

A new necessary and sufficient condition for the simultaneous stabilization of three time-varying linear systems

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ABSTRACT: This paper considers the simultaneous stabilization problems of three discrete-time time-varying linear systems within the framework of nest algebra. Under the transmission condition, a necessary and sufficient condition for the existence of simultaneously stabilizing controllers is given based on the strong representation. Some simultaneous stabilizability criteria are established in terms of single strong representation of only one controller in the transmission condition.

KEYWORDS: simultaneous stabilization, transitivity, nest algebra, strong representation, time-varying system

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INTRODUCTION

The stability for different types of discrete-time time-varying linear systems are studied in Refs. 1–4. The control theory of time-varying linear systems in the framework of nest algebra, which was sparked and developed based on the nest algebra theory in 1980s, is a generalization of H^∞ control in the sense that 2-norm is used to quantify the size of signals. In the context of operator theory, the time-varying linear system can be seen as a (possibly unbounded) infinite-dimensional lower triangular matrix. The stability of the system is equivalent to the boundedness of the infinite-dimensional matrix. This approach is purely operator theoretic and does not use any state space realization. This interpretation of dynamical systems has generated significant research interest over the last decade, as witnessed by recent contributions^{5–7}.

The simultaneous stabilization problem concerns the design of a common controller that would stabilize a finite set of plants, and it was first introduced by Saeks et al⁸ and Viswanadham⁹. The simultaneous stabilization problem for three systems is recognized as one of hard open problems in linear system theory. Few criteria for simultaneous stabilizability appear without any restrictions to the given plants. The traditional necessary and sufficient conditions for simultaneous stabilization of time-varying linear systems are all

characterized in the form of strong representations of all given plants^{10,11}. However, the problem of how to compute strong representations of a general time-varying linear system remains unsolved. Motivated by practical interest, some new simultaneous stabilizability criteria with weaker condition about strong representations of the plants are needed. In 2011, Yu and Han brought out the so-called transitivity in simultaneous stabilization for three time-varying linear systems, which, when satisfied, will lead to characterizations of simultaneous stabilizability and simultaneously stabilizing feedback controllers¹². In other words, the transitivity problem restricts the search for the existence of a simultaneously stabilizing controller to the three plants satisfying the so-called transmission condition. The transmission condition for three given plants L_0, L_1, L_2 is that L_0, L_1 can be simultaneously stabilized by some system C_0 , and L_1, L_2 can be simultaneously stabilized by some system C_1 . It was pointed out that the transmission condition really is a weaker replacement of the strong representation condition in the study of simultaneous stabilization problem¹³. Under the transmission condition, a necessary and sufficient condition for the simultaneous stabilizability of three plants L_0, L_1 , and L_2 was given, and the strong representation of L_2 was avoided¹³. Some sufficient conditions for the simultaneous stabilizability are given in Ref. 14—those

conditions do not employ strong representations of L_0 and L_2 . As an improvement of the previous work, an equivalent condition not depending on the strong representations of L_0 or L_2 will be derived¹⁵. Up to the present, some criteria for simultaneous stabilizability of three plants have been given within framework of nest algebra. While, those results were all established in terms of strong representations of at least one plant, it is natural to ask whether the simultaneous stabilization can be characterized by strong representations of fewer given plants or fewer controllers.

The main goal of this study is to rebuild the simultaneous stabilization criteria of three plants under the transmission condition. Compared with the sufficient conditions in Refs. 12, 14, one of main contributions here is establishing a necessary and sufficient condition to the transitivity of time-varying linear systems. The early equivalent conditions to the simultaneous stability depend on at least one strong representation of the considering plants and one of the controllers^{7,13,15}, while the simultaneous stability criteria depend only on the single strong representation of one controller, which avoids any information about strong representations of each given plant. Additionally, a parametrization for all simultaneous stabilizers is derived under the restrictive transmission condition. In particular, a simultaneous stabilization criterion avoiding any strong representation can be established when one of the stabilizing controllers becomes stable.

PRELIMINARIES

The signal space considered in this paper is the complex separable Hilbert space

$$\mathcal{H} = \left\{ (x_0, x_1, \dots) : x_i \in \mathbb{C}, \sum_{i=0}^{\infty} |x_i| < \infty \right\},$$

and $\mathcal{H}_e = \{(x_0, x_1, \dots) : x_i \in \mathbb{C}\}$ denotes the extended space of \mathcal{H} .

