of positive definite matrices. The Khatri-Rao product\(^6\), as a generalized Hadamard product, for \( A = [A_{ij}] \) and \( B = [B_{ij}] \) in the same block-matrix form, is defined by

\[
A \circ B = [A_{ij}B_{ij}]_{ij}.
\]

In functional analysis aspect, the tensor product of Hilbert space operators can be viewed as an infinite-dimensional extension of the Kronecker product. Mond and Pečarić\(^7\) extended the matrix results of Ando\(^1\) to Hilbert space operators and obtained concavity/convexity theorems associated with positive operator-monotone functions. Ref. 8 extended the notion of tensor product for operators and Tracy-Singh product for matrices to the Tracy-Singh product for Hilbert space operators, and supply its algebraic and order properties. Analytic properties of the Tracy-Singh product were discussed in Ref. 9. Ref. 10 introduced the Khatri-Rao product of Hilbert space operators and gave a relationship between the Khatri-Rao product and the Tracy-Singh product of two operators via isometric selection operators.

In this study, we investigate concavity and convexity of certain maps related to Tracy-Singh products and Khatri-Rao products of operators. The main tools we use are operator means and suitable integral representations of certain operator-monotone functions.
functions. Our results in this paper generalize the results known so far for Tracy-Singh and Khatri-Rao products of matrices and tensor products of operators. Furthermore, we develop new concavity/convexity theorems.

**PRELIMINARIES ON TRACY-SINGH AND KHATRI-RAO PRODUCTS**

Throughout this paper, let \( \mathcal{H}, \mathcal{H}', \mathcal{K}, \) and \( \mathcal{K}' \) be complex Hilbert spaces. When \( \mathcal{H} \) and \( \mathcal{K} \) are Hilbert spaces, the symbol \( \mathbb{B}(\mathcal{X}, \mathcal{Y}) \) stands for the algebra of bounded linear operators from \( \mathcal{X} \) into \( \mathcal{Y} \), and when \( \mathcal{H} = \mathcal{K} \), we write \( \mathbb{B}(\mathcal{H}) \) instead of \( \mathbb{B}(\mathcal{X}, \mathcal{X}) \). The cone of positive operators on \( \mathcal{H} \) is denoted by \( \mathbb{B}(\mathcal{H})^+ \). For self-adjoint operators \( A \) and \( B \) on the same space, the situation \( A \geq B \) means that \( A - B \) is positive. Denote the set of all positive invertible operators on \( \mathcal{H} \) by \( \mathbb{B}(\mathcal{H})^{++} \). If \( A \in \mathbb{B}(\mathcal{H})^{++} \), we write \( A > 0 \). The identity operator and the zero operator are denoted by \( I \) and \( 0 \), respectively.

To define the Tracy-Singh product and the Khatri-Rao product for operators, we decompose

\[
\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}_i', \\
\mathcal{K} = \bigoplus_{l=1}^{q} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{p} \mathcal{K}_k',
\]

where all \( \mathcal{H}_j, \mathcal{H}_i', \mathcal{K}_l, \) and \( \mathcal{K}_k' \) are Hilbert spaces. For each \( j \), let \( U_j : \mathcal{H}_j \to \mathcal{H} \) be the canonical embedding

\[
(0, \ldots, 0, x_j, 0, \ldots, 0) \mapsto x_j.
\]

Similarly, for each \( l \), let \( V_l : \mathcal{K}_l \to \mathcal{K} \) be the canonical embedding. For each \( i \) and \( k \), let \( P_i : \mathcal{H}' \to \mathcal{H}_i' \) and \( Q_k : \mathcal{K}' \to \mathcal{K}_k' \) be the orthogonal projections. Each \( A \in \mathbb{B}(\mathcal{H}, \mathcal{H}') \) and \( B \in \mathbb{B}(\mathcal{K}, \mathcal{K}') \) can be expressed uniquely as operator matrices

\[
A = [A_{ij}]_{i,j=1}^{m,n}, \quad B = [B_{kl}]_{k,l=1}^{p,q},
\]

where \( A_{ij} = P_i A U_j \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i') \) and \( B_{kl} = Q_k B V_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k') \) for each \( i, j, k, l \).

**Definition 1** Let \( A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}') \) and \( B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}, \mathcal{K}') \). We define the Tracy-Singh product of \( A \) and \( B \) to be the bounded linear operator

\[
A \boxtimes B = \left[ [A_{ij} \otimes B_{kl}]_{i,j=1}^{m,n} \right]_{k,l=1}^{p,q},
\]

\[
A \boxtimes B : \bigoplus_{j=1}^{q} \mathcal{H}_j \otimes \mathcal{K}_l \to \bigoplus_{i=1}^{m} \mathcal{H}_i' \otimes \mathcal{K}_k',
\]

When \( m = p \) and \( n = q \), we define the Khatri-Rao product of \( A \) and \( B \) to be the bounded linear operator

\[
A \boxtimes B = [A_{ij} \otimes B_{kl}]_{i,j=1}^{m,n} : \bigoplus_{j=1}^{q} \mathcal{H}_j' \otimes \mathcal{K}_l \to \bigoplus_{i=1}^{m} \mathcal{H}_i' \otimes \mathcal{K}_k',
\]

**Lemma 1** (Refs. 8, 9) Let \( A, B, C, D \) be compatible operators. Then

(i) The map \( (A,B) \to A \otimes B \) is bilinear and jointly continuous.

