Commutator subgroups of Vershik-Kerov groups for infinite symplectic groups

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ABSTRACT: Let $R$ be a commutative ring with identity 1. We describe two kinds of Vershik-Kerov groups for the symplectic case: $\text{Sp}_{\infty}(2,\infty,R)$ and $\text{GSp}_{\infty}(2,\infty,R)$. We also determine the commutator subgroups of these groups over a wide class of commutative rings. For an arbitrary infinite field, we find the bounds for the commutator width of the groups $\text{Sp}_{\infty}(2,\infty,K)$ and $\text{GSp}_{\infty}(2,\infty,K)$.

KEYWORDS: infinite triangular matrices, infinite unitriangular matrices, commutator width, lower central series

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INTRODUCTION

Let $R$ be an associative ring with identity 1. By $\text{GL}_R(\infty,R)$, $\text{GL}_c(\infty,R)$, $\text{GL}_{\infty}(\infty,R)$ we denote the groups of all infinite dimensional (indexed by $\mathbb{N}$) column-finite, row-finite, row-column-finite invertible matrices over $R$, respectively. By $\text{GL}_{\infty}(\infty,R)$ we denote the Vershik-Kerov group which is the subgroup of $\text{GL}_R(\infty,R)$ consisting of matrices having only a finite number of non-zero entries below the main diagonal. The group $\text{GL}_{\infty}(\infty,R)$ stems from asymptotic representation theory which connects functional analysis, algebra, and combinatorial probability theory, and is related to classical groups of infinite dimensions$^1$-$^3$. In recent years, some important subgroups of Vershik-Kerov group have been studied. Gupta and Hołubowski determined the commutator subgroup of Vershik-Kerov group over an infinite field$^4$ and a wide class of associative rings$^5$. Parabolic subgroups of Vershik-Kerov group are described in Refs. 6, 7. Słowik studied the lower central series of subgroups of the Vershik-Kerov group in Ref. 8.

Let $\text{Mat}_n(R)$ be the set of all $n \times n$ matrices over $R$. $\text{Mat}_{\infty}(R)$ stands for the set of all infinite dimensional matrices (indexed by $\mathbb{N}$). Denote by $\text{Mat}_{2,\infty}(R)$ the set $\text{Mat}_2(\text{Mat}_{\infty}(R))$ of $2 \times 2$ matrices with coefficients in $\text{Mat}_{\infty}(R)$. Denote by $\text{Mat}^{\text{fin}}_{2,\infty}(R)$ the set of all the matrices below

$$ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Mat}_{2,\infty}(R) $$

where $A$ is column-finite, $D$ is row-finite and $B$, $C$ are row-column-finite matrices. When $R$ is a commutative ring with identity 1, we define

$$ \text{Sp}^{\text{fin}}_{2,\infty}(R) = \left\{ M \in \text{Mat}^{\text{fin}}_{2,\infty}(R) \mid MHM' = H \right\}, $$

$$ \text{GSp}^{\text{fin}}_{2,\infty}(R) = \left\{ M \in \text{Mat}^{\text{fin}}_{2,\infty}(R) \mid MHM' = \lambda H \right\}, $$

where

$$ H = \begin{pmatrix} O & I \\ -I & 0 \end{pmatrix}, $$

$\lambda \in R^\ast$. $M'$ is the transpose of $M$, $I$ represents the identity matrices, and $O$ the zero matrices. In this paper, we are concerned about the group $\text{Sp}_{\infty}(2,\infty,R)$, which can be viewed as the symplectic case of the Vershik-Kerov group.

Let $R$ be a commutative ring with identity 1 and $\{v_1,\ldots,v_n,v_{n+1},\ldots\}$ a basis of an infinite dimensional (indexed by $\mathbb{N}$) linear space over $R$. By $T(\infty,R)$ we denote the group of all infinite dimensional (indexed by $\mathbb{N}$) upper triangular matrices whose entries on the main diagonal are invertible in $R$. We can find that the elements of $T(\infty,R)$ preserve the complete flag

$$ v_1 \subset \cdots \subset \langle v_1,\ldots,v_n \rangle \subset \langle v_1,\ldots,v_{n+1} \rangle \subset \cdots. $$