Let $\mathcal{B}(\mathcal{H})$ denote the Banach space of bounded linear operators on \mathcal{H} , endowed with the operator norm

$$\|T\| = \sup_{x \in \mathcal{H}, \|x\| \leq 1} \|Tx\|.$$

For $T \in \mathcal{B}(\mathcal{H})$, $\text{Im } T$ denotes the range of T and $\text{Ker } T$ denotes the kernel of T .

For each $n \geq 0$, we denote by P_n the standard truncation projection on \mathcal{H} or \mathcal{H}_e as

$$P_n(x_0, x_1, \dots, x_n, \dots) = (x_0, x_1, \dots, x_n, 0, \dots)$$

with $P_{-1} = 0$ and $P_{\infty} = I$.

$\{P_n : -1 \leq n \leq \infty\}$ is used to define the physical definition of causality. A linear transformation L on \mathcal{H}_e is causal if $P_n L P_n = P_n L$ for each n . L is a (time-varying) linear system if it is a causal linear transformation on \mathcal{H}_e and continuous with respect to the standard seminorm topology¹¹. Indeed, any linear system is an infinite-dimensional lower triangular matrix with respect to the standard basis of \mathcal{H} . A linear system is stable if its restriction to \mathcal{H} is a bounded operator.

Let \mathcal{L} be the algebra of linear systems with respect to the standard addition and multiplication. The set of stable ones, denoted by \mathcal{S} , referred to in the operator theory literature as a nest algebra determined by the discrete nest¹¹ $\{I - P_n : -1 \leq n \leq \infty\}$. \mathcal{S} is a non-self adjoint sub-algebra of the Banach algebra $\mathcal{B}(\mathcal{H})$, closed in the weak operator topology.

The definition of stabilizability and its characterization in terms of strong representations are given as follows.

Consider the feedback configuration contributed by the plant $L \in \mathcal{L}$ and the controller $C \in \mathcal{L}$, where the closed-loop equation is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} I & C \\ -L & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The closed-loop system $\{L, C\}$ is well posed if the internal input $[e_1 \ e_2]^T$ can be expressed as a causal function of the external input $[u_1 \ u_2]^T$. This is equivalent to requiring that $\begin{bmatrix} I & C \\ -L & I \end{bmatrix}$ being invertible.

The inverse is given by the transfer matrix

$$H(L, C) = \begin{bmatrix} (I + CL)^{-1} & -C(I + LC)^{-1} \\ L(I + CL)^{-1} & (I + LC)^{-1} \end{bmatrix}.$$

Definition 1 (1) The closed-loop system $\{L, C\}$ is stable if each entry of $H(L, C)$ belongs to \mathcal{S} .

(2) The plant L is stabilizable if there exists a causal system $C \in \mathcal{L}$ such that $\{L, C\}$ is stable. In this case, C is said to be a stabilizing controller for L .

(3) A family of plants L_0, L_1, \dots, L_n are simultaneously stabilizable if they admits a common stabilizing controller.

Definition 2 [Ref. 16] Let $M, N, \hat{M}, \hat{N} \in \mathcal{S}$ and $L \in \mathcal{L}$.

(1) $[MN]^T$ is a strong right representation of L if $\mathcal{G}(L) = \text{Im}[MN]^T$ and there exist $X, Y \in \mathcal{S}$ such that $[YX][MN]^T = I$.

(2) $[-\hat{N} \hat{M}]$ is a strong left representation of L if $\mathcal{G}(L) = \text{Ker}[-\hat{N} \hat{M}]$ and there exist $\hat{X}, \hat{Y} \in \mathcal{S}$ such that $[-\hat{N} \hat{M}][\hat{X} \hat{Y}]^T = I$, where $\mathcal{G}(L)$ is the graph of L , i.e., $\mathcal{G}(L) = \{[x \ Lx]^T : x \in \mathcal{H}, Lx \in \mathcal{H}\}$.