(ii) \( (A \otimes B)(C \otimes D) = (AC) \otimes (BD) \).

(iii) If \( A \) and \( B \) are invertible, then \( (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \).

(iv) If \( A \) and \( B \) are positive, then \( (A \otimes B)^{\alpha} = A^{\alpha} \otimes B^{\alpha} \) for any \( \alpha > 0 \).

(v) If \( A \geq C \geq 0 \) and \( B \geq D \geq 0 \), then \( A \otimes B \geq C \otimes D \geq 0 \).

(vi) If \( A > 0 \) and \( B > 0 \), then \( A \otimes B > 0 \).

**Lemma 2** (Ref. 9) Let \( A \in \mathbb{B}(\mathcal{H}) \). (i) If \( f \) is an analytic function on a region containing the spectra of \( A \) and \( I \otimes A \), then \( f(I \otimes A) = I \otimes f(A) \).

(ii) If \( f \) is an analytic function on a region containing the spectra of \( A \) and \( A \otimes I \), then \( f(A \otimes I) = f(A) \otimes I \).

**Lemma 3** (Ref. 10) Let \( A \in \mathbb{B}(\mathcal{H}, \mathcal{H}') \) and \( B \in \mathbb{B}(\mathcal{K}, \mathcal{K}') \). If \( A \geq 0 \) and \( B \geq 0 \), then \( A \otimes B \geq 0 \).

**Lemma 4** (Ref. 10) There are isometries \( Z_1 \) and \( Z_2 \) such that

\[
A \boxtimes B = Z_1^* (A \boxtimes B) Z_2 \tag{1}
\]

for all \( A \in \mathbb{B}(\mathcal{H}, \mathcal{H}') \) and \( B \in \mathbb{B}(\mathcal{K}, \mathcal{K}') \). For the case \( \mathcal{H} = \mathcal{H}' \) and \( \mathcal{K} = \mathcal{K}' \), we have \( Z_1 = Z_2 = I \).

**Lemma 5** The Khatri-Rao product of operators is jointly continuous.

Proof: It follows from (1) and the continuity of the Tracy-Singh product (Lemma 1). \( \square \)

For each \( i = 1, \ldots, k \), let \( \mathcal{H}_i \) and \( \mathcal{H}_i' \) be Hilbert spaces and decompose

\[
\mathcal{H}_i = \bigoplus_{r=1}^{n_i} \mathcal{H}_{i,r}, \quad \mathcal{H}_i' = \bigoplus_{s=1}^{m_i} \mathcal{H}_{i,s},
\]

where all \( \mathcal{H}_{i,r} \) and \( \mathcal{H}_{i,s} \) are Hilbert spaces. For a finite number of operator matrices \( A_i \in \mathbb{B}(\mathcal{H}_i, \mathcal{H}_i') \) for \( i = 1, \ldots, k \), we use the following notations,

\[
\bigotimes_{i=1}^{k} A_i = ((A_1 \otimes A_2) \otimes \cdots \otimes A_{k-1}) \otimes A_k,
\]

\[
\bigodot_{i=1}^{k} A_i = ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{k-1}) \boxtimes A_k.
\]
Lemma 6  There are isometries $Z_1$ and $Z_2$
\[
\begin{align*}
\otimes_{i=1}^k A_i &= Z_1^* \left( \prod_{i=1}^k A_i \right) Z_2 \quad (2)
\end{align*}
\]for any $A_i \in \mathcal{B}(\mathcal{H}_i^r, \mathcal{H}_i^s)$, $i = 1, \ldots, k$. If $\mathcal{H}_i$ and $\mathcal{H}_i^r$ are the same space for all $i$, the $Z_1 = Z_2 := Z$.

Proof: We proceed by induction on $k$. If $k = 2$, the property (2) is true by Lemma 4. Suppose that there exist isometries $R_1$ and $R_2$ such that
\[
\begin{align*}
\otimes_{i=1}^{k-1} A_i &= R_1 \left( \prod_{i=1}^{k-1} A_i \right) R_2.
\end{align*}
\]By Lemma 4, there are isometries $S_1, S_2$ such that
\[
\begin{align*}
\otimes_{i=1}^k A_i &= S_1^* \left( \prod_{i=1}^k A_i \right) \otimes A_k S_2.
\end{align*}
\]Then
\[
\begin{align*}
\otimes_{i=1}^k A_i &= S_1^* \left( \prod_{i=1}^k A_i \right) \otimes A_k S_2 \\
&= S_1^* \left[ R_1^* \left( \prod_{i=1}^{k-1} A_i \right) \otimes A_k \right] S_2 \\
&= S_1^* \left[ R_1^* \left( \prod_{i=1}^{k-1} A_i \right) \otimes A_k \right] S_2 \\
&= \left[ \left( \prod_{i=1}^{k-1} A_i \right) \otimes A_k \right] (R_2 \otimes I) S_2 \\
&= \left[ \left( \prod_{i=1}^{k-1} A_i \right) \otimes A_k \right] (R_2 \otimes I) S_2.
\end{align*}
\]Set $Z_1 = (R_1 \otimes I) S_1$ and $Z_2 = (R_2 \otimes I) S_2$. Then $Z_1$ and $Z_2$ are isometries. When $\mathcal{H}_i = \mathcal{H}_i^r$ for all $i = 1, \ldots, k$, we have $Z_1 = Z_2$ from the construction. \(\square\)