For the case of a $2n$-dimensional symplectic space $V$ with a basis $\{u_1,v_1,u_2,v_2,\ldots,u_n,v_n\}$, where $u_k$, $v_k$ ($1 \leq k \leq n$) is a hyperbolic pair, there is an orthogonal direct sum decomposition $V = \langle u_1,v_1 \rangle \perp \langle u_2,v_2 \rangle \perp \cdots \perp \langle u_n,v_n \rangle$. Let $W_k = \langle u_1,\ldots,u_k \rangle$ be
a $k$-dimensional totally isotropic subspace. Then
\[ W_k^\perp = \langle u_1, \ldots, u_k, u_{k+1}, \ldots, u_n, v_k, \ldots, v_n \rangle \] is a $(2n-k)$-dimensional subspace. Thus we can obtain a complete flag of $V$
\[ 0 \subset W_1 \subset \cdots \subset W_n = W_n^\perp \subset \cdots \subset W_1^\perp \subset V. \]
The group preserving the above complete flag should be
\[ \left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \text{Sp}(2n,R) \bigg| A \in T(n,R) \right\}, \]
which is a subgroup of $\text{Sp}(2n,R)$. Here we denote it by $TSp(2n,R)$. If we sequentially select \{u_1, \ldots, u_n, v_n, \ldots, v_1\} as the basis of $V$, we can show that all the elements of $TSp(2n,R)$ are upper triangular invertible matrices.

When we consider the infinite case, the complete flag of $V$ should be
\[ 0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n \subset \cdots, \]
\[ \cdots \subset W_n^\perp \subset W_{n-1}^\perp \subset \cdots \subset W_1^\perp \subset V. \]
And the group preserving this complete flag should be
\[ \left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \text{Sp}_{\infty}^\infty(R) \bigg| A \in T(\infty,R) \right\}. \]
Denote it by $TSp(2,\infty,R)$. Let $UT(\infty,R)$ be the group of all infinite dimensional (indexed by $\mathbb{N}$) upper triangular matrices whose entries on the main diagonal are identities. We can define a subgroup of $TSp(2,\infty,R)$
\[ \left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \text{Sp}_{\infty}^\infty(R) \bigg| A \in UT(\infty,R) \right\} \]
\[ = \text{USp}(2,\infty,R). \]

We can also define an overgroup of $TSp(2,\infty,R)$
\[ \left\{ \begin{pmatrix} A & B \\ O & \lambda(A')^{-1} \end{pmatrix} \in \text{GSp}_{\infty}^\infty(R) \bigg| A \in T(\infty,R) \right\} \]
\[ = \text{TGSp}(2,\infty,R). \]

For an associative ring $R$ with identity 1, we denote by $\text{GL}(n,R)$ the general linear group of $n \times n$ invertible matrices over $R$. $\text{E}(n,R)$ stands for the subgroup of $\text{GL}(n,R)$ generated by all elementary transvections $t_{ij}(\alpha) = I + \alpha E_{ij}$, with $1 \leq i \neq j \leq n$, $\alpha \in R$, where $E_{ij}$ denotes the matrix with 1 at the position $(i,j)$ and zeros elsewhere. When $R$ is a field, we know that the elementary subgroup $\text{E}(n,R)$ coincides with the special linear group $\text{SL}(n,R)$ over $R$. By $\text{GL}(\infty,n,R)$ we denote the subgroup of $\text{GSp}(\infty,\infty,R)$ consisting of all matrices of the form
\[ \begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} \end{pmatrix} \]
where $M_{11} \in \text{GL}(n,R)$, $M_{22} \in T(\infty,R)$. And by $\text{E}(\infty,n,R)$ we denote the subgroup of $\text{GL}(\infty,n,R)$ consisting of all matrices of the same form satisfying $M_{11} \in \text{GL}(n,R)$ and $M_{22} \in \text{UT}(\infty,R)$. It is clear that
\[ \text{GL}(\infty,n,R) \subseteq \text{GL}(\infty,n+1,R), \]
\[ \text{GSp}(\infty,\infty,R) = \bigcup_{n>1} \text{GL}(\infty,n,R), \]
and
\[ \text{E}(\infty,n,R) \subseteq \text{E}(\infty,n+1,R), \]
\[ \text{GSp}(\infty,\infty,R) = \bigcup_{n>1} \text{E}(\infty,n,R). \]