Remark 1 In fact, the strong representation is an alternative, but equivalent, approach to the coprime factorization in stability theory of linear systems. More precisely, $[MN]^T$ is a strong right representation of L if and only if $L = NM^{-1}$ with $M^{-1} \in \mathcal{L}$ and M, N are right-coprime. Similarly, $[-\hat{N} \hat{M}]$ is a strong left representation of L if and only if $L = \hat{M}^{-1}\hat{N}$ with $\hat{M}^{-1} \in \mathcal{L}$ and \hat{M}, \hat{N} are left-coprime.

Let \mathcal{A} be a Banach algebra containing identity and $\mathcal{A}^{n \times m}$ be the set consisting of all the $n \times m$ matrices with the elements in \mathcal{A} .

Definition 3 [Ref. 17] \mathcal{A} is completely finite if every right invertible element $R \in \mathcal{A}^{1 \times n}$ is the first row of an invertible matrix in $\mathcal{A}^{n \times n}$ for all $n \geq 1$. A Banach algebra \mathcal{A} is stably finite if every right (left) invertible element $A \in \mathcal{A}^{n \times n}$ is invertible in $\mathcal{A}^{n \times n}$ for all $n \geq 1$.

Lemma 1 (Ref. 17) If Banach algebra \mathcal{A} is completely finite, then it is stably finite.

Proposition 1 (Ref. 17) The nest algebra \mathcal{S} is completely finite.

The lemmas below are used to formulate the main results shown in the next section.

Lemma 2 (Ref. 15) A linear system $L \in \mathcal{L}$ is stabilizable if and only if L has a strong right representation.

Lemma 3 (Ref. 11) Assume that $[MN]^T$ is a strong right representation of L , and $[VU]^T$ and $[-\hat{U} \hat{V}]$ are the strong left and right representations of C , respectively. Then the followings are equivalent:

- (1) $\{L, C\}$ is stable.
- (2) $\hat{V}M + \hat{U}N$ is invertible in \mathcal{S} .
- (3) $\begin{bmatrix} M & -U \\ N & V \end{bmatrix}$ is invertible in $\mathcal{S}^{2 \times 2}$.

MAIN RESULTS

In this section, we deal with the simultaneous stabilization problem of three given plants L_0, L_1 , and L_2 with restriction of the transmission condition. The following result is an extension of the classical Youla Parametrization Theorem, which is stated in terms of a strong right, but not both, representation.

Lemma 4 A linear system L is stabilizable if and only if L admits a right representation $[MN]^T$ with

$$\begin{bmatrix} M & -\hat{X} \\ N & \hat{Y} \end{bmatrix} \text{invertible in } \mathcal{S}^{2 \times 2} \text{ for some } \hat{X}, \hat{Y} \in \mathcal{S}.$$

A linear system $C \in \mathcal{L}$ stabilizes L if and only if it admits a strong right representation $\begin{bmatrix} \hat{Y} - NQ \\ \hat{X} + MQ \end{bmatrix}$ for some $Q \in \mathcal{S}$.

Proof: According to Lemma 2, L is stabilizable if and only if L has a strong representation $[MN]^T$. By the complete finiteness of \mathcal{S} , $[MN]^T$ is the first column of an invertible matrix in $\mathcal{S}^{2 \times 2}$, $\begin{bmatrix} M & -\hat{X} \\ N & \hat{Y} \end{bmatrix}$. Note that

$$\begin{bmatrix} M & -(\hat{X} + MQ) \\ N & \hat{Y} - NQ \end{bmatrix} = \begin{bmatrix} M & -\hat{X} \\ N & \hat{Y} \end{bmatrix} \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}.$$

By the stable finiteness of \mathcal{S} , it is clear that $\begin{bmatrix} M & -(\hat{X} + MQ) \\ N & \hat{Y} - NQ \end{bmatrix}$ is invertible in $\mathcal{S}^{2 \times 2}$ for every $Q \in \mathcal{S}$. Choose $Q \in \mathcal{S}$ such that $\hat{Y} - NQ$ is invertible in \mathcal{L} , then $\begin{bmatrix} \hat{Y} - NQ \\ \hat{X} + MQ \end{bmatrix}$ is a strong right representation of some linear system $C \in \mathcal{L}$. According to Lemma 3, C is a stabilizing controller for L . The converse is obvious by Lemma 3. \square

In Ref. 13, the authors provide a design method of simultaneously stabilizing controller for three plants L_0, L_1 , and L_2 under the transmission condition, which is characterized in the form of strong representations of the plants L_0, L_1 and the controller C_2 . Next, we give a simpler and useful simultaneous stabilizability criterion of three plants in terms of a strong right representation of only one controller C_2 .