CONVEXITY AND CONVEXITY

In this section, we provide concavity and convexity theorems related to Tracy-Singh products of operators. First of all, recall the following terminologies.

Definition 2  A function $f : (0, \infty) \to (0, \infty)$ is said to be operator-monotone if $f[A] \geq f[B]$ whenever $A \geq B > 0$. Here, $f[A]$ is the (continuous) functional calculus of $f$ defined on the spectrum of $A$.

Definition 3  Let $\mathcal{H}_1, \ldots, \mathcal{H}_k$ be Hilbert spaces. For each $i = 1, \ldots, k$, let $E_i$ be a convex subset of $\overline{\mathcal{B}}(\mathcal{H}_i)$. A function $\phi : E_1 \times \cdots \times E_k \to \overline{\mathcal{B}}(\mathcal{H})$ is said to be concave if
\[
\phi((1-t)A_1 + tB_1, \ldots, (1-t)A_k + tB_k) \\
\leq (1-t)\phi(A_1, \ldots, A_k) + t\phi(B_1, \ldots, B_k)
\]
for any $A_i, B_i \in E_i$ $(i = 1, \ldots, k)$ and $t \in (0, 1)$. A function $\phi$ is convex if $-\phi$ is concave. A map between two convex sets is said to be affine if it preserves convex combinations.

Recall that, for each $t \in (0, 1)$, the $t$-weighted harmonic mean and the $t$-weighted geometric mean of $A, B \in \mathcal{B}(\mathcal{H})^{++}$ is defined respectively by
\[
\begin{align*}
A^t B &= [(1-t)A^{-1} + tB^{-1}]^{-1}, \\
A#_t B &= A^{1/(2(A^{-1/2}BA^{-1/2})^t)} A^{1/2}.
\end{align*}
\]For arbitrary $A, B \in \mathcal{B}(\mathcal{H})^+$, we define the $t$-weighted geometric mean of $A$ and $B$ to be
\[
A#_t B = \lim_{\epsilon \to 0} (A + \epsilon I)#_1(B + \epsilon I),
\]
where the limit is taken in the strong-operator topology.

Lemma 7 (Ref. 11)  For each $t \in [0, 1]$, the map $(A, B) \mapsto A^t B$ is concave on $\mathcal{B}(\mathcal{H})^{++} \times \mathcal{B}(\mathcal{H})^{++}$.

The next lemma gives an integral representation of operator-monotone functions on $(0, \infty)$ in terms of Borel measures on $[0, 1]$.

Lemma 8 (Ref. 12)  Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. Then there is a finite Borel measure $\mu$ on $[0, 1]$ such that
\[
\begin{align*}
f(x) = \int_0^1 x \, d\mu(t), \quad x > 0. \quad (3)
\end{align*}
\]

Theorem 1  Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. If $\phi_1 : \mathcal{B}(\mathcal{H})^{++} \to \mathcal{B}(\mathcal{H})^{++}$ and $\phi_2 : \mathcal{B}(\mathcal{H})^{++} \to \mathcal{B}(\mathcal{H})^{++}$ are concave maps, then the maps
\[
\begin{align*}
(A, B) &\mapsto f(\phi_1(A) \otimes \phi_2(B)^{-1}) \cdot (I \otimes \phi_2(B)), \quad (4) \\
(A, B) &\mapsto f(\phi_1(A)^{-1} \otimes \phi_2(B)) \cdot (\phi_1(A) \otimes I) \quad (5)
\end{align*}
\]
are concave on $\mathcal{B}(\mathcal{H})^{++} \times \mathcal{B}(\mathcal{H})^{++}$.