For infinite dimensional symplectic groups, we can define the Vershik-Kerov groups as follows:
\[ \left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \text{Sp}_{\infty}^{\infty}(R) \bigg| A \in \text{GSp}(\infty,\infty,R) \right\} = \text{Sp}_{\text{VK}}(2,\infty,R), \]
\[ \left\{ \begin{pmatrix} A & B \\ O & \lambda(A')^{-1} \end{pmatrix} \in \text{GSp}_{\infty}^{\infty}(R) \bigg| A \in \text{GSp}(\infty,\infty,R) \right\} = \text{GSp}_{\text{VK}}(2,\infty,R), \]
\[ \left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \text{Sp}_{\infty}^{\infty}(R) \bigg| A \in \text{E}(\infty,\infty,R) \right\} = \text{USp}_{\text{VK}}(2,\infty,R). \]

Now we give some notation which will be used in this paper. For a group $G$ and elements $a$ and $b$ of $G$, we write $[a, b] = a^{-1}b^{-1}ab$ as the commutator of $a$ and $b$. $[G, G]$ stands for the commutator subgroup of $G$ generated by all the commutators of the elements in $G$. Suppose $H$ is a subgroup of $G$, by $[H, G]$ we denote the subgroup of $G$ generated by all commutators $[h, g]$, where $h \in H, g \in G$. The lower central series of $G$ is defined inductively as
\[ \gamma_0(G) = G, \quad \gamma_{n+1}(G) = [\gamma_n(G), G] \quad \text{for} \ n \geq 0. \]

Denote by $c(G)$ the commutator width of $G$, which is the least integer $s$ such that every element of the commutator subgroup of $G$ is the product of at most $s$ commutators. If such an $s$ does not exist, we set $c(G) = \infty$.

The following problem has been discussed in Refs. 4, 5, 7, 8.
Problem 1 Does $E_{VK} (∞, R)$ coincide with the commutator subgroup of $GL_{VK} (∞, R)$?

The above problem was posed by Sushchan-skii at the conference, Groups and Their Actions, Bedlewo 2010. Gupta and Holubowicz gave a positive answer for fields and a wide class of associative rings\(^5\). For the symplectic case, we can investigate the following two problems.

Problem 2 Does $USp_{VK} (2, ∞, R)$ coincide with the commutator subgroup of $GSp_{VK} (2, ∞, R)$?

Problem 3 Does $USp_{VK} (2, ∞, R)$ coincide with the commutator subgroup of $Sp_{VK} (2, ∞, R)$?

Our main results are the following.

Theorem 1 Assume that $R$ is a commutative ring such that $1$ is a sum of two invertible elements. Then the commutator subgroup of $Sp_{VK} (2, ∞, R)$ coincides with the group $USp_{VK} (2, ∞, R)$.

Theorem 2 Assume that $R$ is a commutative ring such that $1$ is a sum of two invertible elements. Then the commutator subgroup of $GSp_{VK} (2, ∞, R)$ coincides with the group $USp_{VK} (2, ∞, R)$.

When we consider these kinds of symplectic groups over an infinite field, we have the following two theorems.

Theorem 3 Assume that $K$ is an infinite field. Then the commutator subgroup of $Sp_{VK} (2, ∞, K)$ coincides with the group $USp_{VK} (2, ∞, K)$ and $c(Sp_{VK} (2, ∞, K)) ≤ 3$.

Theorem 4 Assume that $K$ is an infinite field. Then the commutator subgroup of $GSp_{VK} (2, ∞, K)$ coincides with the group $USp_{VK} (2, ∞, K)$ and $c(GSp_{VK} (2, ∞, K)) ≤ 3$.

To prove the results above, we will use the following important theorems.

Theorem 5 (Ref. 5) Assume that $R$ is an associative ring with a commutative group of invertible elements such that $1$ is a sum of two invertible elements. Then the commutator subgroup of the group $GL_{VK} (∞, R)$ coincides with the group $E_{VK} (∞, R)$.