Theorem 1 Suppose that C_0 stabilizes L_0 and L_1 and C_2 stabilizes L_1 and L_2 . Let $[VU]^T$ be a right representation of C_2 with

$$\begin{bmatrix} V & -\hat{X} \\ U & \hat{Y} \end{bmatrix} \text{invertible in } \mathcal{S}^{2 \times 2}.$$

Set $T_1 = (L_1U + V)^{-1}(L_1\hat{Y} - \hat{X})$. Then L_0, L_1 , and L_2 are simultaneously stabilizable if and only if there exists $Q \in \mathcal{S}$ such that

$$\begin{aligned} & \{L_0U + V + (L_0 - L_1)(I + C_0L_1)^{-1}(C_0V - U)\}^{-1} \\ & \times \{L_0U + V + (L_0 - L_1)(\hat{Y} - UT_1)Q\} \end{aligned} \quad (1)$$

and

$$I + (V + L_2U)^{-1}(L_2 - L_1)(\hat{Y} - UT_1)Q \quad (2)$$

are both invertible in \mathcal{S} . Moreover, $C \in \mathcal{L}$ stabilizes L_0, L_1 , and L_2 if and only if C has a strong right

representation $\begin{bmatrix} V - (\hat{Y} - UT_1)Q \\ U + (\hat{X} + VT_1)Q \end{bmatrix}$ for some $Q \in \mathcal{S}$ such that (1) and (2) are invertible in \mathcal{S} .

Proof: Define the operator

$$T_2 = (L_2U + V)^{-1}(L_2\hat{Y} - \hat{X}). \quad (3)$$

Suppose that $[-\hat{N}_{L_j} \hat{M}_{L_j}]$ is a strong left representation of L_j , $j = 0, 1, 2$. For $i = 1, 2$, it follows from the stability of $\{L_i, C_2\}$ that $\hat{N}_{L_i}U + \hat{M}_{L_i}V$ is invertible in \mathcal{S} , hence

$$T_i = (\hat{N}_{L_i}U + \hat{M}_{L_i}V)^{-1}(\hat{N}_{L_i}\hat{Y} - \hat{M}_{L_i}\hat{X}) \in \mathcal{S}.$$

Assume that C_2 has a strong left representation $[-\hat{N}_{C_2} \hat{M}_{C_2}]$. By the complete finiteness of \mathcal{S} , $[-\hat{N}_{C_2} \hat{M}_{C_2}]$ can be extended to an invertible element of $\mathcal{S}^{2 \times 2}$, say $\begin{bmatrix} Y & X \\ -\hat{N}_{C_2} & \hat{M}_{C_2} \end{bmatrix}$. It follows from the stable finiteness of \mathcal{S} that

$$\begin{bmatrix} Y & X \\ -\hat{N}_{C_2} & \hat{M}_{C_2} \end{bmatrix} \begin{bmatrix} V & -\hat{X} \\ U & \hat{Y} \end{bmatrix} = \begin{bmatrix} YV + XU & -Y\hat{X} + X\hat{Y} \\ 0 & \hat{N}_{C_2}\hat{X} + \hat{M}_{C_2}\hat{Y} \end{bmatrix}$$

is invertible in $\mathcal{S}^{2 \times 2}$, which implies that $\hat{N}_{C_2}\hat{X} + \hat{M}_{C_2}\hat{Y}$ is invertible in \mathcal{S} . Then

$$\begin{aligned} \hat{Y} - UT_i &= \hat{Y} - U(L_iU + V)^{-1}(L_i\hat{Y} - \hat{X}) \\ &= \hat{Y} - C_2(L_iC_2 + I)^{-1}(L_i\hat{Y} - \hat{X}) \\ &= (I + C_2L_i)^{-1}(\hat{Y} + C_2\hat{X}) \\ &= (I + C_2L_i)^{-1}\hat{M}_{C_2}^{-1}(\hat{N}_{C_2}\hat{X} + \hat{M}_{C_2}\hat{Y}) \end{aligned}$$

is invertible in \mathcal{S} . Then one can compute that

$$L_i = (\hat{X} + VT_i)(\hat{Y} - UT_i)^{-1}, \quad i = 1, 2. \quad (4)$$

Note that

$$\begin{bmatrix} \hat{Y} - UT_i & -U \\ \hat{X} + VT_i & V \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} V & -\hat{X} \\ U & \hat{Y} \end{bmatrix} \begin{bmatrix} -T_i & -I \\ I & 0 \end{bmatrix}. \quad (5)$$