Proof: Let $A \in \mathcal{B}(\mathcal{H})^{++}$ and $B \in \mathcal{B}(\mathcal{H})^{++}$. Then $\phi_1(A) > 0$ and $\phi_2(B) > 0$. Lemma 1 implies that $f(\phi_1(A) \otimes \phi_2(B)^{-1})$ and $f(\phi_1(A)^{-1} \otimes \phi_2(B))$ are well-defined operators. By Lemma 8, there is a finite Borel measure $\mu$ on $[0, 1]$ such that (3) holds. Using Bochner integration, we have
\[
\begin{align*}
f(\phi_1(A) \otimes \phi_2(B)^{-1}) \cdot (I \otimes \phi_2(B)) \\
&= \int_0^1 \{((I \otimes I) \mathcal{E}(\phi_1(A) \otimes \phi_2(B)^{-1})) \cdot (I \otimes \phi_2(B))\} \, d\mu(t).
\end{align*}
\]
For each $t \in [0, 1]$, by Lemma 1 we obtain
\[
\{(I \otimes I) !, (\phi_1(A) \otimes \phi_2(B)^{-1})\} \cdot (I \otimes \phi_2(B)) = \left[ (1 - t)(I \otimes I) + t(\phi_1(A) \otimes \phi_2(B)^{-1}) \right]^{-1} \\
\cdot (I \otimes \phi_2(B)) = (I \otimes \phi_2(B))^{-1} \\
\cdot \left[ (1 - t)(I \otimes \phi_2(B))^{-1} + t(\phi_1(A) \otimes I)^{-1} \right]^{-1} = (I \otimes \phi_2(B)) \cdot ((\phi_1(A) \otimes I).
\]
Since the weighted harmonic mean is concave (Lemma 7), so is the map
\[
(A, B) \rightarrow \{(I \otimes I) !, (\phi_1(A) \otimes \phi_2(B)^{-1})\} \cdot (I \otimes \phi_2(B)).
\]
Thus the map (4) is concave. Similarly, the map (5) is concave.

**Remark 1** Since $\phi_1(A) \otimes \phi_2(B)^{-1}$ commutes with $I \otimes \phi_2(B)$, we have
\[
f[\phi_1(A) \otimes \phi_2(B)^{-1}] \cdot (I \otimes \phi_2(B)) = (I \otimes \phi_2(B)) \cdot f[\phi_1(A) \otimes \phi_2(B)^{-1}] .
\]
Similarly,
\[
f[\phi_1(A)^{-1} \otimes \phi_2(A)] \cdot (I \otimes I) = (\phi_1(A) \otimes I) \cdot f[\phi_1(A)^{-1} \otimes \phi_2(B)].
\]

**Example 1** Recall that the function $t \rightarrow t^p$ is operator-monotone for any $0 \leq p \leq 1$. Given two concave maps $\phi_1 : \mathbb{B}(\mathcal{H})^+ \rightarrow \mathbb{B}(\mathcal{H})^+$ and $\phi_2 : \mathbb{B}(\mathcal{H})^+ \rightarrow \mathbb{B}(\mathcal{H})^+$, by Theorem 1 the maps
\[
(A, B) \rightarrow [f[\phi_1(A)^{-1} \otimes \phi_2(B)] \cdot (I \otimes \phi_2(B)),
\]
\[
(A, B) \rightarrow [f[\phi_1(A) \otimes \phi_2(B)^{-1}] \cdot (\phi_1(A) \otimes I)]
\]
are concave on $\mathbb{B}(\mathcal{H})^+ \times \mathbb{B}(\mathcal{H})^+$.

**Corollary 1** Let $f : (0, \infty) \rightarrow (0, \infty)$ be operator-monotone. If $\phi_1 : \mathbb{B}(\mathcal{H})^+ \rightarrow \mathbb{B}(\mathcal{H})^+$ and $\phi_2 : \mathbb{B}(\mathcal{H})^+ \rightarrow \mathbb{B}(\mathcal{H})^+$ are concave maps, then the maps
\[
(A, B) \rightarrow f[\phi_1(A)^{-1} \otimes \phi_2(B)] \cdot (I \otimes \phi_2(B)^{-1}),
\]
\[
(A, B) \rightarrow f[\phi_1(A) \otimes \phi_2(B)^{-1}] \cdot (\phi_1(A) \otimes I)
\]
are convex on $\mathbb{B}(\mathcal{H})^+ \times \mathbb{B}(\mathcal{H})^+$.

**Proof:** Note that the function $g(x) := f(x^{-1})$ is operator-monotone. By Lemma 1, we have
\[
f[\phi_1(A)^{-1} \otimes \phi_2(B)] \cdot (I \otimes \phi_2(B)^{-1}) = g[\phi_1(A)^{-1} \otimes \phi_2(B)]^{-1} (I \otimes \phi_2(B)^{-1}).
\]

**Theorem 1** implies the concavity of the map
\[
(A, B) \rightarrow g[\phi_1(A)^{-1} \otimes \phi_2(B)] \cdot (I \otimes \phi_2(B)) = \{(I \otimes \phi_2(B))^{-1} \cdot f[\phi_1(A)^{-1} \otimes \phi_2(B)]^{-1} = \{(I \otimes \phi_2(B))^{-1} \cdot f[\phi_1(A)^{-1} \otimes \phi_2(B)]^{-1}.
\]
Thus the map (6) is convex. Similarly, the map (7) is convex.

