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A companion of Ostrowski's inequality for complex functions defined on the unit circle and applications

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ABSTRACT: Some companions of Ostrowski's inequality for complex functions defined on the unit circle are proved. Our results in special cases not only recapture known results, but also give a smaller estimator than that of the known results. Applications to a composite quadrature rule and to functions of unitary operators in Hilbert spaces are also considered.

KEYWORDS: Riemann-Stieltjes integral inequalities, unitary operators in Hilbert spaces, quadrature rules

MSC2010: 26D15 41A55

INTRODUCTION

The Riemann-Stieltjes integral plays an important role in mathematics. In Ref. 1, Alomari used $f(x)\{u(\frac{1}{2}[a+b])-u(a)\}+f(a+b-x)\{u(b)-u(\frac{1}{2}[a+b])\}$ to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,\mathrm{d}u(t)$ and proved that

$$|D(f,u;a,b,x)| \leqslant H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r V_a^b(u),$$

for any $x\in[a,\frac{1}{2}(a+b)]$, provided that $f:[a,b]\to\mathbb{R}$ is an $(r-H)$-Hölder type mapping and $u:[a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$, where $V_a^b(f)$ denotes the total variation of f on $[a,b]$, and

$$D(f,u;a,b,x):=\int_a^b f(t)\,\mathrm{d}u(t)-f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right] \quad (1)$$

is the error functional, $H>0$ and $r\in(0,1]$ are given.

If the integrand $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ and the integrator $u(t)=t$, $t\in[a,b]$, then the following companion of Ostrowski's inequalities for functions of bounded variation has been considered by in Ref. 2, in which they obtained the bound

$$\begin{aligned}
&\left|\tfrac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \\
&\leqslant \frac{1}{b-a}\left[(x-a)V_a^x(f)+\left(\frac{a+b}{2}-x\right)V_x^{a+b-x}(f)+(x-a)V_{a+b-x}^b(f)\right] \\
&\leqslant \begin{cases}
\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]V_a^b(f), \text{ and} \\
\left[2\left(\frac{x-a}{b-a}\right)^\alpha+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^\alpha\right]^{1/\alpha}\left[[V_a^x(f)]^\beta+[V_x^{a+b-x}(f)]^\beta+[V_{a+b-x}^b(f)]^\beta\right]^{1/\beta}, \\
\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \text{ and} \\
\frac{x-a+\frac{b-a}{2}}{b-a}\max\{V_a^x(f),V_x^{a+b-x}(f),V_{a+b-x}^b(f)\},
\end{cases}
\end{aligned}$$

for any $x\in[a,\frac{1}{2}(a+b)]$.

In Ref. 3, Dragomir developed Ostrowski's type integral inequality for the complex unit circle $\mathbb{C}(0,1)$.

Theorem 1 *Assume that $f:C(0,1)\to\mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(a)-f(b)|\leqslant H|a-b|^r, \quad (2)$$

for any $a,b\in C(0,1)$, where $H>0$ and $r\in(0,1]$ are given. If $[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to$

$\mathbb{C}$ *is of bounded variation, then*

$$\left|f(\mathrm{e}^{\mathrm{i}s})[u(b)-u(a)]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\right| \leqslant 2^r H \max_{t\in[a,b]}\left|\sin^r\left(\frac{s-t}{2}\right)\right| V_a^b(u), \quad (3)$$

for any $s\in[a,b]$.

For other inequalities for the Riemann-Stieltjes integral, see Refs. 4–19.

Motivated by the above facts, we consider in the present paper the problem of approximating the Riemann-Stieltjes integral $\int_a^b f(\mathrm{e}^{\mathrm{i}s})\,\mathrm{d}u(s)$ by the rule

$$f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right)-u(a)\right] + f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right], \quad (4)$$

where the continuous complex valued function $f : C(0,1)\to\mathbb{C}$ is defined on the complex unit circle $C(0,1)$ and the function $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is of bounded variation.

We denote the error functional

$$T(f,u;a,b;s,t) := f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right)-u(a)\right] + f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t), \quad (5)$$

where $t\in[a,b]$ and f is of $(r-H)$-Hölder type and u is the function of bounded variation.

The outline of this paper is as follows. First, we show some inequalities for the Riemann-Stieltjes integral. Second, we apply them to a composite quadrature rule. Third, the study of applications to unitary operators is discussed.

SOME COMPANIONS OF OSTROWSKI'S TYPE INEQUALITY

The following companions of Ostrowski's inequality for Riemann-Stieltjes integrals hold.

Theorem 2 *Suppose that* $f : C(0,1)\to\mathbb{C}$ *satisfies the following Hölder's type condition:*

$$|f(a)-f(b)|\leqslant H|a-b|^r, \quad (6)$$

for any $a,b\in C(0,1)$*, where* $H>0$ *and* $r\in(0,1]$ *are given. If* $[a,b]\subseteq[0,2\pi]$ *and* $u:[a,b]\to\mathbb{C}$ *is the function of bounded variation, then*

$$\begin{aligned}&\left|f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t) + f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|\\ &\leqslant 2^r H\left\{\max_{t\in[a,((a+b)/2)]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r V_a^{(a+b)/2}(u) + \max_{t\in[(a+b)/2,b]}\left|\sin\left(\frac{a+b-s-t}{2}\right)\right|^r V_{(a+b)/2}^b(u)\right\}\\ &\leqslant 2^r H\sin^r\left[\frac{b-a}{8}+\frac{1}{2}\left|s-\frac{3a+b}{4}\right|\right]V_a^b(u), \quad (7)\end{aligned}$$

for any $s\in[a,\frac{1}{2}(a+b)]$.

Proof: Clearly, we have from Ref. 1

$$\begin{aligned}&f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t) + f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\\ &= \int_{(a+b)/2}^b [f(\mathrm{e}^{\mathrm{i}(a+b-s)})-f(\mathrm{e}^{\mathrm{i}t})]\,\mathrm{d}u(t) + \int_a^{(a+b)/2}[f(\mathrm{e}^{\mathrm{i}s})-f(\mathrm{e}^{\mathrm{i}t})]\,\mathrm{d}u(t), \quad (8)\end{aligned}$$

for any $s\in[a,\frac{1}{2}(a+b)]$.

If $q:[c,d]\to\mathbb{C}$ is a continuous function and $v:[c,d]\to\mathbb{C}$ is the function of bounded variation, then there exists the Riemann-Stieltjes integral $\int_c^d q(t)\,\mathrm{d}v(t)$ and

$$\left|\int_c^d q(t)\,\mathrm{d}v(t)\right|\leqslant \max_{t\in[c,d]}|q(t)|\,V_c^d(v). \quad (9)$$

Applying inequality (9) to identity (8) and using Hölder's type condition (6), we obtain

$$\begin{aligned}&\left|f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t) + f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|\\ &\leqslant \max_{t\in[a,((a+b)/2)]}\left|f(\mathrm{e}^{\mathrm{i}s})-f(\mathrm{e}^{\mathrm{i}t})\right| V_a^{(a+b)/2}(u) + \max_{t\in[(a+b)/2,b]}\left|f(\mathrm{e}^{\mathrm{i}(a+b-s)})-f(\mathrm{e}^{\mathrm{i}t})\right| V_{(a+b)/2}^b(u)\\ &\leqslant H\left\{\max_{t\in[a,((a+b)/2)]}\left|\mathrm{e}^{\mathrm{i}s}-\mathrm{e}^{\mathrm{i}t}\right|^r V_a^{a+b}(u) + \max_{t\in[(a+b)/2,b]}\left|\mathrm{e}^{\mathrm{i}(a+b-s)}-\mathrm{e}^{\mathrm{i}t}\right|^r V_{(a+b)/2}^b(u)\right\}. \quad (10)\end{aligned}$$

From Ref. 3, we have

$$\left|\mathrm{e}^{\mathrm{i}s}-\mathrm{e}^{\mathrm{i}t}\right|^r = 2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r, \quad (11)$$

for any $s, t \in \mathbb{R}$.