According to the stable finiteness of \mathcal{S} and (4), L_i admits a strong right representation $\begin{bmatrix} \hat{Y} - UT_i \\ \hat{X} + VT_i \end{bmatrix}$ with

$$\begin{bmatrix} \hat{Y} - UT_i & -U \\ \hat{X} + VT_i & V \end{bmatrix} \text{invertible in } \mathcal{S}^{2 \times 2}. \text{ Since}$$

$$C_0\hat{X} + C_0VT_1 + \hat{Y} - UT_1 = (I + C_0L_1)(\hat{Y} - UT_1)$$

is invertible in \mathcal{L} , the operator

$$R = (C_0\hat{X} + C_0VT_1 + \hat{Y} - UT_1)^{-1}(C_0V - U) \quad (6)$$

is well-defined. Let $[-\hat{N}_{C_0} \hat{M}_{C_0}]$ be a strong left representation of C_0 . It follows from the stability of $\{L_1, C_0\}$ that

$$R = \{\hat{M}_{C_0}(\hat{X} + VT_1) + \hat{N}_{C_0}(\hat{Y} - UT_1)\}^{-1} \times (\hat{N}_{C_0}V - \hat{M}_{C_0}U) \in \mathcal{S}. \quad (7)$$

Observe that

$$\begin{aligned} V - \hat{X}R - VT_1R &= V - (\hat{X} + VT_1)(C_0\hat{X} + C_0VT_1 + \hat{Y} - UT_1)^{-1}(C_0V - U) \\ &= V - L_1(I + C_0L_1)^{-1}C_0V + L_1(I + C_0L_1)^{-1}U \\ &= (I + L_1C_0)^{-1}(I + L_1C_2)V. \end{aligned}$$

Hence $V - \hat{X}R - VT_1R$ is invertible in \mathcal{L} . It follows from (6) that

$$C_0 = (U + \hat{Y}R - UT_1R)(V - \hat{X}R - VT_1R)^{-1}. \quad (8)$$

Again applying the stable finiteness of \mathcal{S} ,

$$\begin{bmatrix} V - \hat{X}R - VT_1R & -(\hat{X} + VT_1) \\ U + \hat{Y}R - UT_1R & \hat{Y} - UT_1 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \hat{Y} - UT_1 & -U \\ \hat{X} + VT_1 & V \end{bmatrix} \begin{bmatrix} -R & -I \\ I & 0 \end{bmatrix} \quad (9)$$

is invertible in $\mathcal{S}^{2 \times 2}$. Therefore, $\begin{bmatrix} V - \hat{X}R - VT_1R \\ U + \hat{Y}R - UT_1R \end{bmatrix}$ is a strong right representation of C_0 . According to Lemma 3 and Lemma 4, L_0, L_1, L_2 are simultaneously stabilized by a linear system C if and only if C has a strong right representation $\begin{bmatrix} V - (\hat{X} + VT_1)Q \\ U + (\hat{Y} - UT_1)Q \end{bmatrix}$, where $Q \in \mathcal{S}$ satisfies

$$\hat{M}_{L_i}(V - (\hat{X} + VT_1)Q) + \hat{N}_{L_i}(U + (\hat{Y} - UT_1)Q) \quad (10)$$

being invertible in \mathcal{S} for $i = 0, 2$. Since $\hat{M}_{L_2}V + \hat{N}_{L_2}U$ is invertible in \mathcal{S} , for $i = 2$, (10) is invertible in \mathcal{S} if and only if

$$\begin{aligned} (\hat{M}_{L_2}V + \hat{N}_{L_2}U)^{-1}[\hat{M}_{L_2}V - \hat{M}_{L_2}(\hat{X} + VT_1)Q + \hat{N}_{L_2}U + \hat{N}_{L_2}(\hat{Y} - UT_1)Q] \\ = I + (\hat{M}_{L_2}V + \hat{N}_{L_2}U)^{-1}(\hat{N}_{L_2} - \hat{M}_{L_2}L_1)(\hat{Y} - UT_1)Q \\ = I + (V + L_2U)^{-1}(L_2 - L_1)(\hat{Y} - UT_1)Q \end{aligned}$$

Since the parameters a_n and b_n of L_0 are irregular, the computation of strong representation for L_0 is complicated. The previous work in Refs. 14, 15 has demonstrated their applications. Compared with the criteria in Theorem 1, the design method of simultaneous stabilizing controllers in Theorem 1 really is quite simple when the strong representation of a controller in transmission condition is clear.