Theorem 6 (Ref. 5) Assume that $R$ is an associative ring with commutative group of invertible elements such that $1$ is a sum of two invertible elements. Then the commutator subgroup of the group $T (∞, R)$ coincides with the group $UT (∞, R)$ and $c(T (∞, R)) ≤ 3$.

2. Furthermore the lower central series of the group $T (∞, R)$ is

$$
γ_0 (T (∞, R)) = T (∞, R),
γ_k (T (∞, R)) = UT (∞, R), \quad \text{for all } k ≥ 1,
$$

i.e., it stabilizes on the group $UT (∞, R)$.

PROOFS OF THE MAIN RESULTS

We first define the following subgroups. Let

$$(I \begin{pmatrix} B \end{pmatrix} \begin{pmatrix} O \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix}) ∈ Sp_{2,∞}^{{\text{fin}}} (R) \quad | \quad B = B'$$

$= U,$

$$(A \begin{pmatrix} O \end{pmatrix} \begin{pmatrix} (A')^{-1} \end{pmatrix}) ∈ Sp_{2,∞}^{{\text{fin}}} (R) \quad | \quad A ∈ UT (∞, R)$$

$= U T,$

$$(A \begin{pmatrix} O \end{pmatrix} \begin{pmatrix} (A')^{-1} \end{pmatrix}) ∈ Sp_{2,∞}^{{\text{fin}}} (R) \quad | \quad A ∈ T (∞, R)$$

$= T,$

$$(A \begin{pmatrix} O \end{pmatrix} \begin{pmatrix} (A')^{-1} \end{pmatrix}) ∈ Sp_{2,∞}^{{\text{fin}}} (R) \quad | \quad A ∈ E_{VK} (∞, R)$$

$= E_{VK},$

$$(A \begin{pmatrix} O \end{pmatrix} \begin{pmatrix} (A')^{-1} \end{pmatrix}) ∈ Sp_{2,∞}^{{\text{fin}}} (R) \quad | \quad A ∈ GL_{VK} (∞, R)$$

$= GL_{VK}.$

It is clear that $U$ and $UT$ are two subgroups of $USp (2, ∞, R)$ and $U$ is a normal subgroup of $USp (2, ∞, R)$. So it is easy to verify the following lemma.

Lemma 1 $USp (2, ∞, R) = U ∗ UT$.

In the same way, we can obtain the following conclusion.

Lemma 2 $TSp (2, ∞, R) = U ∗ T$. Furthermore, $USp_{VK} (2, ∞, R) = U ∗ E_{VK}$ and $Sp_{VK} (2, ∞, R) = U ∗ GL_{VK}$.

To prove Theorem 1 and Theorem 2, we need to use Lemma 3, Corollary 1 and Theorem 7 which will be proved below.

Lemma 3 For any commutative ring $R$ with 1, every element of the group $U$ can be written as a commutator of $USp (2, ∞, R)$.

Proof: For any element $H$ of $U$ we can write

$$H = \begin{pmatrix} I & X \\ O & I \end{pmatrix},$$
where $X = (x_{ij})$ is a row-column-finite matrix in $\text{Mat}_\infty(R)$ with $X' = X$. Let $J$ be an infinite Jordan matrix

$$J = \begin{pmatrix} 1 & 1 & 1 \\
1 & 1 & 1 \\
\vdots & \vdots & \ddots 
\end{pmatrix}.$$ 

All blank entries are equal to 0. For each $B \in \mathbb{U}$, we will find $X = (x_{ij}) \in \text{Mat}_\infty(R)$ such that

$$\begin{pmatrix} I & X \\
O & I \end{pmatrix}^{-1} \begin{pmatrix} J^{-1} & O \\
O & J' \end{pmatrix}^{-1} \begin{pmatrix} I & X \\
O & I \end{pmatrix} \begin{pmatrix} J^{-1} & O \\
O & J' \end{pmatrix} = \begin{pmatrix} I & X \\
O & I \end{pmatrix} \begin{pmatrix} J X J' - X \\
O & I \end{pmatrix} = \begin{pmatrix} I & B \\
O & I \end{pmatrix}.$$ 