**Theorem 2** Let $f : (0, \infty) \rightarrow (0, \infty)$ be an operator-monotone function. If $\phi_1 : \mathbb{B}(\mathcal{H})^+ \rightarrow \mathbb{B}(\mathcal{H})^+$ is a concave map and $\phi_2 : \mathbb{B}(\mathcal{H})^+ \rightarrow \mathbb{B}(\mathcal{H})^+$ is an affine map, then the maps
\[
(A, B) \rightarrow f[\phi_1(A)^{-1} \otimes \phi_2(B)] \cdot (I \otimes \phi_2(B)),
\]
\[
(A, B) \rightarrow f[\phi_2(B)^{-1} \otimes \phi_1(A)] \cdot (\phi_2(B) \otimes I)
\]
are convex on $\mathbb{B}(\mathcal{H})^+ \times \mathbb{B}(\mathcal{H})^+$.

**Proof:** By Lemma 8, there is a finite Borel measure $\mu$ on $[0, 1]$ such that (3) holds. Then
\[
f[\phi_1(A)^{-1} \otimes \phi_2(B)] \cdot (I \otimes \phi_2(B)) = \int_0^1 \{(I \otimes I) !, (\phi_1(A)^{-1} \otimes \phi_2(B)) \} \cdot (I \otimes \phi_2(B)) \, d\mu(t).
\]
For each $t \in [0, 1]$, it follows from Lemma 1 that
\[
\{(I \otimes I) !, (\phi_1(A)^{-1} \otimes \phi_2(B)) \} = \left[ (1 - t)(I \otimes I) + t(\phi_1(A)^{-1} \otimes \phi_2(B)^{-1}) \right]^{-1}
\]
\[
= \left[ (1 - t)(I \otimes I) + t(\phi_1(A) \otimes \phi_2(B)^{-1}) \right]^{-1}
\]
\[
= (I \otimes \phi_2(B)) \left[ (1 - t)(I \otimes \phi_2(B)) + t(\phi_1(A) \otimes I) \right]^{-1}.
\]
The concavity of the map $(A, B) \rightarrow (1 - t)(I \otimes \phi_2(B)) + t(\phi_1(A) \otimes I)$ and the affinity of the map $(A, B) \rightarrow I \otimes \phi_2(B)$ together yield the convexity of the map
\[
(A, B) \rightarrow (I \otimes \phi_2(B)) \left[ (1 - t)(I \otimes \phi_2(B)) + t(\phi_1(A) \otimes I) \right]^{-1} (I \otimes \phi_2(B))
\]
\[
= \{(I \otimes I) !, (\phi_1(A)^{-1} \otimes \phi_2(B)) \} (I \otimes \phi_2(B)).
\]
Hence the map (8) is convex. Similarly, the map (9) is convex.

**Corollary 2** The maps
\[
(A, B) \rightarrow I \otimes (B \log[B]) - \log[A] \otimes B,
\]
\[
(A, B) \rightarrow (A \log[A]) \otimes I - A \otimes \log[B]
\]
are convex on $\mathbb{B}(\mathcal{H})^+ \times \mathbb{B}(\mathcal{H})^+$. 

www.scienceasia.org
Proof: Using Lemmas 1 and 2, we obtain
\[
I \otimes (B \log[B]) - \log[A] \otimes B = \{ I \otimes \log[B] - \log[A] \otimes I \} \cdot ( I \otimes B ) = \{ \log[I \otimes B] - \log[A \otimes I] \} \cdot ( I \otimes B ) = \log((I \otimes B)(A \otimes I))^{-1} \cdot (I \otimes B) = (I \otimes B) \cdot \log[A^{-1} \otimes B].
\]
Since \( \log x \) is operator-monotone, by Theorem 2 we obtain that the map
\[
(A, B) \mapsto \log(A^{-1} \otimes B) \cdot (I \otimes B)
\]
is convex. Hence the map (10) is convex. Similarly, the map (11) is convex. \( \square \)

Example 2 Let \( \phi_1 : \mathcal{B}(\mathcal{H})^{++} \to \mathcal{B}(\mathcal{H}')^{++} \) be a concave map and \( \phi_2 : \mathcal{B}(\mathcal{H}')^{++} \to \mathcal{B}(\mathcal{H}')^{++} \) an affine map. For any \( 0 < p < 1 \), we have by Theorem 2 that the maps
\[
(A, B) \mapsto [\phi_1(A)^{-1} \otimes \phi_2(B)]^p \cdot (I \otimes \phi_2(B)),
\]
\[
(A, B) \mapsto [\phi_2(B) \otimes \phi_1(A)^{-1}]^p \cdot (\phi_2(B) \otimes I)
\]
are convex on \( \mathcal{B}(\mathcal{H})^{++} \times \mathcal{B}(\mathcal{H}')^{++} \).

We mention that the maps (5), (7), (9) and (11) are extensions of results discussed in Ref. 7.

CONCAVITY THEOREMS FOR TRACY-SINGH AND KHATRI-RAO PRODUCTS

In this section, we present concavity theorems for Tracy-Singh products of operators. Concavity theorems for Khatri-Rao products of operators are established by using the concavity theorems for Tracy-Singh products and the connection between the Khatri-Rao and Tracy-Singh products.

The next result generalizes Corollary 6.2 of Ref. 1 to the case of Tracy-Singh product of operators.