Applying (11) to (10), we deduce

$$\begin{aligned}&\left|f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\right.\\&\qquad\left.+f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|\\&\leqslant 2^r H\left\{\max_{t\in[a,((a+b)/2)]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r V_a^{(a+b)/2}(u)\right.\\&\qquad\left.+\max_{t\in[(a+b)/2,b]}\left|\sin\left(\frac{a+b-s-t}{2}\right)\right|^r V_{(a+b)/2}^b(u)\right\},\end{aligned} \tag{12}$$

for any $s \in [a, \frac{1}{2}(a+b)]$.

We now prove

$$\begin{aligned}&2^r H\left\{\max_{t\in[a,((a+b)/2)]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r V_a^{(a+b)/2}(u)+\right.\\&\qquad\left.+\max_{t\in[(a+b)/2,b]}\left|\sin\left(\frac{a+b-s-t}{2}\right)\right|^r V_{(a+b)/2}^b(u)\right\}\\&\leqslant 2^r H\sin^r\left[\frac{b-a}{8}+\frac{1}{2}\left|s-\frac{3a+b}{4}\right|\right]V_a^b(u),\end{aligned} \tag{13}$$

for any $s \in [a, \frac{1}{2}(a+b)]$.

By $[a,b] \subset [0,2\pi]$ and $s \in [a, \frac{1}{2}(a+b)]$, we have $a+b-s \in [\frac{1}{2}(a+b), b]$. When $t \in [a, \frac{1}{2}(a+b)]$, we have $\frac{1}{2}|t-s| \leqslant \frac{1}{2}|\frac{1}{2}(a+b)-a| \leqslant \frac{1}{2}\pi$. When $t \in [\frac{1}{2}(a+b), b]$, we have $\frac{1}{2}|a+b-s-t| \leqslant \frac{1}{2}|b-\frac{1}{2}(a+b)| \leqslant \frac{1}{2}\pi$. Since the function $\sin(x)$ is monotone increasing on $[0, \frac{1}{2}\pi]$, we have that

$$\begin{aligned}&\max_{t\in[a,((a+b)/2)]}\left|\sin\left(\frac{s-t}{2}\right)\right|\\&=\sin\left(\max_{t\in[a,((a+b)/2)]}\tfrac{1}{2}|t-s|\right)\\&=\sin\left(\tfrac{1}{2}\max\left\{\frac{a+b}{2}-s,s-a\right\}\right)\\&=\sin\left(\frac{b-a}{8}+\frac{1}{2}\left|s-\frac{3a+b}{4}\right|\right),\end{aligned} \tag{14}$$

for any $s \in [a, \frac{1}{2}(a+b)]$. In a similar way, we obtain

$$\begin{aligned}&\max_{t\in[(a+b)/2,b]}\left|\sin\left(\frac{a+b-s-t}{2}\right)\right|\\&=\sin\left(\frac{b-a}{8}+\frac{1}{2}\left|s-\frac{3a+b}{4}\right|\right),\end{aligned} \tag{15}$$

for any $s \in [a, \frac{1}{2}(a+b)]$. Hence we deduce the desired result (13). If we choose $s = \frac{1}{2}(a+b)$ in (7), we recapture Remark 2 of Ref. 3. In particular, if we choose $s = (3a+b)/4$ in (7), then we obtain

$$\begin{aligned}&\left|f\left(\mathrm{e}^{\mathrm{i}((3a+b)/4)}\right)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\right.\\&\qquad\left.+f\left(\mathrm{e}^{\mathrm{i}((a+3b)/4)}\right)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|\\&\leqslant 2^r H\sin^r\left(\frac{b-a}{8}\right)V_a^b(u),\end{aligned} \tag{16}$$

which is more precise than Remark 2 of Ref. 3. □

We can consider the following situation: for any $w, z \in C(0,1)$, the Lipschitz condition $|f(z)-f(w)| \leqslant L|z-w|$ is satisfied, where $f : C(0,1) \to \mathbb{C}$ and $L > 0$. In this case, we can show the sharpness of the corresponding version of (17).

Corollary 1 *Suppose that $f : C(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $C(0,1)$. If $[a,b] \subset [0,2\pi]$ and $u : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then we have*

$$\begin{aligned}&\left|f(\mathrm{e}^{\mathrm{i}((3a+b)/4)})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\right.\\&\qquad\left.+f(\mathrm{e}^{\mathrm{i}((a+3b)/4)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|\\&\leqslant 2L\sin\left(\frac{b-a}{8}\right)V_a^b(u),\end{aligned} \tag{17}$$

where the constant 2 on the right-hand side cannot be replaced by a smaller constant.

Proof: We only need to prove the sharpness of the constant 2. Assume that (17) holds with a constant $C > 0$, that is,

$$\begin{aligned}&\left|f(\mathrm{e}^{\mathrm{i}((3a+b)/4)})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\right.\\&\qquad\left.+f(\mathrm{e}^{\mathrm{i}((a+3b)/4)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|\\&\leqslant CL\sin\left(\frac{b-a}{8}\right)V_a^b(u).\end{aligned} \tag{18}$$

Choose $f : \mathbb{C} \to \mathbb{C}, f(z) = z$ and $u : [0,2\pi] \to \mathbb{R}$ given by

$$u(t) := \begin{cases}0, & 0 \leqslant t < 2\pi,\\ 1, & t = 2\pi.\end{cases} \tag{19}$$

Clearly, f is Lipschitzian with the constant $L = 1$. At the same time, we consider $a = 0$ and $b = 2\pi$. By using integration by parts for the integral, we have

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}\,\mathrm{d}u(t) = \mathrm{e}^{\mathrm{i}t}u(t)|_0^{2\pi} - \mathrm{i}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}u(t)\,\mathrm{d}t = u(2\pi) - u(0) - \mathrm{i}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}u(t)\,\mathrm{d}t \tag{20}$$

and

$$V_0^{2\pi}(u) = 1. \tag{21}$$

Consequently, by (20) and (21), we obtain $C \geqslant 2$. □

Remark 1 Under the assumption Theorem 2 and $u(t) = t$, $t \in [a,b] \subseteq [0,2\pi]$, we have

$$\left|f(\mathrm{e}^{\mathrm{i}s}) + f(\mathrm{e}^{\mathrm{i}(a+b-s)}) - \frac{2}{b-a}\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\right| \leqslant 2^r H\left\{\max_{t\in[a,((a+b)/2)]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r + \max_{t\in[(a+b)/2,b]}\left|\sin\left(\frac{a+b-s-t}{2}\right)\right|^r\right\}, \tag{22}$$

for any $s \in [a, \frac{1}{2}(a+b)]$, provided that $f : C(0,1) \to \mathbb{C}$ satisfies Hölder's type condition (6).