Note that $B' = B$ and $X' = X$. We only need to find $x_{ij}$ for all $i \leq j \in \mathbb{N}$. Comparing entries of two sides of $B = J X J' - X$ we obtain for all $i \in \mathbb{N}$

$$b_{ii} = x_{i+1,j} + x_{i+1,j+1} + x_{i+1,i+1}$$

and for all $k \in \mathbb{N}$

$$b_{i,i+k} = x_{i+1,i+k} + x_{i+1,i+k+1},$$

which is equivalent to

$$x_{i+1,i+1} = b_{ii} - 2x_{i+1,i+1},$$

and

$$x_{i+1,i+k} = b_{i,i+k} - x_{i+1,i+k} - x_{i+1,i+1}.$$ 

We can choose the elements in the first row of $X$ to be arbitrary. Then all the elements in first column of $X$ are obtained from $X' = X$. Next, from the equations above, we can find $x_{22}, x_{23} = x_{32}, x_{24} = x_{22}, x_{25} = x_{32}$, and so on. In this way, row by row and column by column, we can find any element $x_{ij}$ of $X$ in finite number of steps. 

**Lemma 4** Assume that $R$ is a commutative ring such that $1$ is a sum of two invertible elements. Then the commutator subgroup of the group $\mathbb{T}$ coincides with the group $\mathbb{U}$. Furthermore, the lower central series of the group $\mathbb{T}$ is

\[ \gamma_0(\mathbb{T}) = \mathbb{T}, \quad \gamma_k(\mathbb{T}) = \mathbb{U}, \quad \text{for all} \quad k \geq 1, \]

i.e., it stabilizes on the group $\mathbb{U}$.

**Proof:** Note that there exists a group isomorphism from $\mathbb{U} \rtimes \mathbb{T}$ to $\mathbb{UT}$:

\[ A \mapsto \begin{pmatrix} A & O \\
O & (A')^{-1} \end{pmatrix}. \]

From **Theorem 6** we can easily obtain the conclusion.

**Corollary 1** Assume that $R$ is a commutative ring such that $1$ is a sum of two invertible elements. Then the commutator subgroup of the group $\mathbb{G}_L$ coincides with the group $\mathbb{E}_V$.

**Theorem 7** Assume that $R$ is a commutative ring such that $1$ is a sum of two invertible elements. Then the commutator subgroup of $\mathbb{T}$$\mathbb{G}_L$ coincides with $\mathbb{U} \rtimes \mathbb{T}$ and $c(\mathbb{T}) \leq 2$. Furthermore, the lower central series of the group $\mathbb{T}$$\mathbb{G}_L$ is

\[ \gamma_0(\mathbb{T}$$\mathbb{G}_L$) = $\mathbb{T}$$\mathbb{G}_L$, \quad \gamma_k(\mathbb{T}$$\mathbb{G}_L$) = $\mathbb{U}$$\mathbb{T}$, \quad \text{for all} \quad k \geq 1, \]

i.e., it stabilizes on the group $\mathbb{U} \rtimes \mathbb{T}$.

**Proof:** For any two elements in $\mathbb{T}$$\mathbb{G}_L$ having the form

\[ \begin{pmatrix} A_1 & B_1 \\
O & \lambda_1(A_1')^{-1} \end{pmatrix}, \quad \begin{pmatrix} A_2 & B_2 \\
O & \lambda_2(A_2')^{-1} \end{pmatrix}, \]

the commutator is

\[ \begin{pmatrix} [A_1,A_2] & B_3 \\
O & ([A_1,A_2])^{-1} \end{pmatrix}, \]

where $\lambda_1, \lambda_2 \in R^*$, $A_1, A_2 \in \mathbb{T}$ and

\[ B_3 = A_1^{-1} A_2^{-1} (A_1 B_1 + \lambda_2^{-1} B_1 (A_1')^{-1}) \]

\[ - (\lambda_1 A_1^{-1} A_2^{-1} B_2 A_2' + A_1^{-1} B_1 A_1' A_2'). \]

From

\[ [\mathbb{T} \rtimes \mathbb{T}], \mathbb{T}] \subseteq \mathbb{UT} \rtimes \mathbb{T}, \]

we can easily obtain

\[ [\mathbb{T}$$\mathbb{G}_L$, $\mathbb{T}$$\mathbb{G}_L$] \subseteq $\mathbb{U}$$\mathbb{T}$, \quad \text{(1)} \]