Theorem 3 Let \( 0 < p_i \leq 1, \ i = 1, \ldots, k \), be such that \( \sum_{i=1}^{k} p_i \leq 1 \). Then the map
\[
(A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^{k} A_i^{p_i}
\]
is concave on \( \mathcal{B}(\mathcal{H}_1)^{++} \times \cdots \times \mathcal{B}(\mathcal{H}_k)^{++} \).

Proof: We proceed by induction on \( k \). Clearly, the map \( A_1 \mapsto A_1^{p_1} \) is concave. Suppose the assertion is generally true for the case \( k - 1 \). If \( p_k = 0 \), then the map becomes
\[
(A_1, \ldots, A_k) \mapsto ((A_1 \otimes A_2) \otimes \cdots \otimes A_{k-1}) \otimes I,
\]
which is concave. If \( p_k = 1 \), then \( p_i = 0 \) for all \( i = 1, \ldots, k - 1 \) and the map is clearly concave. Now suppose \( 0 < p_k < 1 \). By the induction assumption, the map
\[
\phi(A_1, \ldots, A_{k-1}) = \bigotimes_{i=1}^{k-1} A_i^{p_i/(1-p_k)}
\]
is concave. By applying Theorem 1 with \( f(x) = x^{p_k} \), the map
\[
(A_1, \ldots, A_k) \mapsto \phi(A_1, \ldots, A_{k-1})^\ast \otimes A_k \cdot (\phi(A_1, \ldots, A_{k-1}) \otimes I)
\]
is concave. We obtain the concavity of the map (12),
\[
f(\phi(A_1, \ldots, A_{k-1})^{-1} \otimes A_k)(\phi(A_1, \ldots, A_{k-1}) \otimes I)
\]
\[
= (\phi(A_1, \ldots, A_{k-1})^{-p_k} \otimes A_k^{p_k})(\phi(A_1, \ldots, A_{k-1}) \otimes I)
\]
\[
= \phi(A_1, \ldots, A_{k-1})^{-p_k} \otimes A_k^{p_k} = \bigotimes_{i=1}^{k} A_i^{p_i}
\]
A special case of Theorem 3 is when \( k = 2 \).

Corollary 3 For each \( r \in (0, 1) \), the map
\[
(A, B) \mapsto A^{1-r} \otimes B^r
\]
is concave on \( \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H}')^+ \).

Proof: Theorem 3 implies that the map (13) is concave on \( \mathcal{B}(\mathcal{H})^{++} \times \mathcal{B}(\mathcal{H}')^{++} \). Since the Tracy-Singh product is jointly continuous (Lemma 1), this map is also concave on \( \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H}')^+ \). \( \square \)

Next, we develop concavity theorems for Khatri-Rao products of operators.

Theorem 4 Let \( 0 \leq p_i \leq 1, \ i = 1, \ldots, k \), be such that \( \sum_{i=1}^{k} p_i \leq 1 \). Then the map
\[
(A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^{k} A_i^{p_i}
\]
is concave on \( \mathcal{B}(\mathcal{H}_1)^{++} \times \cdots \times \mathcal{B}(\mathcal{H}_k)^{++} \).

Proof: From Lemma 6, the map \( X \mapsto Z^* X Z \), taking the Tracy-Singh product \( \bigotimes_{i=1}^{k} A_i \) into the Khatri-Rao product \( \bigotimes_{i=1}^{k} A_i \), is linear and preserves positivity. Recall that the composition between a linear map and a concave map results in a concave map. Since the map \( (A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^{k} A_i^{p_i} \) is concave by Theorem 3, we have the concavity of the map is concave. We obtain the concavity of the map from (12), since
\[
(A_1, \ldots, A_k) \mapsto Z^* \bigotimes_{i=1}^{k} A_i^{p_i} Z = \bigotimes_{i=1}^{k} A_i^{p_i}
\]
\( \square \)
Corollary 4 For each \( r \in (0, 1) \), the map
\[
(A, B) \mapsto A^{-r} \otimes B^r,
\]
is concave on \( \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+ \).

Proof: It follows from Theorem 4 when \( k = 2 \) together with the continuity of the Khatri-Rao product, Lemma 5.

CONVEXITY THEOREMS FOR TRACY-SINGH
AND KHATRI-RAO PRODUCTS

In this section, we establish convexity theorems for Tracy-Singh products and Khatri-Rao products of operators. Weighted arithmetic/geometric/harmonic means of operators serve as useful tools.

Lemma 9 (Ref. 13) Let \( A_i, B_i \in \mathcal{B}(\mathcal{H})^+, 1 \leq i \leq k \). Then
\[
\bigotimes_{i=1}^k A_i \otimes \bigotimes_{i=1}^k B_i = \bigotimes_{i=1}^k (A_i \otimes B_i).
\]

Theorem 5 Let \( \phi_i, i = 1, \ldots, k, \) be a concave map from \( \mathcal{B}(\mathcal{H})^+ \to \mathcal{B}(\mathcal{H})^+ \). Then the map
\[
(A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^k \phi_i(A_i)^{-1}
\]
is convex on \( \mathcal{B}(\mathcal{H}_1)^+ \times \cdots \times \mathcal{B}(\mathcal{H}_k)^+ \).