Remark 2 If $f : C(0,1) \to \mathbb{C}$ satisfies (6), and $p : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$, then we have

$$\left|f(\mathrm{e}^{\mathrm{i}s})\int_a^{(a+b)/2} p(t)\,\mathrm{d}t + f(\mathrm{e}^{\mathrm{i}(a+b-s)})\int_{(a+b)/2}^b p(t)\,\mathrm{d}t - \int_a^b f(\mathrm{e}^{\mathrm{i}t})p(t)\,\mathrm{d}t\right| \leqslant 2^r H\left\{\max_{t\in[a,((a+b)/2)]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r \int_a^{(a+b)/2} |p(t)|\,\mathrm{d}t + \max_{t\in[(a+b)/2,b]}\left|\sin\left(\frac{a+b-s-t}{2}\right)\right|^r \int_{(a+b)/2}^b |p(t)|\,\mathrm{d}t\right\}. \tag{23}$$

Theorem 3 *On the circle $C(0,1)$, suppose that $f : C(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$. If $[a,b] \subseteq [0,2\pi]$ and the function $u : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$ on $[a,b]$, then*

$$\left|f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right) - u(a)\right] - \int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t) + f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b) - u\left(\frac{a+b}{2}\right)\right]\right| \leqslant 32KL\sin^2\left(\frac{b-a}{8}\right), \tag{24}$$

for any $s \in [a, \frac{1}{2}(a+b)]$.

Proof: If $w : [a,b] \to \mathbb{C}$ is a Riemann integrable function and the function $v : [a,b] \to \mathbb{C}$ is M-Lipschitzian, then there exists the Riemann-Stieltjes integral $\int_a^b w(t)\,\mathrm{d}v(t)$ and

$$\left|\int_a^b w(t)\,\mathrm{d}v(t)\right| \leqslant M\int_a^b |w(t)|\,\mathrm{d}t. \tag{25}$$

Making use of (25), we have from (8) that

$$\left|f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right) - u(a)\right] - \int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t) + f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b) - u\left(\frac{a+b}{2}\right)\right]\right| \leqslant KL\left\{\int_a^{(a+b)/2} \left|\mathrm{e}^{\mathrm{i}s} - \mathrm{e}^{\mathrm{i}t}\right|\,\mathrm{d}t + \int_{(a+b)/2}^b \left|\mathrm{e}^{\mathrm{i}(a+b-s)} - \mathrm{e}^{\mathrm{i}t}\right|\,\mathrm{d}t\right\}, \tag{26}$$

for any $s \in [a, \frac{1}{2}(a+b)]$. By (11), for any $t, s \in \mathbb{R}$, we have $|\mathrm{e}^{\mathrm{i}s} - \mathrm{e}^{\mathrm{i}t}| = 2|\sin(\frac{1}{2}(s-t))|$. Then

$$\int_a^{(a+b)/2} |\mathrm{e}^{\mathrm{i}s} - \mathrm{e}^{\mathrm{i}t}|\,\mathrm{d}t = 8\left[\sin^2\left(\frac{s-a}{4}\right) + \sin^2\left(\frac{\frac{a+b}{2} - s}{4}\right)\right] \leqslant 16\sin^2\left(\frac{b-a}{8}\right), \tag{27}$$

for any $s \in [a, \frac{1}{2}(a+b)]$. In the similar way, we have

$$\int_{(a+b)/2}^b \left|\mathrm{e}^{\mathrm{i}(a+b-s)} - \mathrm{e}^{\mathrm{i}t}\right|\,\mathrm{d}t \leqslant 16\sin^2\left(\frac{b-a}{8}\right), \tag{28}$$

for any $s \in [a, \frac{1}{2}(a+b)]$. Plugging (27) and (28) into (26), we complete the proof of (24). □

Remark 3 Under the assumptions in Theorem 3,

$$\left|f(\mathrm{e}^{\mathrm{i}((3a+b)/4)})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\right.$$
$$\left.+f(\mathrm{e}^{\mathrm{i}((3b+a)/4)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leqslant 32KL\sin^2\left(\frac{b-a}{16}\right). \quad (29)$$

Remark 4 If $u(t)=t, t\in[a,b]$, then

$$\left|f(\mathrm{e}^{\mathrm{i}s})+f(\mathrm{e}^{\mathrm{i}(a+b-s)})-\frac{2}{b-a}\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\right|$$
$$\leqslant \frac{32L}{b-a}\left[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{\frac{a+b}{2}-s}{4}\right)\right]$$
$$\leqslant \frac{64L}{b-a}\sin^2\left(\frac{b-a}{8}\right), \quad (30)$$

for any $s\in[a,\frac{1}{2}(a+b)]$ and

$$\left|f(\mathrm{e}^{\mathrm{i}((3a+b)/4)})+f(\mathrm{e}^{\mathrm{i}((3b+a)/4)})-\frac{2}{b-a}\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\right|$$
$$\leqslant \frac{64L}{b-a}\sin^2\left(\frac{b-a}{16}\right). \quad (31)$$

Remark 5 If $w:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is essentially bounded on $[a,b]$ and $f:C(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $C(0,1)$, then for any $s\in[a,\frac{1}{2}(a+b)]$,

$$\left|f(\mathrm{e}^{\mathrm{i}s})\int_a^{(a+b)/2} w(t)\,\mathrm{d}t-\int_a^b f(\mathrm{e}^{\mathrm{i}t})w(t)\,\mathrm{d}t\right.$$
$$\left.+f(\mathrm{e}^{\mathrm{i}(a+b-s)})\int_{(a+b)/2}^b w(t)\,\mathrm{d}t\right|$$
$$\leqslant 32L\|w\|_\infty\sin^2\left(\frac{b-a}{8}\right), \quad (32)$$

where $\|w\|_\infty := \operatorname{ess\,sup}_{t\in[a,b]}|w(t)|$.

Theorem 4 *On the circle $C(0,1)$, suppose that $f:C(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$. If $u:[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then for any $s\in[a,\frac{1}{2}(a+b)]$,*

$$\left|f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\right.$$
$$\left.+f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leqslant L\int_a^{(a+b)/2}\operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,\mathrm{d}t$$
$$+L\int_{(a+b)/2}^b\operatorname{sgn}(a+b-s-t)\cos\left(\frac{a+b-s-t}{2}\right)u(t)\,\mathrm{d}t$$
$$+2L\sin\left(\frac{s-a}{2}\right)[u(b)-u(a)]. \quad (33)$$

Proof: If $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then there exists the Riemann-Stieltjes integral $\int_a^b p(t)\,\mathrm{d}v(t)$ and

$$\left|\int_a^b p(t)\,\mathrm{d}v(t)\right|\leqslant\int_a^b|p(t)|\,\mathrm{d}v(t). \quad (34)$$

