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Stability of an alternative functional equation related to Jensen's functional equation

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ABSTRACT: Given an integer $\lambda \neq 1$, we verify the Hyers-Ulam stability of the alternative Jensen's functional equations $f(xy^{-1})-2f(x)+\lambda f(xy)=0$ where f is a mapping from a 2-divisible group to a Banach space and λ is an integer.

KEYWORDS: alternative equation, Jensen's functional equation

MSC2010: 39B82 39B72

INTRODUCTION

The alternative Cauchy functional equations have been widely studied. For instance, Kannappan and Kuczma[1] studied the solutions of the alternative Cauchy functional equations of the form

$$(f(x+y)-af(x)-bf(y))(f(x+y)-f(x)-f(y))=0, \quad (1)$$

where f is a function from an abelian group to a commutative integral domain and of characteristic zero. Ger[2] extended (1) to the alternative functional equation

$$(f(x+y)-af(x)-bf(y))(f(x+y)-cf(x)-\mathrm{d}f(y))=0.$$

Forti[3] then established the general solution of the alternative Cauchy functional equations

$$(cf(x+y)-af(x)-bf(y)-d)(f(x+y)-f(x)-f(y))=0.$$

Nakmahachalasint[4] first studied the solutions of an alternative Jensen's functional equations of the form

$$f(x)\pm 2f(xy)+f(xy^2)=0 \quad (2)$$

on a semigroup which extended the work in Refs. 5, 6 on the classical Jensen's functional equation

$$f(xy^{-1})-2f(x)+f(xy)=0 \quad (3)$$

on a group. Nakmahachalasint[7] also investigated the Hyers-Ulam stability of the alternative Jensen's functional equation (2) in the class of mappings from 2-divisible abelian groups to Banach spaces.

Given an integer $\lambda \neq 1$, Srisawat, Kitisin and Nakmahachalasint studied the solution of the alternative Jensen's functional equation of the form[8]

$$f(xy^{-1})-2f(x)+f(xy)=0 \quad \text{or} \quad f(xy^{-1})-2f(x)+\lambda f(xy)=0 \quad (4)$$

when f is a function from a group to a uniquely divisible abelian group, but the stability problem has not yet been investigated. This paper aims to prove the Hyers-Ulam stability of the alternative Jensen's functional equation (4) when f is a mapping from a 2-divisible abelian group $(G,\cdot)$ to a Banach space $(E,\|\cdot\|)$. In other words, for every $\varepsilon \geqslant 0$, we show that there exist $\delta_1,\delta_2 \geqslant 0$ such that if a mapping $f: G \to E$ satisfies the inequalities

$$\|f(xy^{-1})-2f(x)+f(xy)\| \leqslant \delta_1 \quad \text{or} \quad \|f(xy^{-1})-2f(x)+\lambda f(xy)\| \leqslant \delta_2 \quad (5)$$

for all $x,y \in G$, then there exists a unique Jensen's mapping $J: G \to E$ with

$$\|f(x)-J(x)\| \leqslant \varepsilon$$

for all $x \in G$.

AUXILIARY LEMMAS

Let $(G,\cdot)$ be a group and $(E,\|\cdot\|)$ be a Banach space. Given an integer λ and a function $f: G \to E$, for every pair $x,y \in G$ we define

$$\mathscr{F}_y^{(\lambda)}(x) := \|f(xy^{-1})-2f(x)+\lambda f(xy)\|.$$

For $\delta_1, \delta_2 \geqslant 0$ and $\lambda \neq 1$, we write

$$\mathscr{S} f_y^{(\lambda)}(x) := \left(\mathscr{F}_y^{(1)}(x) \leqslant \delta_1 \quad \text{or} \quad \mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2 \right),$$

$$\begin{aligned} \mathscr{M}_{\delta_1,\delta_2}^{\lambda} := {} & (8+19|\lambda|+14\lambda^2+3|\lambda^3|)\delta_1 \\ & +(61+88|\lambda|+31\lambda^2+3|\lambda^3|)\delta_2 \end{aligned}$$

and we denote the statement

$$\mathscr{A}_{(G,E)}^{(\lambda)} := \{f : G \to E \mid \mathscr{S} f_y^{(\lambda)}(x) \text{ for all } x, y \in G\}.$$

We first prove two lemmas concerning $\mathscr{S} f_y^{(\lambda)}(x)$.