From **Lemma 1**, $\mathbb{U} \rtimes \mathbb{T} = \mathbb{U} \rtimes \mathbb{T}$. Thus we know that for all elements $G$ in $\mathbb{U} \rtimes \mathbb{T}$, there exists a unique $H$ in $\mathbb{U}$ and $K$ in $\mathbb{UT}$ such that $G = HK$. For each $G \in \mathbb{U} \rtimes \mathbb{T}$, we can write

\[ H = \begin{pmatrix} I & B \\
O & I \end{pmatrix}, \quad K = \begin{pmatrix} A & O \\
O & (A')^{-1} \end{pmatrix}. \]
where $A \in \text{UT}(\infty, R)$ and $B = B' = B_0A' = A'B_0$, $H \in \mathcal{U}$, and $K \in \mathcal{U} T$. From Lemma 3 we know $H$ is a commutator of $\text{USp}(2, \infty, R)$. And from Lemma 4 we obtain that $K$ can be written as a product of two commutators of $T$. So each element in $\text{USp}(2, \infty, R)$ can be written as a product of three commutators of $\text{USp}(2, \infty, R)$.

$$G = \begin{pmatrix} A & B_0 \\ O & (A')^{-1} \end{pmatrix} = \begin{pmatrix} I & B \\ O & I \end{pmatrix} \begin{pmatrix} A & O \\ O & (A')^{-1} \end{pmatrix} = HK,$$

where $A \in \text{UT}(\infty, R)$ and $B = B' = B_0A' = A'B_0, H \in \mathcal{U}$, and $K \in \mathcal{U} T$. From Lemma 3 we know $H$ is a commutator of $\text{USp}(2, \infty, R)$. And from Lemma 4 we obtain that $K$ can be written as a product of two commutators of $T$. So each element in $\text{USp}(2, \infty, R)$ can be written as a product of three commutators of $\text{USp}(2, \infty, R)$.

$$\text{USp}(2, \infty, R) \subseteq [\text{USp}(2, \infty, R), \text{USp}(2, \infty, R)] \subseteq [\text{USp}(2, \infty, R), \text{TGSp}(2, \infty, R)] \subseteq [\text{TGSp}(2, \infty, R), \text{TGSp}(2, \infty, R)].$$

Then the lower central series of $\text{TGSp}(2, \infty, R)$ is

$$\gamma_0(\text{TGSp}(2, \infty, R)) = \text{TGSp}(2, \infty, R),$$
$$\gamma_1(\text{TGSp}(2, \infty, R)) = \text{USp}(2, \infty, R),$$
$$\gamma_2(\text{TGSp}(2, \infty, R)) = [\gamma_1(\text{TGSp}(2, \infty, R)), \text{TGSp}(2, \infty, R)] = \text{USp}(2, \infty, R),$$

and so on.

When we chose $\lambda_1 = \lambda_2 = 1$, the two elements

$$\begin{pmatrix} A_1 & B_1 \\ O & \lambda_1(A_1')^{-1} \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ O & \lambda_2(A_2')^{-1} \end{pmatrix}$$

in $\text{TGSp}(2, \infty, R)$ are also two elements in the group $\text{TSp}(2, \infty, R)$. Then (1) and (2) in the proof of Theorem 7 are changed to

$$[\text{TSp}(2, \infty, R), \text{TSp}(2, \infty, R)] \subseteq \text{USp}(2, \infty, R)$$

and

$$\text{USp}(2, \infty, R) \subseteq [\text{USp}(2, \infty, R), \text{USp}(2, \infty, R)] \subseteq [\text{USp}(2, \infty, R), \text{TSp}(2, \infty, R)] \subseteq [\text{TSp}(2, \infty, R), \text{TSp}(2, \infty, R)],$$

respectively. So

$$\text{USp}(2, \infty, R) = [\text{USp}(2, \infty, R), \text{USp}(2, \infty, R)] = [\text{USp}(2, \infty, R), \text{TSp}(2, \infty, R)] = [\text{TSp}(2, \infty, R), \text{TSp}(2, \infty, R)].$$

And the lower central series of $\text{TSp}(2, \infty, R)$ is

$$\gamma_0(\text{TSp}(2, \infty, R)) = \text{TSp}(2, \infty, R),$$
$$\gamma_1(\text{TSp}(2, \infty, R)) = \text{USp}(2, \infty, R),$$
$$\gamma_2(\text{TSp}(2, \infty, R)) = [\gamma_1(\text{TSp}(2, \infty, R)), \text{TSp}(2, \infty, R)] = \text{USp}(2, \infty, R),$$

Thus we can obtain the following corollary.