Making use of (34), we have from (8) that

$$\left|f(\mathrm{e}^{\mathrm{i}s})\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\right.$$
$$\left.+f(\mathrm{e}^{\mathrm{i}(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leqslant\int_{(a+b)/2}^b\left|f(\mathrm{e}^{\mathrm{i}(a+b-s)})-f(\mathrm{e}^{\mathrm{i}t})\right|\,\mathrm{d}u(t)$$
$$+\int_a^{(a+b)/2}\left|f(\mathrm{e}^{\mathrm{i}s})-f(\mathrm{e}^{\mathrm{i}t})\right|\,\mathrm{d}u(t)$$
$$\leqslant L\int_{(a+b)/2}^b\left|\mathrm{e}^{\mathrm{i}(a+b-s)}-\mathrm{e}^{\mathrm{i}t}\right|\,\mathrm{d}u(t)$$
$$+L\int_a^{(a+b)/2}\left|\mathrm{e}^{\mathrm{i}s}-\mathrm{e}^{\mathrm{i}t}\right|\,\mathrm{d}u(t), \quad (35)$$

for any $s\in[a,\frac{1}{2}(a+b)]$. By (11), for any $t,s\in\mathbb{R}$, we have $|\mathrm{e}^{\mathrm{i}s}-\mathrm{e}^{\mathrm{i}t}|=2|\sin(\frac{1}{2}(s-t))|$. Then

$$\int_a^{(a+b)/2}|\mathrm{e}^{\mathrm{i}s}-\mathrm{e}^{\mathrm{i}t}|\,\mathrm{d}u(t)$$
$$=2\int_a^s\sin\left(\frac{s-t}{2}\right)\mathrm{d}u(t)+2\int_s^{(a+b)/2}\sin\left(\frac{t-s}{2}\right)\mathrm{d}u(t), \quad (36)$$

for any $s\in[a,\frac{1}{2}(a+b)]\subseteq[0,2\pi]$. Using integration by parts for the Riemann-Stieltjes integral, we have

$$\int_a^s\sin\left(\frac{s-t}{2}\right)\mathrm{d}u(t)$$
$$=-\sin\left(\frac{s-a}{2}\right)u(a)+\frac{1}{2}\int_a^s\cos\left(\frac{s-t}{2}\right)u(t)\,\mathrm{d}t \quad (37)$$

and

$$\int_s^{(a+b)/2} \sin\left(\frac{t-s}{2}\right) \mathrm{d}u(t) = \sin\left(\frac{\frac{a+b}{2}-s}{2}\right) u\left(\frac{a+b}{2}\right) - \frac{1}{2}\int_s^{(a+b)/2} \cos\left(\frac{t-s}{2}\right) u(t)\,\mathrm{d}t. \quad (38)$$

Hence, by (36), (37) and (38), we have

$$\begin{aligned}\int_a^{(a+b)/2} \left|e^{is}-e^{it}\right| \mathrm{d}u(t) &= 2\left[\sin\left(\frac{\frac{a+b}{2}-s}{2}\right) u\left(\frac{a+b}{2}\right) - \sin\left(\frac{s-a}{2}\right) u(a)\right] \\ &\quad + \int_a^{(a+b)/2} \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right) u(t)\,\mathrm{d}t. \quad (39)\end{aligned}$$

In the similar way, we have

$$\begin{aligned}\int_{(a+b)/2}^{b} \left|e^{i(a+b-s)}-e^{it}\right| \mathrm{d}u(t) &= 2\left[\sin\left(\frac{s-a}{2}\right) u(b) - \sin\left(\frac{\frac{a+b}{2}-s}{2}\right) u\left(\frac{a+b}{2}\right)\right] \\ &\quad + \int_{(a+b)/2}^{b} \operatorname{sgn}(a+b-s-t)\cos\left(\frac{a+b-s-t}{2}\right) u(t)\,\mathrm{d}t. \quad (40)\end{aligned}$$

By (35), (39) and (40), (33) is proved. □

Remark 6 If we choose $s = \frac{1}{2}(a+b)$ in (33), we recapture Theorem 5 of Ref. 3.

Corollary 2 *Under the assumptions of Theorem 4, for any $s \in [a, \frac{1}{2}(a+b)]$,*

$$\begin{aligned}&\left| f(e^{is})\left[u\left(\frac{a+b}{2}\right)-u(a)\right] - \int_a^b f(e^{it})\,\mathrm{d}u(t) + f(e^{i(a+b-s)})\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right| \\ &\leqslant 2L\sin\left(\frac{s-a}{2}\right)[u(b)-u(a)+u(s)-u(a+b-s)] \\ &\quad + 2L\sin\left(\frac{\frac{a+b}{2}-s}{2}\right)[u(a+b-s)-u(s)] \\ &=: B(s), \quad (41)\end{aligned}$$

where

$$B(s) \leqslant 2L\sin\left[\frac{b-a}{8} + \frac{1}{2}\left|s - \frac{3a+b}{4}\right|\right][u(b)-u(a)]$$

and

$$4L\sin\left[\frac{b-a}{8}\right]\cos\left[\frac{1}{2}\left(s-\frac{3a+b}{4}\right)\right] \times\left[\frac{u(b)-u(a)}{2} + \left|u(a+b-s)-u(s)+\frac{u(a)-u(b)}{2}\right|\right]. \quad (42)$$

In particular,

$$\begin{aligned}&\left| f\left(e^{i((3a+b)/4)}\right)\left[u\left(\frac{a+b}{2}\right)-u(a)\right] - \int_a^b f(e^{it})\,\mathrm{d}u(t) + f\left(e^{i((a+3b)/4)}\right)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]\right| \\ &\leqslant L\int_a^{(a+b)/2} \operatorname{sgn}\left(\frac{3a+b}{4}-t\right)\cos\left(\frac{\frac{3a+b}{4}-t}{2}\right) u(t)\,\mathrm{d}t \\ &\quad + L\int_{(a+b)/2}^{b} \operatorname{sgn}\left(\frac{a+3b}{4}-t\right)\cos\left(\frac{\frac{a+3b}{4}-t}{2}\right) u(t)\,\mathrm{d}t \\ &\quad + 2L\sin\left(\frac{b-a}{8}\right)[u(b)-u(a)] \\ &=: M, \quad (43)\end{aligned}$$

where

$$M \leqslant 2L\sin\left(\frac{b-a}{8}\right)[u(b)-u(a)].$$

Proof: Since $0 < b-a < 2\pi$, we have $0 \leqslant \frac{1}{2}(|s-t|) \leqslant \frac{1}{2}(\frac{1}{2}(a+b)-a) \leqslant \frac{1}{2}\pi$ for any $s, t \in [a, \frac{1}{2}(a+b)]$. If $s \in [a, \frac{1}{2}(a+b)]$ and $t \in [\frac{1}{2}(a+b), b]$, then $0 \leqslant \frac{1}{2}(a+b-s-t) \leqslant \frac{1}{2}(a+b-a-\frac{1}{2}(a+b)) \leqslant \frac{1}{4}(b-a) \leqslant \frac{1}{2}\pi$. Hence we have $\cos\frac{1}{2}(|s-t|) \geqslant 0$ for any $s, t \in [a, \frac{1}{2}(a+b)]$; $\cos\frac{1}{2}(|a+b-s-t|) \leqslant 0$ for any $s \in [a, \frac{1}{2}(a+b)]$, $t \in [\frac{1}{2}(a+b), b]$. Using the fact that u is monotonic nondecreasing on $[a, b]$,

$$\begin{aligned}&\int_a^s \cos\left(\frac{s-t}{2}\right) u(t)\,\mathrm{d}t \leqslant 2u(s)\sin\left(\frac{s-a}{2}\right), \\ &\int_s^{(a+b)/2} \cos\left(\frac{s-t}{2}\right) u(t)\,\mathrm{d}t \geqslant 2u(s)\sin\left(\frac{\frac{a+b}{2}-s}{2}\right), \\ &\int_{(a+b)/2}^{a+b-s} \cos\left(\frac{a+b-s-t}{2}\right) u(t)\,\mathrm{d}t \leqslant 2u(a+b-s)\sin\left(\frac{\frac{a+b}{2}-s}{2}\right), \\ &\int_{a+b-s}^{b} \cos\left(\frac{a+b-s-t}{2}\right) u(t)\,\mathrm{d}t \geqslant 2u(a+b-s)\sin\left(\frac{s-a}{2}\right). \quad (44)\end{aligned}$$