Lemma 1 *Let $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$ and $x, y \in G$. Given $\alpha \geqslant 0$, if $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$ and $\|f(xy)\| \leqslant \alpha$, then*

$$\mathscr{F}_y^{(1)}(x) \leqslant \delta_2 + (1+|\lambda|)\alpha. \tag{6}$$

Proof: Assume that $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$ and $\|f(xy)\| \leqslant \alpha$. Thus

$$\begin{aligned} \|f(xy^{-1}) - 2f(x)\| &\leqslant \mathscr{F}_y^{(\lambda)}(x) + \|-\lambda f(xy)\| \\ &\leqslant \delta_2 + |\lambda|\alpha. \end{aligned} \tag{7}$$

By $\|f(xy)\| \leqslant \alpha$ and (7), we obtain

$$\begin{aligned} \mathscr{F}_y^{(1)}(x) &\leqslant \|f(xy^{-1}) - 2f(x)\| + \|f(xy)\| \\ &\leqslant \delta_2 + (1+|\lambda|)\alpha. \end{aligned}$$

□

Lemma 2 *Let $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$ and $x, y \in G$. If $\mathscr{F}_y^{(1)}(x) > \delta_1$, then $\|f(xy^{-1}) - f(xy)\| \leqslant 2\delta_2$.*

Proof: Assume that $\mathscr{F}_y^{(1)}(x) > \delta_1$. The alternatives in $\mathscr{S} f_{y^{-1}}^{(\lambda)}(x)$ and $\mathscr{S} f_y^{(\lambda)}(x)$ give

$$\mathscr{F}_{y^{-1}}^{(\lambda)}(x) \leqslant \delta_2, \qquad \mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2, \tag{8}$$

respectively. Eliminating $f(x)$ from (8), we obtain

$$\|(1-\lambda)(f(xy^{-1}) - f(xy))\| \leqslant 2\delta_2.$$

Since $|1-\lambda| \geqslant 1$, we must have

$$\|f(xy^{-1}) - f(xy)\| \leqslant 2\delta_2$$

as desired. □

Next, we prove four lemmas concerning $\mathscr{S} f_{y^2}^{(\lambda)}(x)$.

Lemma 3 *Let $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$ and $x, y \in G$.*

(i) *If $\mathscr{F}_{y^2}^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(xy) > \delta_1$, then $\mathscr{F}_{y^2}^{(1)}(x) \leqslant 6\delta_2$.*

(ii) *If $\mathscr{F}_{y^2}^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(xy^{-1}) > \delta_1$, then $\mathscr{F}_{y^2}^{(1)}(x) \leqslant 6\delta_2$.*

Proof: Case (i). Assume that $\mathscr{F}_{y^2}^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(xy) > \delta_1$. By Lemma 2, we obtain

$$\begin{aligned} \|f(xy^{-2}) - f(xy^2)\| &\leqslant 2\delta_2, \\ \|f(x) - f(xy^2)\| &\leqslant 2\delta_2, \end{aligned}$$

respectively. From the above inequality, we obtain

$$\|f(xy^{-2}) - 2f(x) + f(xy^2)\| \leqslant 6\delta_2.$$

Hence $\mathscr{F}_{y^2}^{(1)}(x) \leqslant 6\delta_2$.

Case (ii). The proof is as in case (i) after replacing y by y^{-1}. □

Lemma 4 *Let $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$ and $x, y \in G$. If $\mathscr{F}_{y^2}^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(x) \leqslant \delta_1$, then*

$$\mathscr{F}_{y^2}^{(1)}(x) \leqslant 6\max\{\delta_1, \delta_2\}.$$

Proof: Assume that $\mathscr{F}_{y^2}^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(x) \leqslant \delta_1$. We first consider the alternatives in $\mathscr{S} f_y^{(\lambda)}(xy^{-1})$ and $\mathscr{S} f_y^{(\lambda)}(xy)$ as follows. If $\mathscr{F}_y^{(1)}(xy^{-1}) > \delta_1$ or $\mathscr{F}_y^{(1)}(xy) > \delta_1$, then Lemma 3 gives $\mathscr{F}_{y^2}^{(1)}(x) \leqslant 6\delta_2$. Then we assume that $\mathscr{F}_y^{(1)}(xy^{-1}) \leqslant \delta_1$ and $\mathscr{F}_y^{(1)}(xy) \leqslant \delta_1$. Thus

$$\mathscr{F}_{y^2}^{(1)}(x) \leqslant \mathscr{F}_y^{(1)}(xy^{-1}) + 2\mathscr{F}_y^{(1)}(x) + \mathscr{F}_y^{(1)}(xy) \leqslant 6\delta_1.$$