**Corollary 2** Assume that $R$ is a commutative ring such that 1 is a sum of two invertible elements. Then the commutator subgroup of $\text{TSp}(2, \infty, R)$ coincides with the group $\text{USp}(2, \infty, R)$ and $c(\text{TSp}(2, \infty, R)) \leq 3$. Furthermore, the lower central series of the group $\text{TSp}(2, \infty, R)$ is

$$\gamma_0(\text{TSp}(2, \infty, R)) = \text{TSp}(2, \infty, R),$$
$$\gamma_k(\text{TSp}(2, \infty, R)) = \text{USp}(2, \infty, R), \quad \forall k \geq 1,$$

i.e., it stabilizes on the group $\text{USp}(2, \infty, R)$.

Now we finish the proof of Theorem 1 and Theorem 2. Proof: Using the method in Theorem 7, we can easily obtain

$$[\text{GSp}_V(2, \infty, R), \text{GSp}_V(2, \infty, R)] \subseteq \text{USp}_V(2, \infty, R)$$

and

$$[\text{Sp}_V(2, \infty, R), \text{Sp}_V(2, \infty, R)] \subseteq \text{USp}_V(2, \infty, R),$$

which are similar to (1). From Lemma 2 we know that $\text{USp}_V(2, \infty, R) = U \rtimes \mathcal{E}_V$. So for each $G \in \text{USp}_V(2, \infty, R)$, there exists a decomposition

$$G = \begin{pmatrix} I & B \\ O & I \end{pmatrix} \begin{pmatrix} A & O \\ O & (A')^{-1} \end{pmatrix},$$

where

$$\begin{pmatrix} I & B \\ O & I \end{pmatrix} \in U, \quad \begin{pmatrix} A & O \\ O & (A')^{-1} \end{pmatrix} \in \mathcal{E}_V.$$

Then from Lemma 3 and Corollary 1 we obtain

$$[\text{GSp}_V(2, \infty, R), \text{GSp}_V(2, \infty, R)] \subseteq \text{USp}_V(2, \infty, R)$$

and

$$[\text{Sp}_V(2, \infty, R), \text{Sp}_V(2, \infty, R)] \subseteq \text{USp}_V(2, \infty, R).$$

Thus we obtain the conclusions of Theorem 1 and Theorem 2. □

To prove Theorem 3 and Theorem 4, Corollary 3 of Lemma 5 will be used. Next we show Lemma 5 (which is also proved in Ref. 9 in a different way) and two corollaries.
Lemma 5 Assume that $R$ is an associative ring with an infinite field $K$ in the centre $Z(R)$ of $R$. Every element $C \in UT(\infty, R)$ is a commutator of $T(\infty, R)$.

Proof: Let $A = \text{diag}(a_1, a_2, \ldots, a_n, \ldots)$ be a diagonal matrix with pairwise distinct non-zero elements $a_1, a_2, \ldots, a_n, \ldots$ of $K$ in its diagonal. We will find $X = (x_{ij}) \in UT(\infty, R)$ such that $C = X^{-1}A^{-1}AX$. Since every unitriangular matrix is invertible, this equation is equivalent to

$$AXC =XA.$$

We use induction on $n = j - i$ (i.e., $n$ is a number of the superdiagonal of $X$ above the main diagonal). When $n = 1$, comparing the $(i, i+1)$ entries of both sides of the matrix equation we can obtain

$$a_i(c_{i,i+1} + x_{i,i+1}) = a_{i+1}x_{i,i+1},$$

which implies

$$(a_{i+1} - a_i)x_{i,i+1} = a_i c_{i,i+1}.$$

Now we suppose that $x_{ij}$ for all $j - i < n$ has been found. Comparing the $(i, i+n)$ entries of both sides of the matrix equation we obtain