Applying (44)–(33), we have proved (41). From the elementary property stating that

$$\alpha x + \beta y \leqslant \max\{\alpha, \beta\}(x+y),$$

where $\alpha,\beta,x,y \geqslant 0$, we can obtain the bounds for $B(s)$. The details are omitted. □

A COMPOSITE QUADRATURE RULE

In this section, we use the results from the previous sections to approximate the Riemann-Stieltjes integral $\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)$, in terms of the Riemann-Stieltjes integral $\int_a^b f(t)\,\mathrm{d}u(t)$.

We consider the following partition of the interval $[a,b]$

$$\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the intermediate points $\xi_k \in [x_k, \frac{1}{2}(x_k+x_{k+1})]$, where $0 \leqslant k \leqslant n-1$. We define $h_k := x_{k+1}-x_k, 0 \leqslant k \leqslant n-1$ and the norm of the partition Δ_n is $\nu(\Delta_n) = \max\{h_k : 0 \leqslant k \leqslant n-1\}$.

We define the quadrature rule

$$\begin{aligned} O_n(f,u,\Delta_n,\xi) := &\sum_{k=0}^{n-1} f(\mathrm{e}^{\mathrm{i}\xi_k})\Big[u\Big(\frac{x_{k+1}+x_k}{2}\Big) - u(x_k)\Big] \\ &+ \sum_{k=0}^{n-1} f(\mathrm{e}^{\mathrm{i}(x_k+x_{k+1}-\xi_k)})\Big[u(x_{k+1}) - u\Big(\frac{x_k+x_{k+1}}{2}\Big)\Big], \end{aligned} \tag{45}$$

where $f : C(0,1) \to \mathbb{C}$ is a continuous function and $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$. Define the remainder $R_n(f,u,\Delta_n,\xi)$ in approximating the Riemann-Stieltjes integral $\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)$ by $O_n(f,u,\Delta_n,\xi)$. Then

$$\int_a^b f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t) = O_n(f,u,\Delta_n,\xi) + R_n(f,u,\Delta_n,\xi). \tag{46}$$

We provide a priory bounds for $R_n(f,u,\Delta_n,\xi)$ in several instances of f and u as above in the following result.

Proposition 1 *Assume that $f : C(0,1) \to \mathbb{C}$ satisfies Hölder's type condition (6). If $[a,b] \subseteq [0,2\pi]$ and $u : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then for any partition $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm $\nu(\Delta_n) \leqslant \pi$, we have the error bound*

$$\begin{aligned} &|R_n(f,u,\Delta_n,\xi)| \\ &\leqslant 2^r H \sum_{k=0}^{n-1} \sin^r\Big[\frac{x_{k+1}-x_k}{8} + \frac{1}{2}\Big|\xi_k - \frac{3x_k+x_{k+1}}{4}\Big|\Big] V_{x_k}^{x_{k+1}}(u) \\ &\leqslant 2^r H \sum_{k=0}^{n-1} \sin^r\Big(\frac{x_{k+1}-x_k}{4}\Big) V_{x_k}^{x_{k+1}}(u) \\ &\leqslant \frac{1}{2^r} H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r V_{x_k}^{x_{k+1}}(u) \\ &\leqslant \frac{1}{2^r} H \nu^r(\Delta_n) V_a^b(u), \end{aligned} \tag{47}$$

for any intermediate points $\xi_k \in [x_k, \frac{1}{2}(x_k+x_{k+1})]$, where $0 \leqslant k \leqslant n-1$.

Proof: Since $\nu(\Delta_n) \leqslant \pi$, then on using (7) on each interval $[x_k, x_{k+1}]$ and for any intermediate points $\xi_k \in [x_k, \frac{1}{2}(x_k+x_{k+1})]$ where $0 \leqslant k \leqslant n-1$, we have

$$\begin{aligned} &\Big| f(\mathrm{e}^{\mathrm{i}\xi_k})\Big[u\Big(\frac{x_k+x_{k+1}}{2}\Big) - u(x_k)\Big] + \\ &\quad f(\mathrm{e}^{\mathrm{i}(x_k+x_{k+1}-\xi_k)})\Big[u(x_{k+1}) - u\Big(\frac{x_k+x_{k+1}}{2}\Big)\Big] \\ &\quad - \int_{x_k}^{x_{k+1}} f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u(t)\Big| \\ &\leqslant 2^r H \sin^r\Big[\frac{x_{k+1}-x_k}{8} + \frac{1}{2}\Big|\xi_k - \frac{3x_k+x_{k+1}}{4}\Big|\Big] V_{x_k}^{x_{k+1}}(u) \\ &\leqslant 2^r H \sin^r\Big(\frac{x_{k+1}-x_k}{4}\Big) V_{x_k}^{x_{k+1}}(u) \\ &\leqslant \frac{1}{2^r} H (x_{k+1}-x_k)^r V_{x_k}^{x_{k+1}}(u). \end{aligned} \tag{48}$$

Summing over k from 0 to $n-1$ in (48) and using the generalized Delta inequality, we deduce (47). □

Remark 7 If we choose $\xi_k = \frac{1}{2}(\lambda_k + \lambda_{k+1})$, we recapture Corollary 1 of Ref. 3. In particular, if we choose $\xi_k = (3x_k + x_{k+1})/4$, then we obtain $|R_n(f,u,\Delta_n,\xi)| \leqslant (1/4^r) H \nu^r(\Delta_n) V_a^b(u)$, which is more precise than Proposition 4 of Ref. 3.