□

Lemma 5 *Let $f \in \mathscr{A}_{(G,E)}^{(-1)}$ and $x, y \in G$. If $\mathscr{F}_y^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(xy) \leqslant \delta_1$, then*

$$\|f(xy^2)\| \leqslant \max\{3\delta_1 + 6\delta_2, 5\delta_1 + 4\delta_2\}. \tag{9}$$

Proof: Assume that $\mathscr{F}_y^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(xy) \leqslant \delta_1$. From $\mathscr{F}_y^{(1)}(x) > \delta_1$, Lemma 2 gives

$$\|f(xy^{-1}) - f(xy)\| \leqslant 2\delta_2. \tag{10}$$

By $\mathscr{F}_y^{(1)}(x) > \delta_1$, the alternatives in $\mathscr{S} f_y^{(-1)}(x)$ and $\mathscr{S} f_{y^{-1}}^{(-1)}(x)$ give $\mathscr{F}_y^{(-1)}(x) \leqslant \delta_2$ and $\mathscr{F}_{y^{-1}}^{(-1)}(x) \leqslant \delta_2$, respectively. Hence

$$\|2f(x)\| \leqslant \frac{1}{2}\left(\mathscr{F}_y^{(-1)}(x) + \mathscr{F}_{y^{-1}}^{(-1)}(x) \right) \leqslant \delta_2. \tag{11}$$

From (11) and $\mathscr{F}_y^{(1)}(xy) \leqslant \delta_1$ we obtain

$$\|4f(xy) - 2f(xy^2)\| \leqslant 2\delta_1 + \delta_2. \quad (12)$$

Next, we will consider the following two possible cases in $\mathscr{S}f_y^{(-1)}(xy^2)$.

Case (i). Assume that $\mathscr{F}_y^{(-1)}(xy^2) \leqslant \delta_2$. Eliminating $f(xy^2)$ from (12) and $\mathscr{F}_y^{(-1)}(xy^2) \leqslant \delta_2$, we have

$$\|3f(xy) + f(xy^3)\| \leqslant 2\delta_1 + 2\delta_2. \quad (13)$$

By (10) and (13), we obtain

$$\|f(xy^{-1}) + 2f(xy) + f(xy^3)\| \leqslant 2\delta_1 + 4\delta_2 \quad (14)$$

and

$$\|f(xy^{-1}) - 4f(xy) - f(xy^3)\| \leqslant 2\delta_1 + 4\delta_2. \quad (15)$$

Consider $\mathscr{S}f_{y^2}^{(-1)}(xy)$ as follows. The alternative $\mathscr{F}_{y^2}^{(1)}(xy) \leqslant \delta_1$ and (14) give

$$\|4f(xy)\| \leqslant 3\delta_1 + 4\delta_2, \quad (16)$$

while the alternative $\mathscr{F}_{y^2}^{(-1)}(xy) \leqslant \delta_2$ and (15) give

$$\|2f(xy)\| \leqslant 2\delta_1 + 5\delta_2. \quad (17)$$

By (12), (16) and (17), we obtain

$$\|f(xy^2)\| \leqslant 3\delta_1 + 6\delta_2.$$

Case (ii). Assume that $\mathscr{F}_y^{(1)}(xy^2) \leqslant \delta_1$. Eliminating $f(xy^2)$ from (12) and $\mathscr{F}_y^{(1)}(xy^2) \leqslant \delta_1$, we have

$$\|3f(xy) - f(xy^3)\| \leqslant 3\delta_1 + \delta_2. \quad (18)$$

By (10) and (18), we obtain

$$\|f(xy^{-1}) - 4f(xy) + f(xy^3)\| \leqslant 3\delta_1 + 3\delta_2 \quad (19)$$

and

$$\|f(xy^{-1}) + 2f(xy) - f(xy^3)\| \leqslant 3\delta_1 + 3\delta_2, \quad (20)$$

follows. The alternative $\mathscr{F}_{y^2}^{(1)}(xy) \leqslant \delta_1$ and (19) give

$$\|2f(xy)\| \leqslant 4\delta_1 + 3\delta_2, \quad (21)$$

while the alternative $\mathscr{F}_{y^2}^{(-1)}(xy) \leqslant \delta_2$ and (20) give

$$\|4f(xy)\| \leqslant 3\delta_1 + 4\delta_2. \quad (22)$$

By (12), (21) and (22), we obtain

$$\|f(xy^2)\| \leqslant 5\delta_1 + 4\delta_2.$$

From the two cases, we have (9) as desired. □

Lemma 6 *Let $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$ and let $x, y \in G$. If $\mathscr{F}_{y^2}^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(x) > \delta_1$, then*

$$\mathscr{F}_{y^2}^{(1)}(x) \leqslant \mathscr{M}_{\delta_1,\delta_2}^{\lambda}. \quad (23)$$