Proof: Let \( t \in [0, 1] \). By continuity, we may assume that \( A_i \) and \( B_i \) are positive invertible operators. Applying Lemmas 1 and 9 and the arithmetic-geometric means inequality for operators, we have
\[
\left[ \bigotimes_{i=1}^k \phi_i((1-t)A_i + tB_i)^{-1} \right]^r \leq \bigotimes_{i=1}^k ((1-t)\phi_i(A_i) + t\phi_i(B_i))^{-1} \leq \bigotimes_{i=1}^k (\phi_i(A_i) \otimes \phi_i(B_i))^{-1} \leq (1-t) \bigotimes_{i=1}^k \phi_i(A_i)^{-1} + t \bigotimes_{i=1}^k \phi_i(B_i)^{-1}.
\]

Hence the map (15) is convex.

Corollary 5 Let \( 0 < p_i \leq 1, i = 1, \ldots, k \). Then the map
\[
(A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^k A_i^{-p_i}
\]
is convex on \( \mathcal{B}(\mathcal{H}_1)^+ \times \cdots \times \mathcal{B}(\mathcal{H}_k)^+ \).

Proposition 1 Let \( 0 \leq p_i \leq 1, i = 1, \ldots, k, \) and \( 1 \leq q \leq 2 \) be such that \( \sum_{i=1}^k p_i \geq q-1 \). Then the map
\[
(A_1, \ldots, A_{k+1}) \mapsto \left( \bigotimes_{i=1}^k A_i^{-p_i} \right) \otimes A_{k+1}^q
\]
is convex on \( \mathcal{B}(\mathcal{H}_1)^+ \times \cdots \times \mathcal{B}(\mathcal{H}_{k+1})^+ \).

Proof: By Theorem 3, the map
\[
(A_1, \ldots, A_{k+1}) \mapsto \left( \bigotimes_{i=1}^k A_i^q \right) \otimes A_{k+1}^{2-q}
\]
is convex on \( \mathcal{B}(\mathcal{H}_1)^+ \times \cdots \times \mathcal{B}(\mathcal{H}_{k+1})^+ \). Clearly, the map
\[
(A_1, \ldots, A_{k+1}) \mapsto \left( \bigotimes_{i=1}^k \right) I \otimes A_{k+1}
\]
is affine. It follows from Lemma 1 that the map
\[
(A_1, \ldots, A_{k+1}) \mapsto \left( \bigotimes_{i=1}^k A_i^q \right) \otimes A_{k+1}^{2-q}
\]
is convex.

Theorem 6 For each \( r \in (0, 1) \), the maps
\[
(A, B) \mapsto A^{-r} \otimes B^{1+r},
\]
\[
(A, B) \mapsto A^{1+r} \otimes B^{-r}
\]
are convex on \( \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+ \).

Proof: The convexity of the map (16) follows from Proposition 1. By continuity, we may assume that \( A \) and \( B \) are invertible. Lemma 1 implies that
\[
A^{1+r} \otimes B^{-r} = (A' \otimes B^{-r})(A \otimes I) = (A \otimes B^{-1})'(A \otimes I).
\]

It follows from Lemmas 1 and 8 that
\[
A^{1+r} \otimes B^{-r} = \int_0^1 ((I \otimes I)_t (A \otimes B^{-1})) d\mu(t) (A \otimes I)
\]
\[
= \int_0^1 ((I \otimes I) + t(A \otimes B^{-1}))^{-1} (A \otimes I) d\mu(t)
\]
\[
= \int_0^1 ((I \otimes I) + t(A' \otimes B))^{-1} (A \otimes I) d\mu(t)
\]
\[
= \int_0^1 (A \otimes I)((I \otimes I) + t(I \otimes B))^{-1} (A \otimes I) d\mu(t).
\]
Since the map $A \mapsto A^{-1}$ is convex and the map $(A,B) \mapsto (1-t)(A \otimes I) + t(I \otimes B)$ is affine, the map $(A,B) \mapsto (1-t)(A \otimes I) + t(I \otimes B) \otimes I (I \otimes B)$ is convex. Thus the map $(A,B) \mapsto A^{1+r} \otimes B^{-r}$ is convex.

**Proposition 2** Let $\phi_i, i = 1, \ldots, k$, be convex maps from $B(\mathcal{H})^{++}$ to $B(\mathcal{H}^{''})^{++}$. Then the map

$$(A_1, \ldots, A_k) \mapsto \mathbf{A} \phi_i(A_i)^{-1}$$

is convex on $B(\mathcal{H}_1)^{++} \times \cdots \times B(\mathcal{H}_k)^{++}$.

**Proof:** It follows from Lemma 6 and Theorem 5. $\square$

**Corollary 6** Let $0 < p_i \leq 1$ for each $i = 1, \ldots, k$. Then the map

$$(A_1, \ldots, A_k) \mapsto \mathbf{A}_i^{1-p_i}$$

is convex on $B(\mathcal{H}_1)^{++} \times \cdots \times B(\mathcal{H}_k)^{++}$.