Corollary 3 *Under the assumption of Proposition 1 and $\xi_k = (3x_k + x_{k+1})/4$, we define the special quadrature rule by*

$$\begin{aligned} &T_n(f,u,\Delta_n) \\ &:= \sum_{k=0}^{n-1} f\Big(\mathrm{e}^{\mathrm{i}((3x_k+x_{k+1})/4)}\Big)\Big[u\Big(\frac{x_k+x_{k+1}}{2}\Big) - u(x_k)\Big] \\ &+ \sum_{k=0}^{n-1} f\Big(\mathrm{e}^{\mathrm{i}((x_k+3x_{k+1})/4)}\Big)\Big[u(x_{k+1}) - u\Big(\frac{x_k+x_{k+1}}{2}\Big)\Big] \end{aligned} \tag{49}$$

and the error $E_n(f,u,\Delta_n)$ by

$$\int_a^b f(e^{it})\,du(t) = T_n(d,u,\Delta_n) + E_n(f,u,\Delta_n). \quad (50)$$

Then we have the error bounds

$$\begin{aligned}
&|E_n(f,u,\Delta_n)| \\
&\leqslant 2^r H \sum_{k=0}^{n-1} \sin^r\left(\frac{x_{k+1}-x_k}{8}\right) V_{x_k}^{x_{k+1}}(u) \\
&\leqslant \frac{1}{4^r} H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r V_{x_k}^{x_{k+1}}(u) \\
&\leqslant \frac{1}{4^r} H v^r(\Delta_n) V_a^b(u). \qquad (51)
\end{aligned}$$

In the following result, we consider the case of both integrator and integrand being Lipschitzian.

Proposition 2 *Under the assumption of Theorem 3, for any partition $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, we have the error bound*

$$\begin{aligned}
&|E_n(f,u,\Delta_n,\xi)| \\
&\leqslant 16KL \sum_{k=0}^{n-1} \left[\sin^2\left(\frac{\xi_k - x_k}{4}\right) + \sin^2\left(\frac{\frac{x_k+x_{k+1}}{2} - \xi_k}{4}\right)\right] \\
&\leqslant 32KL \sum_{k=0}^{n-1} \sin^2\left(\frac{x_{k+1}-x_k}{8}\right) \\
&\leqslant \tfrac{1}{2} KL \sum_{k=0}^{n-1} (x_{k+1}-x_k)^2 \\
&\leqslant \tfrac{1}{2} KL(b-a)v(\Delta_n) \qquad (52)
\end{aligned}$$

for any $\xi_k \in [x_k, \frac{1}{2}(x_k + x_{k+1})]$, where $0 \leqslant k \leqslant n-1$. In particular,

$$\begin{aligned}
&|E_n(f,u,\Delta_n)| \\
&\leqslant 32KL \sum_{k=0}^{n-1} \sin^2\left(\frac{x_{k+1}-x_k}{16}\right) \\
&\leqslant \tfrac{1}{8} KL \sum_{k=0}^{n-1} (x_{k+1}-x_k)^2 \\
&\leqslant \tfrac{1}{8} KL(b-a)v(\Delta_n). \qquad (53)
\end{aligned}$$

The proof is similar to Theorem 3 and the details are omitted.

Proposition 3 *Under the assumption of Theorem 4, for any quadrature $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm $v(\Delta_n) \leqslant \pi$ and $\xi_k \in [x_k, \frac{1}{2}(x_k + x_{k+1})]$, we have the error bound*

$$\begin{aligned}
&|R_n(f,u,\Delta_n,\xi)| \\
&\leqslant 2L \sum_{k=0}^{n-1} \sin\left(\frac{\xi_k - x_k}{2}\right)[u(x_{k+1}) - u(x_k)] \\
&\quad + L \sum_{k=0}^{n-1} \int_{x_k}^{(x_k+x_{k+1})/2} \operatorname{sgn}(\xi_k - t) \cos\left(\frac{\xi_k - t}{2}\right) u(t)\,dt \\
&\quad + L \sum_{k=0}^{n-1} \int_{(x_k+x_{k+1})/2}^{x_{k+1}} \operatorname{sgn}(x_k + x_{k+1} - \xi_k - t) \\
&\qquad \times \cos\left(\frac{x_k + x_{k+1} - \xi_k - t}{2}\right) u(t)\,dt \\
&\leqslant 2L \sum_{k=0}^{n-1} \left[\sin\left(\frac{\xi_k - x_k}{2}\right)[u(x_{k+1}) - u(x_k) + u(\xi_k) \right. \\
&\quad \left. - u(x_k + x_{k+1} - \xi_k)]\right] + 2L \sum_{k=0}^{n-1} \left[\sin\left(\frac{x_k + x_{k+1}}{4} \right.\right. \\
&\quad \left.\left. - \frac{\xi_k}{2}\right)[u(x_k + x_{k+1} - \xi_k) - u(\xi_k)]\right] \\
&\leqslant 2L \sum_{k=0}^{n-1} \sin\left[\frac{x_{k+1}-x_k}{8} \right. \\
&\quad \left. + \tfrac{1}{2}\left|\xi_k - \frac{3x_k + x_{k+1}}{4}\right|\right][u(x_{k+1}) - u(x_k)] \\
&\leqslant 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_{k+1}-x_k}{4}\right)[u(x_{k+1}) - u(x_k)] \\
&\leqslant \frac{L}{2} \sum_{k=0}^{n-1} (x_{k+1}-x_k)[u(x_{k+1}) - u(x_k)] \\
&\leqslant \frac{\Delta_n L}{2}[u(b) - u(a)], \qquad (54)
\end{aligned}$$

where $0 \leqslant k \leqslant n-1$. In particular,

$$\begin{aligned}
&|E_n(f,u,\Delta_n)| \\
&\leqslant 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_{k+1}-x_k}{8}\right)[u(x_{k+1}) - u(x_k)] \\
&\leqslant \frac{L}{4} \sum_{k=0}^{n-1} (x_{k+1}-x_k)[u(x_{k+1}) - u(x_k)] \\
&\leqslant \frac{L}{4} v(\Delta_n)[u(b) - u(a)]. \qquad (55)
\end{aligned}$$

The proof is similar to Corollary 2 and details are omitted.

Remark 8 If we choose $\xi_k = \frac{1}{2}(\lambda_k + \lambda_{k+1})$, we recapture Proposition 3 of Ref. 3. If we choose $\xi_k = (3\lambda_k + \lambda_{k+1})/4$, we have $|E_n(f,u,\Delta_n)| \leqslant \frac{1}{4} L v(\Delta_n)[u(b) - u(a)]$, which is more precise than Proposition 3 of Ref. 3.

APPLICATIONS FOR FUNCTIONS OF UNITARY OPERATORS

In Ref. 3, the author used inequality (3) to give estimates of a unitary operator. In this section, we apply our previous inequality (7) to give estimates of unitary operators U defined on complex Hilbert spaces.

We recall here some basic facts on unitary operators and spectral families. We say that the bounded linear operator $U : H \to H$ on the Hilbert space H is unitary if $U^* = U^{-1}$.

It is well known that if U is a unitary operator, there exists a family of projections $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the spectral family of U with the following properties[20]:

(i) $E_\lambda \leqslant E_\mu$ for $0 \leqslant \lambda \leqslant \mu \leqslant 2\pi$;
(ii) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the identity operator on H);
(iii) $E_{\lambda+0} = E_\lambda$ for $0 \leqslant \lambda < 2\pi$;
(iv) $U = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\lambda}\,\mathrm{d}E_\lambda$, where the integral is of Riemann-Stieltjes type.