Proof: From $\mathscr{F}_y^{(1)}(x) > \delta_1$, we obtain

$$\|f(xy^{-1}) - f(xy)\| \leqslant 2\delta_2 \quad (24)$$

by Lemma 2. We will consider the alternatives in $\mathscr{S}f_y^{(\lambda)}(xy)$ as follows. If $\mathscr{F}_y^{(1)}(xy) > \delta_1$, then Lemma 3 give $\mathscr{F}_{y^2}^{(1)}(x) \leqslant 6\delta_2$ which satisfies (23). Thus we assume that $\mathscr{F}_y^{(1)}(xy) \leqslant \delta_1$. First, suppose that $\lambda = -1$. From $\mathscr{F}_y^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(xy) \leqslant \delta_1$, by Lemma 5, we obtain

$$\|f(xy^2)\| \leqslant \max\{3\delta_1 + 6\delta_2, 5\delta_1 + 4\delta_2\}.$$

Second, suppose that $\lambda \neq -1$. Since $\mathscr{F}_y^{(1)}(x) > \delta_1$, the alternatives in $\mathscr{S}f_y^{(\lambda)}(x)$ gives $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$. By (24) and $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$, we obtain

$$\|2f(x) - (1+\lambda)f(xy)\| \leqslant 3\delta_2. \quad (25)$$

Eliminating $f(x)$ from (25) and $\mathscr{F}_y^{(1)}(xy) \leqslant \delta_1$, we obtain

$$\|(3-\lambda)f(xy) - 2f(xy^2)\| \leqslant 2\delta_1 + 3\delta_2. \quad (26)$$

Next, we will consider the following two possible cases in $\mathscr{S}f_y^{(\lambda)}(xy^{-1})$.

Case (i). Assume that $\mathscr{F}_y^{(1)}(xy^{-1}) > \delta_1$. Since $\mathscr{F}_{y^2}^{(1)}(x) > \delta_1$ and $\mathscr{F}_y^{(1)}(xy^{-1}) > \delta_1$, Lemma 3 gives $\mathscr{F}_{y^2}^{(1)}(x) \leqslant 6\delta_2$ which satisfies (23).

Case (ii). Assume that $\mathscr{F}_y^{(1)}(xy^{-1}) \leqslant \delta_1$. We eliminate $f(xy^{-1})$ and $f(x)$ from (24), (25) and $\mathscr{F}_y^{(1)}(xy^{-1}) \leqslant \delta_1$ to obtain

$$\|2f(xy^{-2}) - (3-\lambda)f(xy)\| \leqslant 2\delta_1 + 11\delta_2. \quad (27)$$

From $\mathscr{F}_{y^2}^{(1)}(x) > \delta_1$, the alternative $\mathscr{S}f_{y^2}^{(\lambda)}(x)$ gives $\mathscr{F}_{y^2}^{(\lambda)}(x) \leqslant \delta_2$. Then we eliminate $f(x)$ and $f(xy^2)$ from (25), (26) and $\mathscr{F}_{y^2}^{(\lambda)}(x) \leqslant \delta_2$ to obtain

$$\begin{aligned}\|2f(xy^{-2}) - (2-\lambda+\lambda^2)f(xy)\| \\ \leqslant 2|\lambda|\delta_1 + (8+3|\lambda|)\delta_2. \quad (28)\end{aligned}$$

From (27) and (28), we obtain

$$\|(1-\lambda^2)f(xy)\| \leqslant (2+2|\lambda|)\delta_1 + (19+3|\lambda|)\delta_2. \quad (29)$$

Since $|1-\lambda^2| \geqslant 1$, (29) simplifies to

$$\|f(xy)\| \leqslant (2+2|\lambda|)\delta_1 + (19+3|\lambda|)\delta_2. \quad (30)$$

From (26) and (30), we use $|\lambda-3| \leqslant |\lambda|+3$ to obtain

$$\|f(xy^2)\| \leqslant (8+11|\lambda|+3\lambda^2)\delta_1 + (60+28|\lambda|+3\lambda^2)\delta_2. \quad (31)$$

From the above two cases, we obtain (31). Hence by Lemma 1 we obtain (23) as desired. □

HYERS-ULAM STABILITY

In this section, we will prove the Hyers-Ulam stability of the alternative Jensen's functional equation (4). The following lemma is crucial for the main theorem.