**Proof:** It follows from Proposition 2 by putting $\phi_i(A_i) = A_i^{p_i}$ for each $i$. $\square$

**Proposition 3** For each $r \in (0,1)$, the maps

$$(A,B) \mapsto A^{-r} \otimes B^{1+r},$$

$$(A,B) \mapsto A^{1+r} \otimes B^{-r}$$

are convex on $B(\mathcal{H})^{++} \times B(\mathcal{H})^{''}$.

**Proof:** It follows from Lemma 4 and Theorem 6. $\square$

Recall that the Moore-Penrose inverse of an operator $T \in B(\mathcal{H}, \mathcal{H}^{'})$ is the operator $T^+ \in B(\mathcal{H}, \mathcal{H})$ satisfying the conditions $TT^+T = T$, $T^+TT^+ = T$, $(TT^+)^* = TT^+$, and $(T^+T)^* = T^+T$. It is well known that $T^+$ exists if and only if the range of $T$ is closed.

**Lemma 10 (Ref. 15)** Let

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

be a self-adjoint operator. Suppose that $T_{11}$ has a closed range. Then $T \geq 0$ if and only if $T_{11} \succ 0$, $T_{12} = T_{11}^{1/2}T_{12}$ and $T_{22} \geq T_{21}^{1/2}T_{12}$.

Recall that for any interval $J$, a continuous function $f : J \rightarrow \mathbb{R}$ is convex if and only if $f(x + h) + f(x - h) - 2f(x) \geq 0$ for all $x \in J$ and $h > 0$ such that $x \pm h \in J$.

**Theorem 7** Let $A \in B(\mathcal{H})^+$ and $B \in B(\mathcal{H})^+$ have closed ranges. Then the operator-valued function

$$\phi : [-1, 1] \rightarrow B(\mathcal{H}_1 \otimes \mathcal{H}_2),$$

$$\phi(t) = A^{1+t} \otimes B^{-1-t} + A^{-1-t} \otimes B^{1+t}$$

is convex on $[-1,1]$, decreasing on $[-1,0]$, increasing on $[0,1]$, attains minimality at $t = 0$, and attains maximality at $t = -1, 1$.

**Proof:** Let $s \in [-1,1]$ and $t > 0$ be such that $s \pm t \in [-1,1]$. Consider the operator matrices

$$T_1 = \begin{bmatrix} A^{1+s+t} & A^{1+s} \\ A^{1+s} & A^{1+s-t} \end{bmatrix},$$

$$T_2 = \begin{bmatrix} A^{1-s-t} & A^{1-s} \\ A^{1-s} & A^{1+s-t} \end{bmatrix},$$

$$T_3 = \begin{bmatrix} B^{1+s+t} & B^{1+s} \\ B^{1+s} & B^{1+s-t} \end{bmatrix},$$

$$T_4 = \begin{bmatrix} B^{1-s-t} & B^{1-s} \\ B^{1-s} & B^{1+s-t} \end{bmatrix}.$$
Corollary 7 Let $A \in \mathcal{B}(\mathcal{H})^+$ and $B \in \mathcal{B}(\mathcal{H})^+$ have closed ranges. Then the parameterization

$$
\psi : [0, 1) \to \mathcal{B}\left(\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{H}_i\right),
$$

$$
\psi(t) = A^t \boxplus B^{1-t} + A^{1-t} \boxplus B^t
$$

is convex on $[0, 1)$, decreasing on $[0, 1/2)$, increasing on $[1/2, 1)$, attains minimality at $t = 1/2$, and attains maximality at $t = 0, 1$.

Proof: Let $f : [0, 1) \to [-1, 1]$ be defined by $f(t) = 2t - 1$. Then $\psi = \phi \circ f$ where $\phi$ is given by (18).

Now, the desired results follow from Theorem 7 by using $f((0, 1)) = [-1, 1]$, $f((0, 1/2)) = [-1, 0]$, $f([1/2, 0)) = [0, 1]$, and $f(1/2) = 0$.

As a consequence, we obtain an operator version of the arithmetic-geometric mean inequality as follows.

Corollary 8 Let $A \in \mathcal{B}(\mathcal{H})^+$ and $B \in \mathcal{B}(\mathcal{H})^+$ have closed ranges. For any $t \in [1/2, 1)$, we have

$$
2(A^{1/2} \boxplus B^{1/2}) \leq A^t \boxplus B^{1-t} + A^{1-t} \boxplus B^t \leq A \boxplus B,
$$

where $\boxplus$ denotes the Khatri-Rao sum defined by $A \boxplus B = A \boxtimes I + I \boxtimes B$.

We mention that Theorem 5, Corollary 5, and Proposition 1 generalize the matrix results involving Tracy-Singh products provided in Ref. 5.

Acknowledgements: The first author expresses his gratitude towards Thailand Research Fund for providing the Royal Golden Jubilee PhD Scholarship, grant No. PHD60K0225 to support his PhD study.

REFERENCES