Furthermore, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying these requirements for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : C(0,1) \to \mathbb{C}$ on the complex unit circle $C(0,1)$, we have

$$f(U) = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\lambda})\,\mathrm{d}E_\lambda, \tag{56}$$

where the integral is taken in the Riemann-Stieltjes sense. In particular, we have

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\lambda})\,\mathrm{d}\langle E_\lambda x,\ y\rangle \tag{57}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f(\mathrm{e}^{\mathrm{i}\lambda})\right|^2 \mathrm{d}\|E_\lambda x\|^2 = \int_0^{2\pi} \left|f(\mathrm{e}^{\mathrm{i}\lambda})\right|^2 \mathrm{d}\langle E_\lambda x,\ x\rangle, \tag{58}$$

for any vector $x, y \in H$. We consider the following partition of the interval $[0, 2\pi]$:

$$\Delta_n : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 2\pi$$

and the intermediate points $\xi_k \in [\lambda_k, \frac{1}{2}(\lambda_k + \lambda_{k+1})]$, where $0 \leqslant k \leqslant n-1$. We define $h_k := \lambda_{k+1} - \lambda_k$, $0 \leqslant k \leqslant n-1$ and the norm of the partition Δ_n is $\nu(\Delta_n) = \max\{h_k : 0 \leqslant k \leqslant n-1\}$.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U, then

$$\begin{aligned} O_n(f, U, \Delta_n, \xi; x, y) &:= \sum_{k=0}^{n-1} f(\mathrm{e}^{\mathrm{i}\xi_k}) \left\langle \left(E_{(\lambda_k+\lambda_{k+1})/2} - E_{\lambda_k}\right)x,\ y\right\rangle \\ &+ \sum_{k=0}^{n-1} f(\mathrm{e}^{\mathrm{i}(x_{k+1}+x_k-\xi_k)}) \left\langle \left(E_{\lambda_{k+1}} - E_{(\lambda_k+\lambda_{k+1})/2}\right)x, y\right\rangle \end{aligned} \tag{59}$$

and

$$\begin{aligned} T_n(f, U, \Delta_n; x, y) &:= \sum_{k=0}^{n-1} f\left(\mathrm{e}^{\mathrm{i}(3\lambda_k+\lambda_{k+1})/4}\right) \left\langle \left(E_{(\lambda_k+\lambda_{k+1})/2} - E_{\lambda_k}\right)x,\ y\right\rangle \\ &+ \sum_{k=0}^{n-1} f\left(\mathrm{e}^{\mathrm{i}(\lambda_k+3\lambda_{k+1})/4}\right) \left\langle \left(E_{\lambda_{k+1}} - E_{(\lambda_k+\lambda_{k+1})/2}\right)x, y\right\rangle, \end{aligned} \tag{60}$$

where $x, y \in H$.

Theorem 5 *With the above assumptions for U, $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, Δ_n with $\nu(\Delta_n) \leqslant \pi$ and if $f : C(0,1) \to \mathbb{C}$ satisfies Hölder's type condition (6), then we have the representation*

$$\langle f(U)x, y\rangle = O_n(f, U, \Delta_n, \xi; x, y) + R_n(f, U, \Delta_n, \xi; x, y) \tag{61}$$

with the error $R_n(f, U, \Delta_n, \xi; x, y)$ which satisfies the bounds

$$\begin{aligned} &|R_n(f, U, \Delta_n, \xi; x, y)| \\ &\leqslant 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{\lambda_{k+1} - \lambda_k}{8} + \frac{1}{2}\left|\xi_k - \frac{3\lambda_k + \lambda_{k+1}}{4}\right|\right] V_{\lambda_k}^{\lambda_{k+1}}(\langle E_{(\cdot)}x, y\rangle) \\ &\leqslant \frac{1}{2^r} H \sum_{k=0}^{n-1} (\lambda_{k+1} - \lambda_k)^r V_{\lambda_k}^{\lambda_{k+1}}(\langle E_{(\cdot)}x, y\rangle) \\ &\leqslant \frac{H}{2^r} \nu^r(\Delta_n)\|x\|\|y\|, \end{aligned} \tag{62}$$

for any $x, y \in H$ and the intermediate points $\xi_k \in [\lambda_k, \frac{1}{2}(\lambda_k + \lambda_{k+1})]$, where $0 \leqslant k \leqslant n-1$. In particular, we have

$$\langle f(U)x, y\rangle = T_n(f, U, \Delta_n; x, y) + E_n(f, U, \Delta_n; x, y), \tag{63}$$

with the error

$$\begin{aligned}|E_n(f,U,\Delta_n;x,y)| &\leqslant 2^r H\sum_{k=0}^{n-1}\sin^r\left(\frac{\lambda_{k+1}-\lambda_k}{8}\right)V_{\lambda_k}^{\lambda_{k+1}}(\langle E_{(\cdot)}x,y\rangle)\\ &\leqslant \frac{H}{4^r}\nu^r(\Delta_n)\|x\|\|y\|, \end{aligned}\tag{64}$$

for any vector $x,y\in H$.

Proof: For any $x,y\in H$, we define $u(\lambda):=\langle E_\lambda x,y\rangle$, $\lambda\in[0,2\pi]$. We know that u is of bounded variation, and from Ref. 3,

$$V_0^{2\pi}(u)=:V_0^{2\pi}(\langle E_{(\cdot)}x,y\rangle)\leqslant\|x\|\|y\|.\tag{65}$$

By (7) and (65), (62) can be proved. □

Remark 9 If we choose $\xi_k=\frac{1}{2}(\lambda_k+\lambda_{k+1})$, we recapture Theorem 6 of Ref. 3. If we choose $\xi_k=(3\lambda_k+\lambda_{k+1})/4$, we have $|E_n(f,U,\Delta_n;x,y)|\leqslant(H/4^r)\nu^r(\Delta_n)\|x\|\|y\|$, which is more precise than Theorem 6 of Ref. 3.

Remark 10 In the case when the partition reduces to the whole interval $[0,2\pi]$, then making use of (7), for any $s\in[0,\pi]$ and any vectors $x,y\in H$, we have the bound

$$\begin{aligned}\Big|f(e^{is})\langle E_\pi x,y\rangle+f(e^{i(2\pi-s)})\langle(1_H-E_\pi)x,y\rangle\\ -\langle f(U)x,y\rangle\Big|\\ \leqslant 2^r H\sin^r\left[\frac{\pi}{4}+\frac{1}{2}\left|s-\frac{\pi}{2}\right|\right]V_0^{2\pi}(\langle E_{(\cdot)}x,y\rangle).\end{aligned}\tag{66}$$

If we obtain $s=\frac{1}{2}\pi$, we obtain the best inequality

$$\begin{aligned}|f(i)\langle E_\pi x,y\rangle+f(-i)\langle(1_H-E_\pi)x,y\rangle\\ -\langle f(U)x,y\rangle|\\ \leqslant 2^{r/2}HV_0^{2\pi}(\langle E_{(\cdot)}x,y\rangle)\leqslant 2^{r/2}H\|x\|\|y\|,\end{aligned}\tag{67}$$

for any vectors $x,y\in H$.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U. Depending only one vector $x\in H$, we can introduce the following sums

$$\begin{aligned}\tilde{O}_n(f,U,\Delta_n,\xi;x) &:=\sum_{k=0}^{n-1}f(e^{i\xi_k})\left\langle\left(E_{(\lambda_k+\lambda_{k+1})/2}-E_{\lambda_k}\right)x,\ x\right\rangle\\ &+\sum_{k=0}^{n-1}f(e^{i(\lambda_k+\lambda_{k+1}-\xi_k)})\left\langle\left(E_{\lambda_{k+1}}-E_{(\lambda_k+\lambda_{k+1})/2}\right)x,x\right\rangle.\end{aligned}\tag{68}$$