CONCLUSION

This paper has dealt with the simultaneous stabilization problem of three discrete time-varying systems from the transitivity version. This simultaneous stabilizability criteria extends the results in Refs. 14, 15 by requiring only the strong representation of a controller in the transmission condition. It is noted that the main results can cover all the discrete-time time-varying linear system with the realization described by the difference equations

$$\begin{aligned}x_{n+1} &= a_n x_n + b_n u_n, & x_0 &= 0, \\y_n &= c_n x_n + d_n u_n,\end{aligned}$$

where $x_n, u_n, y_n \in \mathbb{C}^k$ are the state, the input, and the output of the plant, a_n, b_n, c_n, d_n are complex matrices of appropriate dimensions. Clearly, the time-varying linear system is the operator from $u = (u_0, u_1, \dots, u_n, \dots)$ to $y = (y_0, y_1, \dots, y_n, \dots)$ with matrix representation

$$\begin{bmatrix} d_0 & & & & \\ c_1 b_0 & d_1 & & & \\ c_2 a_1 b_0 & c_2 b_1 & d_2 & & \\ c_3 a_2 a_1 b_0 & c_3 a_2 b_1 & c_3 b_2 & d_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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REFERENCES

- Zhou B (2016) On asymptotic stability of linear time-varying systems. *Automatica* **68**, 266–76.
- Zhou B, Zhao T (2017) On asymptotic stability of discrete-time linear time-varying systems. *IEEE Trans Automat Contr* **62**, 4274–81.
- Chen G, Yang Y (2016) New stability conditions for a class of linear time varying systems. *Automatica* **71**, 342–7.
- Zhou B, Luo WW (2018) Improved Razumikhin and Krasovskii stability criteria for time-varying stochastic time-delay systems. *Automatica* **91**, 256–69.
- Feintuch A (2012) On the strong stabilization of slowly time-varying linear systems. *Systems Control Lett* **61**, 112–6.
- Cantoni M, Jonsson U, Khong SZ (2013) Robust stability analysis for feedback interconnections of time-varying linear systems. *SIAM J Control Optim* **51**, 353–79.
- Liu L, Lu YF (2014) Stabilizability, representations and factorizations for the time-varying linear system. *System Control Lett* **66**, 58–64.
- Saeks R, Murray J (1982) Fractional representation algebraic geometry, and the simultaneous stabilization problem. *IEEE Trans Automat Contr* **27**, 895–903.
- Vidyasagar M, Viswanadham N (1982) Algebraic design techniques for reliable stabilization. *IEEE Trans Automat Contr* **27**, 1085–95.
- Lu YF, Xu XP (2008) The stabilization problem for discrete time-varying linear systems. *Systems Control Lett* **57**, 936–9.
- Feintuch A (1998) *Robust Control Theory in Hilbert Space*, Springer-Verlag.
- Yu TQ (2011) The transitivity in simultaneous stabilization. *System Control Lett* **60**, 1–6.
- Liu L, Lu YF (2013) Necessary and sufficient conditions to the transitivity in simultaneous stabilization of time-varying systems. *IET Control Theory A* **7**, 1834–42.
- Wang HZ, Yu TQ, Xiao JM (2016) Simultaneously stabilising controllers for time-varying linear systems within the framework of nest algebras. *Int J Control* **89**, 1632–40.
- Liu L, Lu YF (2015) Transitivity in simultaneous stabilization for a family of time-varying systems. *System Control Lett* **78**, 55–62.
- Dale WN, Smith MC (1993) Stabilizability and existence of system representations for discrete-time time-varying systems. *SIAM J Control Optim* **31**, 1538–57.
- Davidson KR, Ji YQ (2009) Topological stable rank of nest algebras. *P Lon Math Soc* **98**, 652–78.