Lemma 7 *Let $(G,\cdot)$ be a 2-divisible group. If $f \in \mathscr{A}^{(\lambda)}_{(G,E)}$, then $\mathscr{F}^{(1)}_y(x) \leqslant \mathscr{M}^{\lambda}_{\delta_1,\delta_2}$ for all $x, y \in G$.*

Proof: Let $f \in \mathscr{A}^{(\lambda)}_{(G,E)}$ and $x, y \in G$. Since G is a 2-divisible group, there exists $z \in G$ such that $y = z^2$. Considering the alternatives in $\mathscr{S}f^{(\lambda)}_{z^2}(x)$ and $\mathscr{S}f^{(\lambda)}_{z}(x)$, the proof is complete by Lemma 4 and Lemma 6. □

It should be remarked that the 2-divisibility of the group $(G,\cdot)$ is important. In fact, Srisawat[8], Kitisin and Nakmahachalasint proved that (4) is equivalent to (3) when the domain of f is a 2-divisible group. For $\lambda = -3$, (4) becomes

$$f(xy^{-1}) - 2f(x) + f(xy) = 0 \quad \text{or}$$
$$f(xy^{-1}) - 2f(x) - 3f(xy) = 0. \quad (32)$$

However, when the domain of f is not a 2-divisible group, (32) does not need to be equivalent to (3) as illustrated by the following example.

Example 1 Given $a \in E\backslash\{0\}$. Let $f : \mathbb{Z} \to E$ be a mapping such that

$$f(n) = (-1)^n a \text{ for all } n \in \mathbb{Z}.$$

We will first prove that f satisfies (32). Given $n, m \in \mathbb{Z}$. If m is odd, then we see that $n-m$ and $n+m$ have the same parity whereas n and $n+m$ have the opposite. Hence $f(n-m) - 2f(n) - 3f(n+m) = 0$. Otherwise, if m is even, then $n-m$, n, $n+m$ all have the same parity, i.e., $f(n-m)-2f(n)+f(n+m) = 0$. Next, we will show that f does not satisfy (3). It should be noted that $f(0)-2f(1)+f(2) = 4a$. From $a \neq 0$, we obtain $4a \neq 0$. Thus f satisfies (32) but f does not satisfy (3).

Next, we will prove the Hyers-Ulam stability of the alternative Jensen's functional equation (4) by the so-called direct method. The stability results of Jensen's functional equation can be found, for instance, in Ref. 9.

Theorem 1 *Let $(G,\cdot)$ be a 2-divisible group. If $f \in \mathscr{A}^{(\lambda)}_{(G,E)}$, then there exists a unique Jensen's mapping $J : G \to E$ satisfying (3) with $J(0) = f(0)$ such that*

$$\|f(x) - J(x)\| \leqslant 2\mathscr{M}^{\lambda}_{\delta_1,\delta_2} \quad \forall x \in G.$$

Furthermore, the mapping J is given by

$$J(x) = f(0) + \lim_{n\to\infty} \frac{1}{2^n}(f(x^{2^n}) - f(0)) \quad \forall x \in G.$$

Proof: Assume that $f \in \mathscr{A}^{(\lambda)}_{(G,E)}$. By Lemma 7, we obtain $\mathscr{F}^{(1)}_y(x) \leqslant \mathscr{M}^{\lambda}_{\delta_1,\delta_2}$ for all $x, y \in G$, i.e.,

$$\|f(xy^{-1}) - 2f(x) + f(xy)\| \leqslant \mathscr{M}^{\lambda}_{\delta_1,\delta_2}.$$

We define a function $\tilde{f} : G \to E$ by

$$\tilde{f}(x) = f(x) - f(0) \quad \text{for all } x \in G.$$

It can be observed that $\tilde{f}(0) = 0$. Then for each $x, y \in G$, we have

$$\left\| \tfrac{1}{2}\big(\tilde{f}(xy) + \tilde{f}(xy^{-1})\big) - \tilde{f}(x) \right\| \leqslant \tfrac{1}{2}\mathscr{M}^{\lambda}_{\delta_1,\delta_2}. \quad (33)$$