Theorem 6 *If* $f:C(0,1)\to\mathbb{C}$ *is Lipschitzian with the constant* $L>0$ *on the circle* $C(0,1)$, Δ_n *with* $\nu(\Delta_n)\leqslant\pi$ *and* U, $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ *are defined above, then we have the representation*

$$\langle f(U)x,x\rangle=\tilde{O}_n(f,U,\Delta_n,\xi;x)+\tilde{R}_n(f,U,\Delta_n,\xi;x)\tag{69}$$

with the error $\tilde{R}_n(f,U,\Delta_n,\xi;x)$ *which satisfies the bounds*

$$\begin{aligned}\left|\tilde{R}_n(f,U,\Delta_n,\xi;x)\right| &\leqslant 2L\sum_{k=0}^{n-1}\sin\left[\frac{\lambda_{k+1}-\lambda_k}{8}+\frac{1}{2}\left|\xi_k-\frac{3\lambda_k+\lambda_{k+1}}{4}\right|\right]\langle(E_{\lambda_{k+1}}-E_{\lambda_k})x,x\rangle\\ &\leqslant\frac{\nu(\Delta_n)}{2}L\|x\|^2,\end{aligned}\tag{70}$$

for any vectors $x\in H$ *and the intermediate points* $\xi_k\in[\lambda_k,\frac{1}{2}(\lambda_k+\lambda_{k+1})]$, *where* $0\leqslant k\leqslant n-1$. *In particular, we have*

$$\langle f(U)x,x\rangle=\tilde{T}_n(f,U,\Delta_n;x)+\tilde{E}_n(f,U,\Delta_n;x)\tag{71}$$

with the error

$$\begin{aligned}\left|\tilde{E}_n(f,U,\Delta_n;x)\right| &\leqslant\frac{1}{4}L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\langle(E_{\lambda_{k+1}}-E_{\lambda_k})x,\ x\rangle\\ &\leqslant\frac{1}{4}L\nu(\Delta_n)\|x\|^2,\end{aligned}\tag{72}$$

for any $x\in H$.

Proof: The proof follows from Proposition 3 applied for the monotonic nondecreasing function $u(t):=\langle E_t x,x\rangle$, $t\in[0,2\pi]$. □

Remark 11 If we choose $\xi_k=\frac{1}{2}(\lambda_k+\lambda_{k+1})$, we recapture Theorem 7 of Ref. 3. If we choose $\xi_k=(3\lambda_k+\lambda_{k+1})/4$, we have $|\tilde{E}_n(f,U,\Delta_n;x)|\leqslant\frac{1}{4}L\nu(\Delta_n)\|x\|^2$, which is more precise than Theorem 7 of Ref. 3.

Remark 12 In the case when the partition reduces to the whole interval $[0,2\pi]$, then by the above result, we obtain

$$\Big|f(e^{is})\langle E_\pi x,x\rangle+f(e^{i(2\pi-s)})\langle(1_H-E_\pi)x,x\rangle-\langle f(U)x,x\rangle\Big|$$

$$\leqslant L\int_0^{\pi}\operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)\langle E_t x,\ x\rangle\,\mathrm{d}t$$
$$+L\int_{\pi}^{2\pi}\operatorname{sgn}(2\pi-s-t)\cos\left(\frac{2\pi-s-t}{2}\right)\langle E_t x,x\rangle\,\mathrm{d}t$$
$$+2L\sin\left(\frac{s}{2}\right)\|x\|^2, \quad (73)$$

for any $s\in[0,\pi]$ and $x\in H$.

Example 1 We choose two complex functions as follows to provide some simple examples for the inequalities above.

(a) Consider the power function $f : C\backslash\{0\}\to\mathbb{C}, f(z)=z^m$ where m is a non-zero integer. Then, clearly, for any z, w belonging to the unit circle $C(0,1)$, we have the inequality

$$|f(z)-f(w)|\leqslant|m|\,|z-w|,$$

which shows that f is Lipschitzian with the constant $L=|m|$ on the circle $C(0,1)$. Then from (63), for any unitary operator U, we obtain

$$\left|\mathrm{e}^{ims}\langle E_\pi x,y\rangle+\mathrm{e}^{im(2\pi-s)}\langle(1_H-E_\pi)x,y\rangle-\langle U^m x,y\rangle\right|$$
$$\leqslant 2\,|m|\sin\left[\frac{\pi}{4}+\tfrac{1}{2}\left|s-\frac{\pi}{2}\right|\right]V_0^{2\pi}(\langle E_{(\cdot)}x,y\rangle), \quad (74)$$

for any vectors $x,y\in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U. If we obtain $s=\frac{1}{2}\pi$, then the best inequality is

$$|\mathrm{i}^m\langle E_\pi x,y\rangle+(-\mathrm{i})^m\langle(1_H-E_\pi)x,y\rangle-\langle U^m x,y\rangle|$$
$$\leqslant\sqrt{2}\,|m|\,V_0^{2\pi}(\langle E_{(\cdot)}x,y\rangle)\leqslant\sqrt{2}\,|m|\,\|x\|\,\|y\|, \quad (75)$$

for any vectors $x,y\in H$.

(b) For $a\neq\pm1$, 0, consider the function $f : C(0,1)\to\mathbb{C}$, $f_a(z)=1/(1-az)$. From Ref. 3, we have

$$|f_a(z)-f_a(w)|\leqslant\frac{|a|}{(1-|a|)^2}\,|z-w|, \quad (76)$$

for any $z,w\in C(0,1)$, showing that the function f_a is Lipschitzian with the constant $L_a=|a|/(1-|a|)^2$ on the circle $C(0,1)$. Then from (63), for any unitary operator U, we obtain

$$|(1-a\,\mathrm{e}^{\mathrm{i}s})^{-1}\langle E_\pi x,y\rangle-\langle(1_H-aU)^{-1}x,y\rangle$$
$$+(1-a\,\mathrm{e}^{\mathrm{i}(2\pi-s)})^{-1}\langle(1_H-E_\pi)x,y\rangle|$$
$$\leqslant\frac{2\,|a|}{(1-|a|)^2}\sin\left[\frac{\pi}{4}+\tfrac{1}{2}\left|s-\frac{\pi}{2}\right|\right]V_0^{2\pi}(\langle E_{(\cdot)}x,y\rangle), \quad (77)$$

for any vectors $x,y\in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U. If we obtain $s=\frac{1}{2}\pi$, then the best inequality is

$$|(1-a\mathrm{i})^{-1}\langle E_\pi x,y\rangle-\langle(1_H-aU)^{-1}x,y\rangle$$
$$+(1+a\mathrm{i})^{-1}\langle(1_H-E_\pi)x,y\rangle|$$
$$\leqslant\frac{\sqrt{2}\,|a|}{(1-|a|)^2}V_0^{2\pi}(\langle E_{(\cdot)}x,y\rangle)\leqslant\frac{\sqrt{2}\,|a|}{(1-|a|)^2}\,\|x\|\,\|y\|, \quad (78)$$

for any vectors $x,y\in H$.

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