$$a_i(c_{i,i+n} + x_{i,i+1}c_{i+1,i+n} + x_{i,i+2}c_{i+2,i+n} + \cdots + x_{i,i+n-1}c_{i+n-1,i+n} + x_{i,i+n}) = a_{i+n}x_{i,i+n}$$

which is equivalent to

$$(a_{i+n} - a_i)x_{i,i+n} = a_i(c_{i,i+n} + x_{i,i+1}c_{i+1,i+n} + x_{i,i+2}c_{i+2,i+n} + \cdots + x_{i,i+n-1}c_{i+n-1,i+n}).$$

Thus we can find $x_{i,i+n}$ for all $i \in \mathbb{N}$. □

Corollary 3 Assume that $K$ is an infinite field. Then every element $C \in UT(\infty, K)$ is a commutator of $T(\infty, K)$.

Corollary 4 Assume that $K$ is an infinite field. Then the commutator subgroup of $\text{Sp}_N(2, \infty, K)$ coincides with the group $\text{USp}_N(2, \infty, K)$ and $\text{c(TSp}_N(2, \infty, K)) \leq 2$. Furthermore the lower central series of the group $T\text{Sp}_N(2, \infty, K)$ is

$$\gamma_0(\text{Sp}_N(2, \infty, K)) = \text{USp}_N(2, \infty, K),$$

$$\gamma_k(\text{Sp}_N(2, \infty, K)) = \text{USp}_N(2, \infty, K), \quad \forall k \geq 1,$$

i.e., it stabilizes on the group $\text{USp}_N(2, \infty, K)$. Now we finish the proof of Theorem 3 and Theorem 4. Proof: From Theorem 1 and Theorem 2, we know that the commutator subgroup of $\text{Sp}_N(2, \infty, K)$ coincides with $\text{USp}_N(2, \infty, K)$. So does the commutator subgroup of $\text{GSp}_N(2, \infty, K)$.

Next we will determine the commutator width of $\text{Sp}_N(2, \infty, K)$ and $\text{GSp}_N(2, \infty, K)$.

From Lemma 2 we know that every element of $\text{USp}_N(2, \infty, K)$ can be written as a product of an element of $U$ and an element of $E_{N,K}$. Each element of $E_{N,K}(\infty, K)$ has the following decomposition:

$$\begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} \end{pmatrix} = \begin{pmatrix} I_n & M_{12} \\ O & M_{22} \end{pmatrix} \begin{pmatrix} M_{11} & O \\ O & I \end{pmatrix},$$

where $M_{11} \in E(n, K)$ and $M_{11} \in UT(\infty, K)$. From Theorems 1 and 2 of Ref. 10, we know that for any field $K$ except $\mathbb{F}_2$ and $\mathbb{F}_3$, every element of $\text{SL}(n, K)$ (coinciding with $E(n, K)$) is a commutator of $\text{GL}(n, K)$. Note that

$$\begin{pmatrix} A_1 & O \\ O & I \end{pmatrix}, \begin{pmatrix} A_2 & O \\ O & I \end{pmatrix} = \begin{pmatrix} [A_1,A_2] & O \\ O & I \end{pmatrix},$$

and the matrix

$$\begin{pmatrix} M_{11} & O \\ O & I \end{pmatrix}$$

is a commutator. From Corollary 3 it follows that

$$\begin{pmatrix} I_n & M_{12} \\ O & M_{22} \end{pmatrix}$$

is a commutator of $\text{GL}(n, K)$. So

$$\begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} \end{pmatrix}$$

is a product of at most 2 commutators. Note that there is a group isomorphism from $E_{N,K}(\infty, R)$ to $\mathbb{B}_{VK}$:

$$A \mapsto \begin{pmatrix} A & O \\ O & (A')^{-1} \end{pmatrix},$$

and any element of $E_{N,K}$ is a product of at most 2 commutators. Finally, from Lemma 3, every element of $\text{USp}_N(2, \infty, K)$ can be written as a product of at most 3 commutators. □

REFERENCES