Put $y = x$ in (33). Using $\tilde{f}(0) = 0$ we obtain

$$\left\| \frac{\tilde{f}(x^2)}{2} - \tilde{f}(x) \right\| \leqslant \tfrac{1}{2}\mathscr{M}^{\lambda}_{\delta_1,\delta_2}. \quad (34)$$

For each positive integer n and each $x \in G$, we apply (34) to obtain

$$\left\| \frac{\tilde{f}(x^{2^n})}{2^n} - \tilde{f}(x) \right\| = \left\| \sum_{i=1}^{n} \left(\frac{\tilde{f}(x^{2^i})}{2^i} - \frac{\tilde{f}(x^{2^{i-1}})}{2^{i-1}} \right) \right\| \leqslant \left(1 - \frac{1}{2^n}\right)\mathscr{M}^{\lambda}_{\delta_1,\delta_2}. \quad (35)$$

Consider the sequence $\{2^{-n}f(x^{2^n})\}$. For all positive integers m, n and every $x \in X$, we use (35) to obtain

$$\left\| \frac{\tilde{f}(x^{2^{n+m}})}{2^{n+m}} - \frac{\tilde{f}(x^{2^n})}{2^n} \right\| = \frac{1}{2^n} \left\| \frac{\tilde{f}(x^{2^n\cdot 2^m})}{2^m} - \tilde{f}(x^{2^n}) \right\| \leqslant \frac{1}{2^n}\left(1 - \frac{1}{2^m}\right)\mathscr{M}^{\lambda}_{\delta_1,\delta_2}.$$

Hence $\{2^{-n}f(x^{2^n})\}$ is a Cauchy sequence. We can define a function $\tilde{J} : G \to E$ by

$$\tilde{J}(x) = \lim_{n\to\infty} \frac{\tilde{f}(x^{2^n})}{2^n} \quad \forall x \in G.$$

Replacing x by x^{2^n} and y by y^{2^n} in (33), we obtain

$$\left\|\tfrac{1}{2}\big(\tilde{f}(x^{2^n}y^{2^n})+\tilde{f}(x^{2^n}y^{-2^n})\big)-\tilde{f}(x^{2^n})\right\| \leqslant \tfrac{1}{2}\mathscr{M}^{\lambda}_{\delta_1,\delta_2}. \tag{36}$$

Next, multiplying (36) by 2^{-n} and taking $n \to \infty$, we obtain

$$\tilde{J}(xy)+\tilde{J}(xy^{-1})-2\tilde{J}(x)=0.$$

From (35), as $n \to \infty$, we have

$$\|\tilde{f}(x)-\tilde{J}(x)\| \leqslant \mathscr{M}^{\lambda}_{\delta_1,\delta_2} \qquad \forall x \in G.$$

To show the uniqueness of $\tilde{J}$, let $\mathscr{J} : G \to E$ satisfy $\mathscr{J}(0)=0$ and

$$\|\tilde{f}(x)-\mathscr{J}(x)\| \leqslant \mathscr{M}^{\lambda}_{\delta_1,\delta_2} \qquad \forall x \in G.$$

For every positive integer n, we have

$$\tilde{J}(x^{2^n}) = 2^n\tilde{J}(x), \qquad \mathscr{J}(x^{2^n}) = 2^n\mathscr{J}(x).$$

Hence

$$\begin{aligned}
&\|\mathscr{J}(x)-\tilde{J}(x)\| \\
&\quad = \left\|\frac{1}{2^n}\big(\tilde{J}(x^{2^n})-\tilde{f}(x^{2^n})\big)-\frac{1}{2^n}\big(\tilde{f}(x^{2^n})-\mathscr{J}(x^{2^n})\big)\right\| \\
&\quad \leqslant \frac{1}{2^n}\left\|\tilde{f}(x^{2^n})-\tilde{J}(x^{2^n})\right\|+\frac{1}{2^n}\left\|\tilde{f}(x^{2^n})-\mathscr{J}(x^{2^n})\right\| \\
&\quad \leqslant \frac{2}{2^n}\mathscr{M}^{\lambda}_{\delta_1,\delta_2}.
\end{aligned} \tag{37}$$

As $n \to \infty$ in (37), we have $\mathscr{J}(x) = \tilde{J}(x)$ for all $x \in G$. By defining a function $J : G \to E$ by $J(x) = \tilde{J}(x)+f(0)$ for all $x \in G$, the proof is complete. □

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